CLASSIFICATION OF SIMPLY CONNECTED COHOMOGENEITY ONE MANIFOLDS IN LOWER DIMENSIONS

A Thesis by

Sara Elizabeth Muzzy

Bachelor of Science, Wichita State University, 2015

Submitted to the Department of Mathematics, Statistics, and Physics
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
the requirements for the degree of
Master of Science

May 2017

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

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Catherine Searle, Committee Chair

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Mark Walsh, Committee Member

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Phil Parker, Committee Member

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Rajeev Nair, Committee Member

ABSTRACT

A cohomogeneity one manifold is a topological manifold with an effective topological action of a compact Lie group whose quotient is one dimensional. Cohomogeneity one manifolds were introduced by Mostert in 1957, however Mostert's original structure theorem had two omissions. The first omission was corrected by Neumann in 1967 but the second omission was not corrected until 2015 by Galaz-Garcia and Zarei. In this paper we will examine the revised structure theorems for cohomogeneity one manifolds and compile all work done by Parker, Neumann, Mostert, Hoelscher, Galaz-Garcia and Zarei on the equivariant classification of closed, simply connected cohomogeneity one manifolds in dimensions up to 7.

ACKNOWLEDGEMENTS

My success required endless guidance and assistance and therefore also belongs to many others. Foremost I respect and thank Dr.Catherine Searle for having as much patience and wisdom as she has mathematical knowledge. She believed in me even when I did not believe in myself. My success is also shared by my beloved husband, Ryan. I could not ask for a more supportive partner and this would not have been possible without him. Finally, I must give thanks to my beautiful children, Aiden and Emmy. They are my inspiration and motivation. They sacrificed for this just as I did and I hope this makes them proud.

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CHAPTER 1

Introduction

There is a long history, in mathematics, of studying spaces which have "large" symmetries. One motivation for understanding spaces with "large" symmetries lies in the fact that manifolds with positive sectional curvature admit large isometry groups, in other words they have "large" symmetries. Currently there are few known examples of such manifolds. By understanding spaces with "large" symmetries, the hope is to discover techniques to construct new examples of positive curvature. Homogeneous spaces with positive curvature were classified in the 70's by work of Berger [4], Berard Bergery [3], Wallach [37] and Aloff and Wallach [36]. A next logical step to take is to study manifolds on which a groups acts isometrically with a one-dimensional quotient. These are called cohomogeneity one manifolds.

One definition of a "large" action is one with low cohomogeneity. The cohomogeneity of a G-action on a manifold, M, is defined to be the dimension of the quotient space, M/G. From this perspective, cohomogeneity one actions can be thought of as the second "largest" symmetric actions, where the "largest" symmetric action is a transitive action, as the orbit space is 0-dimensional. Cohomogeneity one manifolds were first introduced by Mostert in 1957 [27]. An interesting question to consider is how big the class of cohomogeneity one manifolds is.

Such manifolds were classified in dimensions 4 and lower by work of Neumann in 1957 [28] and Parker in 1986 [31], using the original structure theorem from Mostert's 1957 paper, "On a Compact Lie Group Acting on a Manifold." In 2007, Hoelscher classified simply connected cohomogeneity one manifolds in dimensions 5, 6, and 7 as and also corrected omissions in dimension 4 [19]. In Neumann's classification in dimension 3 [28] he also corrects one of two omissions in Mostert's original structure theorem. The second omission was corrected in 2015 by Galaz-Garcia and Zarei [14]. This resulted in correcting omissions in dimension 5, 6, and 7.

The goal of this thesis is to introduce the readers to cohomogeneity one manifolds in an accessible way and to compile the above classifications in an organized manner. We will begin by offering an introduction to transformation groups and some basics on the general theory of G-spaces. For further background information on G-spaces, the reader is encouraged to consult, "Introduction to Compact Transformation Groups" by Bredon [5]. In chapter 2, the reader will find preliminary material specifically on cohomogeneity one manifolds, as well as structure information, including, but not limited to, the most recent structure theorem by Galaz-Garcia and Zarei [14]. This most recent structure theorem includes all corrections made to the original theorem of Mostert. The proof of the Galaz-Garcia and Zarei theorem [14] can also be found in this section. In Chapter 3 the reader will find each dimension from $2-7$ classified, individually, for simply connected cohomogeneity one manifolds. Each section in Chapter 3 will end with a table of all of the possible group diagrams of the given manifolds. We also include an appendix with all such tables.

CHAPTER 2

Preliminaries

The following definitions, propositions, corollaries, and theorems will serve as a brief review of all of the background material required to understand the work in later chapters. The definitions and notation follow from Lee [23] and Bredon [5], which will serve as valuable resources, should the reader need further explanation of any preliminary work.

2.1 Transformation Groups

In the theory of transformation groups we consider symmetries of objects such as topological spaces, topological manifolds, polyhedra, etc. More generally, one may look at subgroups of the full groups of symmetries. In the following section we will review the basics of transformation groups, including; group actions, equivariant maps, isometry groups, orbits, orbit spaces, and Lie groups.

2.1.1 Group Actions

In this section, we present the basics of group actions. We begin with the definition of a topological group.

Definition 2.1.1. *A* ***topological group*** *is a Hausdorff space, G, together with a continuous multiplication map, $m : G \times G \to G$ with $(g, h) \mapsto gh$ for all $g, h \in G$, which makes G into a group and such that the map $i : G \to G$, with $g \mapsto g^{-1}$, is continuous. For $g \in G$, there is a left translation map $L_g : G \to G$ defined by $L_g(h) = gh$. Then $L_{gh} = L_g L_h$ and $L_{g^{-1}} = (L_g)^{-1}$ and each L_g is a homeomorphism of G onto itself.*

Let G be a topological group, X a Hausdorff topological space and $\Theta : G \times X \to X$ a map such that,

i) $\Theta(g, \Theta(h, x)) = \Theta(gh, x)$ for all $g, h \in G$ and $x \in X$

ii) $\Theta(e, x) = x$ for all $x \in X$ where $\{e\}$ is the identity of G.

Definition 2.1.2. *An* ***action*** *is a map* Θ *of* G *on* X. *The space* X *together with a given action of* Θ *of* G *is called a* G*-space. We will consider an action of this kind to be a left group action and* X *a left* G*-space. An analogous notion of right group action will be useful on a few occasions where* X *is a right* G*-space. Consider* $g(x)$ *for* $\Theta(g, x)$, *then we can simplify the conditions above to:*

i) $(g(hx)) = (gh)(x)$

ii) $e(x) = x$

We can also say that G acts on X if there is a continuous homomorphism

$$\theta : G \to \text{Homeo}(X)$$

where Homeo(X) denotes the group under composition of all homeomorphism of X onto itself. The kernel of this homomorphism θ will be called the *kernel of the action* Θ denoted by

$$\ker(\Theta) = \{g \in G \mid g(x) = x \text{ for all } x \in X\}.$$

This is a normal subgroup of G and is closed in G. Therefore the quotient group, $\bar{G} = G/\ker(\Theta)$, is another group by the First Isomorphism Theorem for Groups and it induces an action of $\bar{G}$ on X. This leads us to define the following types of actions, with which we will work exclusively.

Definition 2.1.3. ***Effective Action*** *The action of* G *on* M *is effective if the kernel of the action is equal to the identity. In other words, only the identity acts as the identity.*

Definition 2.1.4. ***Almost Effective Action*** *The action of* G *on* M *is almost effective if the kernel of the action is finite.*

There are various other types of actions which merit special consideration and which we define next.

Definition 2.1.5. ***Free Action*** *The action of G on M is* free *if $G_p = \{e\}$ for all $p \in M$.*

In other words, the only element which fixes anything is that which fixes everything. The study of free actions is related to covering space theory and, more generally to the the study of fiber bundles, which can be seen in work of Edwards [12].

Definition 2.1.6. ***Almost Free Action*** *The action of G on M is* almost free *is G_p is finite for all $p \in M$.*

Definition 2.1.7. ***Semi-Free Action*** *The action of G on M is* semifree *if G_p is $\{e\}$ or G for all $p \in M$.*

In other words, for a semi-free action, for each point in the space, either the entire group or only a single element can act as the identity element. Therefore the isotropy groups are either trivial or the entire group.

Definition 2.1.8. ***Transitive Action*** *The action of G on M is transitive is $G \cdot p = M$ for some $p \in M$*

An action which is transitive is equivalent to the action having only one orbit. Dodson and Parker give the following example in [10] of a transitive action which leads us to the definition of a homogeneous space.

Example 2.1.9. *If a compact G acts* transitively *on a Hausdorff X then, for all $x \in X$*

$$X/G \cong G/G_x.$$

In this case X is called a ***homogeneous space*** for G because for all $x, y \in X$ there is a homeomorphism $f : X \to X$ such that $f(x) = y$.

2.1.2 Equivariant Maps and Isotropy Groups

An ***equivariant map*** $\varphi : X \to Y$ between G-spaces is a map which commutes with the group actions. That is

$$\varphi(g(x)) = g(\varphi(x)) \text{ for all } g \in G, \ x \in X.$$

If φ is a homeomorphism, then φ is an ***equivalence*** of G-spaces. Two actions are essentially the same if they differ only by an automorphism of G. We can say that two G-spaces X and Y are ***weakly equivalent*** if there is an automorphism α of G and a homeomorphism $\varphi : X \to Y$ with

$$\varphi(g(x)) = \alpha(g)(\varphi(x)) \text{ for all } g \in G, \ x \in X.$$

Let X be a G-space and let $x \in X$. The set of elements of G leaving x fixed is a closed subgroup of G given by

$$G_x = \{g \in G \mid g(x) = x\}.$$

Regard G_x as the ***isotropy group*** of G at x. A point x in a G-space X is said to be a ***fixed point*** when $G_x = G$. We denote the subspace of fixed points of G on X by

$$X^G = \{x \in X \mid g(x) = x \text{ for all } g \in G\}.$$

We also denote this set by $\text{Fix}(X, G)$. Note that the kernel of an action is just $\cap_{x \in X} G_x$. We can now rephrase our earlier definition of an effective action. An action is ***effective*** if each $g \neq \{e\}$ in G moves at least one point.

2.1.3 Orbits and Orbit Spaces

Orbits and isotropy groups are closely related. In fact, there is an inverse relationship between permutations that fix an element of a group and the orbits of that element.

In the following section we will explore the basics of orbits and orbit space.

If X is a G-space and $x \in X$, then the subspace of x under G,

$$G(x) = \{g(x) \in X \mid g \in G\},$$

is called an ***orbit***.

That is, $G(x)$ is the set of all images of x under the action by elements of G. If $g(x) = h(y)$ for some $g, h \in G$ and $x, y \in X$, then for any $g' \in G$, $g'(x) = g'g^{-1}g(x) = g'g^{-1}h(y) \in G(y)$ such that $G(x) \subset G(y)$. Conversely, using the same method, we have that

$G(y) \subset G(x)$. Hence the orbits $G(x)$ and $G(y)$ of any two points $x, y \in X$ are either equal or disjoint. Therefore, the sets of orbits partition X.

Let $X/G = X^*$ denote the set whose elements are the orbits $G(x)$ of G on X. Let $\pi : X \to X/G$ denote the natural map taking x into its orbit x^*, that is, $\pi(x) = x^*$. Then X/G is endowed with the quotient topology, $U \subset X/G$ is open if and only if $\pi^{-1}(U)$ is open in X, and X/G is called the ***orbit space*** of X with respect to G. The ***cohomogeneity*** of the action is defined to be the dimension of the orbit space.

Notice $G(x) \cong G/G_x$ where G/G_x is the ***isotropy subgroup***. Since isotropy subgroups for points in the same orbit are conjugate, we will say that the orbit $G(x)$ has *orbit type* G/H and ***isotropy type*** H, where H is conjugate to G_x. Orbits will be either principal, exceptional or singular, depending on the relative size of their isotropy subgroups.

Definition 2.1.10. ***Principal orbits*** *correspond to those orbits with the smallest possible isotropy subgroup and each principal orbit has the same dimension.*

Definition 2.1.11. ***Exceptional orbits*** *correspond to orbits where its isotropy subgroup is a finite extension of the principal isotropy subgroup.*

Definition 2.1.12. ***Singular orbits*** *correspond to orbits where its isotropy subgroup is of strictly larger dimension than that of the principal isotropy subgroup.*

2.1.4 Lie Groups

The previous definitions apply to topological groups but we can expand these notions to the idea of Lie groups. Suppose M is a topological space. We say that M is a ***topological manifold*** of dimension n if it has the following properties:

i) M is a Hausdorff space.

ii) M is second countable, that is, there exists a countable basis for the topology of M.

iii) M is locally Euclidean of dimension n. Each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 2.1.13. *A **chart** on M is a pair (U, φ) where U is an open subset of M and $\varphi : U \to V$ where U is a homeomorphism from U to an open subset $V = \varphi(U) \subseteq \mathbb{R}^n$. Each point $p \in M$ is contained in the domain of some chart (U, φ).*

Definition 2.1.14. *An **atlas**, $\mathcal{A}$, for M is a collection of charts whose domains cover M. We can call $\mathcal{A}$ smooth if any two charts in $\mathcal{A}$ are smoothly compatible with each other and $\mathcal{A}$ is maximal if it is not properly contained in any larger smooth atlas. If M is a topological manifold, a smooth structure on M is a maximal smooth atlas.*

Hence, a *smooth manifold* is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M. We are now ready to define a Lie group.

Definition 2.1.15. *A **Lie group** is a smooth manifold, G, with a group structure, such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ given by*

$$(g, h) \mapsto gh \qquad g \mapsto g^{-1}$$

are both smooth.

The following are three simple examples of Lie Groups.

Example 2.1.16. *The circle* $\boldsymbol{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ *is a smooth manifold and a group under complex multiplication. Multiplication and inversion have the smooth coordinate expressions* $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$ *and* $\theta \mapsto -\theta$. *Hence* $\boldsymbol{S}^1$ *is a Lie group, called the circle group.*

Example 2.1.17. *The* n*-torus* $\boldsymbol{T}^n = \boldsymbol{S}^1 \times \cdots \times \boldsymbol{S}^1$ *is a product of copies of the circle and is therefore an* n*-dimensional abelian Lie group.*

Example 2.1.18. *An interesting Lie group is* S^3, *as unit quarternions. The Lie group* $SU(2) \cong S^3$. *This can be seen by parametrizing the matrices in* $SU(2)$, *however will we see this another way. First consider how we define the complex numbers,* $\mathbb{C} = \{a + b\imath \mid a, b \in \mathbb{R}$, *where* $\imath$ *is a completely new object which is not found in* $\mathbb{R}$. *We are also able to define multiplication with complex numbers using the relation,* $\imath^2 = -1$. *We can define the* quaternions, $\mathbb{H} \cong \mathbb{R}^4$, *in a similar way. Let* $\imath, \jmath$ *and* k *be distinct elements which, again, can not be found in* $\mathbb{R}$ *and define the following relations,* $\imath^2 = \jmath^2 = k^2 = -1$ *and* $\imath\jmath = -\jmath\imath = k$. *Therefore we can now define,*

$$\mathbb{H} = \{a + b\imath + c\jmath + dk \mid a, b, c, d \in \mathbb{R}\},$$

where multiplication can be defined by distribution terms and using our relations on $\imath, \jmath$ *and* k. *Next we will define the norm in the following way,* $|\cdot|: \mathbb{H} \to \mathbb{R} \geq 0$ *by,*

$$| a + b\imath + c\jmath + dk |^2 = a^2 + b^2 + c^2 + d^2.$$

We find that $| q_1 q_2 | = | q_1 | \, | q_2 |$, *for all* q_1 *and* $q_2 \in \mathbb{H}$. *Therefore the norm is multiplicative and so*

$$S^3 = \{q \in \mathbb{H} \mid | q | = 1\}$$

is closed under multiplication. We also find the inverse of $q = a + b\imath + c\jmath + dk$ *is the conjugate,* $\bar{q} = a - b\imath - c\jmath - dk$ *which means* S^3 *is a group. It has the usual smooth structure that comes from identifying it as a hyper-surface in* $\mathbb{R}^4 \cong \mathbb{H}$. *The multiplication and inverse maps are smooth and therefore we have shown this is a Lie group.*

In Example 2.1.18 we showed $SU(2) \cong S^3$, note that $SU(2)$ double covers $SO(3) \cong \mathbb{R}P^3$ and both cover the *Poincaré homology spheres*, defined here below.

Definition 2.1.19. *The **Poincaré homology sphere**, $\mathbb{P}^3$, is a homogeneous space for the Lie groups $SU(2)$ and $SO(3)$. We can write $\mathbb{P}^3 \approx SU(2)/I^* \approx SO(3)/I$, where $I*$ is the binary icosahedral group and I is the icosahedral group [37].*

One of the most important applications of Lie groups is that of actions by Lie groups on manifolds.

The following is a list of important Lie groups which will appear in later chapters;

- $O(n)$, the orthogonal group: real orthogonal matrices
- $SO(n)$, special orthogonal group: real orthogonal matrices with determinant $= 1$
- $Spin(n)$, spin group: the double cover of $SO(n)$
- $Sp(n)$, compact symmetric group: quaternionic $n \times n$ unitary matrices
- $U(n)$, unitary group: complex $n \times n$ unitary matrices
- $SU(n)$, special unitary group: complex $n \times n$ unitary matrices with determinant $= 1$

In the following section, unless otherwise stated, all manifolds are assumed to be closed, connected and compact and all actions are assumed to be locally smooth and effective.

2.2 General Theory of G-Spaces

The theory of compact transformation groups can be considered a generalization of the theory of fiber bundles. In the following section we will review fiber bundles and important theorems related to fiber bundles which will be useful for work in later chapters, as well as the general theory of G-spaces.

2.2.1 Tubes and Slices

For the remainder of this thesis, we will assume G is a compact topological group.

Definition 2.2.1. *A* ***cross-section*** *for the orbit map* $\pi\colon X \to X/G$ *is a continuous map* $\sigma : X/G \to X$ *such that* $\pi \circ \sigma$ *is the identity on* X/G.

A group action, $X \times G \to X$, has ***slices*** if projection onto the orbit space $p\colon X \to X/G$ has ***local sections***. That is, around every $x \in X/G$ there is a neighborhood U and a continuous map

$$s\colon U \to E\colon p \circ s = id\colon U \to U.$$

Theorem 2.2.2. *[5] Let* X *be a* G*-space with* G *compact. Let* C *be a closed subset of* X *touching each orbit in exactly one point. Then the map* $\sigma : X/G \to X$ *defined by* $\sigma(x^*) = G(x) \cap C$ *is a cross-section. Conversely, the image of a cross-section is closed in* X.

Proof. We will show that σ is continuous. Let $A \subset C$ be closed. Then $\sigma^{-1}(A) = \pi(A)$ is closed. For the converse, let $C = \sigma(X/G)$ and let $\{x_\alpha\}$ be a net in C converging to $x \in X$. Then $\lim \pi(x_\alpha) = \pi(x)$ such that $x = \lim x_\alpha = \lim \sigma\pi(x_\alpha) = \sigma\pi(x) \in C$ and hence C is closed. □

Consider the ***twisted product***, $G \times_H A$, where H is a compact closed subgroup of G, A is a left H-space and G is a right H-space. H acts on $G \times A$ by $(h, (g, a)) \mapsto (gh^{-1}, ha)$ and $G \times_H A$ is the orbit space of the H-action. The H-orbit of (g, a) will be denoted by $[g, a]$, so that $[g, a] = [g', a']$ if and only if there is a $h \in H$ with $g' = gh^{-1}$ and $a' = ha$. We define a G-action on $G \times_H A$ by putting

$$g'[g,a] = [g'g,a].$$

If we want to find the isotropy group at a point of the twisted product $G \times_H A$, we look at points of the form $[e,a]$. Then $g \in G_{[e,a]}$ if and only if $[e,a] = g[e,a] = [g,a]$. Hence, for some $h \in H$, we have $(h^{-1}, h(a)) = (g,a)$, that is, $g \in H_a$. We find that $G_{[e,a]} = H_a$ in $G \times_H A$.

Let X be a G-space with G compact and let $P \subset X$ be an orbit of type G/H. Then a ***tube*** about P is a G-equivariant homeomorphism

$$\varphi : G \times_H A \to X$$

into an open neighborhood of P in X, where A is some space on which H acts. Every G-orbit in $G \times_H A$ passes through a point of the form $[e,a]$. Letting $a \in A$ such that $\varphi[e,a] \in P$. Let $x = \varphi[e,a]$ so that $P = G(x)$. Then by the above discussion, $G_x = G_{[e,a]} = H_a \subset H$ and since G_x is conjugate to H, by assumption (since P is an orbit of type G/H) we have $G_x = H_a = H$. Therefore such a point $a \in A$ is fixed under H.

Definition 2.2.3. *Let $x \in X$, a G-space. Let $x \in S \subset X$ be such that $G_x(S) = S$. Then S is called a* ***slice*** *at x if the map $G \times_{G_x} S \to X$ taking $[g,s] \mapsto gs$ is a tube about $G(x)$.*

We now recall the following theorem without proof from [25] about the existence of tubes (see also [5]).

Theorem 2.2.4. *[25] Let G be a compact Lie group and X be a G-space which is completely regular, then there exists a tube about each of its points.*

Note that the existence of a tube is equivalent to the existence of a slice and Theorem 2.2.4 is often referred to as the Slice Theorem. It is a valuable tool in the theory of transformation groups since it allows one to explicitly describe the structure of an action in a neighborhood of a given orbit. As we will see in later chapters, the slice theorem will allow us to directly recover our cohomogeneity one manifold, M, from a description of the quotient space M/G and the principal and singular isotropy subgroups.

Here below we include a theorem from [5] which gives us various equivalent characterizations of a slice.

Theorem 2.2.5. *[5] Let G be a topological transformation group of a Hausdorff space X . A subspace S of X is called a slice at a point $x \in S$ if the following conditions hold*

i) S is invariant under the stabilizer G_x of X

ii) the union, $G(S)$, of all orbits intersecting S is an open neighborhood of the orbit $G(x)$ of X

iii) if $G \times_{G_x} S$ is the homogeneous fiber space over G/G_x with fiber S, then the equivariant mapping $\varphi : G \times_{G_x} S \to X$, which is uniquely defined by the condition that its restriction to the fiber S over G_x is the identity mapping $S \to S$, is a homeomorphism of $G \times_{G_x} S$ onto $G(S)$.

.

2.2.2 Fiber Bundles

Let G be a G-space acting effectively on a space F. Let B be a connected space with base point $b \in B$ and $p : E \to B$ a continuous map.

Definition 2.2.6. *A **fiber bundle** is a map p with fiber F, base space B, and total space X satisfying the following:*

i) $p^{-1}(b) = F$

ii) $p : E \to B$ is surjective

iii) For every point $x \in B$, there exists a neighborhood U of x in B and a fiber preserving homeomorphism $\varphi : p^{-1}(U) \to U \times F$ called a local trivialization of X over U such that the diagram commute:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle proj} & \\ & U & \end{array}$$

Given a fiber bundle $p : E \to B$ with fiber, F, the space B is called the base space, the space E is called the total space, and this is commonly denoted by a triple (F, E, B). Fiber bundles and fibrations encode topological and geometric information about the spaces over which they are defined.

A fiber bundle with base space B and fiber F can be viewed as a parametrized family of objects, each homeomorphic to F, where the family is parametrized by points in B.

An important case of fiber bundles are principal bundles. ***Principal bundles*** are essentially parametrized families of topological groups, often Lie groups. They are more simple in the sense that the fiber $F = G$. A ***principal*** G***-bundle***, where G is a G-space, is a fiber bundle $p : P \to B$ together with a right action $P \times G \to P$, such that G preserves the fibers of P and acts freely and transitively on them. Since the group action preserves the fibers of p and acts transitively, it follows that the orbits of the G-action are these fibers and the orbit space P/G is homeomorphic to the base space X.

The following are fundamental results on fiber bundles. The first of which is originally due to A. Gleason [16]; its proof can be found in work of Steenrod [35].

Theorem 2.2.7. *[7] Let E be a smooth manifold, having a free, smooth G-action, where G is a compact Lie group. Then the action has slices. In particular, the projection map $p : E \to E/G$ defines a principal G-bundle.*

The following is one of the earlier results in fiber bundle theory, appearing originally in H. Samelson's thesis [33].

Corollary 2.2.8. *[7] Let G be a Lie group, and let $H < G$ be a compact subgroup. Then the projection onto the orbit space, $p : G \to G/H$, is a principal H-bundle.*

Principal bundles define other fiber bundles in the presence of group actions. Namely, suppose $p : E \to B$ be a principal G-bundle and F is a space with a right group action. Then the product space, $E \times F$, has the "diagonal" group action, $(e, f)g = (eg, fg)$. Consider the orbit space, $E \times_G F = (E \times F)/G$. Then the induced projection map, $p : E \times_G F \to B$, is a locally trivial fibration with fiber F.

For example we have the following important class of fiber bundles.

Proposition 2.2.9. *[7] Let G be a compact Lie group and $K < H < G$ be closed subgroups. Then the projection map of coset spaces, $p : G/K \to G/H$, is a locally trivial fibration with fiber H/K.*

Proof. Observe that $G/K \cong G \times_H H/K$ where H acts on H/K in the natural way. Moreover the projection map $p : G/K \to G/H$ is the projection which can be viewed as the projection $G/K = G \times_H H/K \to G/H$ and so is the H/K-fiber bundle induced by the H-principal bundle $G \to G/H$ via the action of H on the coset space H/K. □

The following lemma can be found in work of Bredon [5], Steenrod [35], and Husemoller [20] and will be necessary in future proofs.

Lemma 2.2.10. *Suppose that $p : X \to B$ is a bundle with fiber F and a structure group K. Suppose that F is also a left G-space and that the actions of G and K commute. Then there exists a unique G-action on X covering the trivial action on B such that each chart $\phi : F \times U \to p^{-1}(U)$ is equivariant, where G acts on $F \times U$ by $(g, (f, u)) \mapsto (gf, u))$.*

Proof. Let θ denote the G-action. The action is defined by the equivariance of the charts and therefore it suffices to prove that it is independent of the choice of chart over U. In other words, we need to show that each $\phi^{-1}\psi : F \times U \to F \times U$ is equivariant, but,

$$g(\phi^{-1}\psi(f,u)) = g(f \cdot \theta(u), u)$$

$$= (g(f \cdot \theta(u)), u)$$

$$((gf) \cdot \theta(u), u)$$

$$= \phi^{-1}\psi(gf, u) = \phi^{-1}\psi(g(f,u)).$$

□

CHAPTER 3

Cohomogeneity One Manifolds

3.1 Introduction to Cohomogeneity One Manifolds

In the following chapter we will review basics of cohomogeneity one manifolds. We will start with a more formal definition of cohomogeneity one manifold.

Definition 3.1.1. *Let M be a topological n-manifold with a topological action of a compact connected Lie group G. The action is of cohomogeneity one if the orbit space is one-dimensional or, equivalently, if there exists an orbit of dimension $n-1$. A topological manifold with a topological action of cohomogeneity one is a* ***cohomogeneity one manifold****.*

Lemma 3.1.2. *[27] If M is a compact manifold, then the orbit space, M/G, is the circle S^1, or the interval, I. The only singular or exceptional orbits of the action map to the boundary points of I.*

To prove this lemma, it is enough to prove the following special case from a lemma of [5], which relies on compactness and the slice theorem.

Lemma 3.1.3. *[5] Let M^n be an n-dimensional G-manifold with principal orbits of dimension d. If $n-d \leq 1$ then M/G is a manifold (with boundary)*

Proof. Let $l = n - d$ be the dimension of the orbit space. We will examine the structure of M/G by induction on l.

If we consider a linear tube in M around an orbit G/K, we find it will have the form,

$$G \times_K V, (G \times_K V)^* = V^*.$$

This V^* is an open cone over S^*, where S is the sphere in V.

The dim M/G= dim $V^* =$ dim $S^* + 1$, since coning over an object increases the dimension by 1.

If $l = 0$ then M/G is a discrete set of points. If M is a sphere then M/G is one or two points. Therefore when $l = 1$, which is the desired case, then M/G has the local structure of the open cone over one or two points. The cone over one point is a half-open interval, and as such, is a manifold with boundary. The cone over two points is an open interval, and therefore it is a manifold without boundary. □

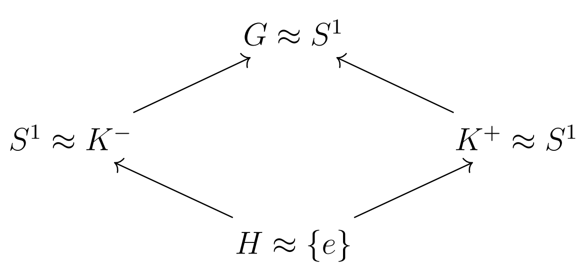
We can now prove Lemma 3.1.3.

Lemma 3.1.3. To prove the first statement, note that from the classification of 1-manifolds (see, for example, the Manifold Atlas [24]), the only compact, connected 1-dimensional manifolds are I or S^1. To prove the second statement, note that if there are singular or exceptional orbits, then the action of the isotropy group must be transitive on the normal sphere to the orbit and hence by its image is a single point. Since the orbit space near the image of the singular or exceptional orbit is a cone over the image of its normal sphere, we see that such orbits map to boundary points.

□

When the orbit space is homeomorphic to to an interval the resulting cohomogeneity one manifold, a convenient way to describe the manifold and the action is to use the group diagram, shown below. However, a single cohomogeneity one manifold may have different diagrams.

When the orbit space is an interval we can picture this group diagram in the following way.



where,

$$\begin{array}{ccccc} & & G \approx S^1 & & \\ & \nearrow & & \nwarrow & \\ S^1 \approx K^- & & & & K^+ \approx S^1 \\ & \nwarrow & & \nearrow & \\ & & H \approx \{e\} & & \end{array}$$

Definition 3.1.4. *Two group diagrams are called* ***equivalent*** *if they determine the same cohomogeneity one manifold up to equivariant diffeomorphism.*

The following lemma from Grove, Wilking, and Ziller defines when two group diagrams are equivalent.

Lemma 3.1.5. *[17] Two group diagrams (G, H, K^-, K^+) and $(\tilde{G}, \tilde{H}, \tilde{K^-}, \tilde{K^+})$ and are equivalent if and only if after possibly switching the roles of K^- and K^+, the following holds: There exist elements $b \in G$ and $a \in N(H)_c$, where $N(H)_c$ is the identity component of the normalizer of H, with $\tilde{K^-} = bK^-b^{-1}$, $\tilde{H} = bHb^{-1}$ and $\tilde{K^+} = abK^+b^{-1}a^{-1}$.*

There are cases where a manifold, M, could have different cohomogeneity one actions. For example, there are many distinct cohomogeneity one actions on spheres of dimension ≤ 3. A special case is when a *proper normal subgroup* $G_1 \subset G$ acts on the manifold with the same orbits. In this case the action is called ***reducible***. Otherwise the action is called ***non-reducible***.

3.2 Structure

In this section we will discuss the structure, first for $M/G = I$ and then for $M/G = S^1$.

$M/G = I$.

If $\pi : M \to M/G$ is the ***orbit projection***, inverse images of the interior points are the ***regular orbits*** and $B_- = \pi^{-1}(0)$ and $B_+ = \pi^{-1}(L)$, where $M/G = I = [0, L]$, are the two non-regular orbits. Let some point, $x_- \in B_-$ and let $\gamma : [0, L] \to M$ be a minimal geodesic from B_- to B_+, parametrized by arc length which will will say starts at X_-. The geodesic is orthogonal to B_- and so is also orthogonal to all orbits. Now define $X_+ = \gamma(L) \in B_+$, $x_0 = \gamma(\frac{L}{2})$ and let $K^{\pm} = G_{x_{\pm}}$ be the isotropy groups at $x_{\pm}$ and $H = G_{x_0} = G_{\gamma(t)}$, $0 < t < L$, the principal isotropy group. Therefore $B_{\pm} = G \cdot x_{\pm} = G/K^{\pm}$ and $G \cdot \gamma(t) = G/H$ for $0 < t < L$.

The slice theorem gives us the description of the tubular neighborhoods

$$D(B_-) = \pi^{-1}([0, \frac{L}{2}])$$

and

$$D(B_+) = \pi^{-1}([\frac{L}{2}, L])$$

of the non-principal orbits: $D(B_{\pm}) = G \times_{K_{\pm}} D^{l_{\pm}}$, where $D^{l_{\pm}}$ are disks in the normal tangent space. Therefore we can decompose our manifold into two disk bundles, $G \times_{K^{\pm}} D_{\pm}$, glued along their common boundary.

Let G be a compact Lie group acting on M by cohomogeneity one with subgroups $H \subset K^{\pm} \subset G$ and let $K^{\pm}/H = S^{l_{\pm}-1}$ be spheres. This is the group diagram mentioned earlier, (G, H, K^-, K^+). It is known that a transitive action of a compact Lie group K on a sphere, S^{l-1} is conjugate to a linear action, see Ziller [38] . Therefore we can assume that $K^{\pm}$ acts linearly on $S^{l_{\pm}}$ with isotropy group the principal subgroup $H \subset K^{\pm}$ and thus extends to a linear action on the bounding disk $D_{l_{\pm}}$. Therefore the group diagram gives us the following disk bundle decomposition of M,

$$M = G \times_{k^-} D_{l_-} \cup_{G/H} G \times_{K_+} D_{l_+}.$$

These structure properties are important as they reduce classifying these types of manifolds to finding subgroups of compact groups with certain properties, which can be a less complicated problem. The original structure theorem by Mostert in 1957 had two omissions. The first was discovered and corrected by Neumann in 1967, however, the second omission was not corrected until 2015 when it was corrected by Galaz-García and Zarei. The second omission in Mostert's theorem originated in the claim that an integral homology sphere that is also a homogeneous space for a compact Lie group but be a standard sphere. This is actually a corollary by Mostert. The exception to this is the Poincaré homology sphere, $\mathbb{P}^3$. The Poincaré homology sphere is a homogeneous space for the Lie groups $SU(2)$ and $SO(3)$, that is, $\mathbb{P}^3 = SU(2)/I^* = SO(3)/I$, where I^* is the binary icosahedral group and I is the icosahedral group. It was Bredon [6] who proved the Poincaré homology sphere is the only integral homology sphere that is also a homogeneous space other than the usual spheres . The following theorem by Galaz-García and Zarei address the case Mostert missed where the normal spheres to the singular orbits can be Poincaré homology spheres. In this section we will state the structure theorem and proceed to give background needed to prove the structure theorem, which will be done at the end of the chapter.

Theorem 3.2.1. *[14] Let M be a closed topological manifold with an (almost) effective topological G action of cohomogeneity one with principal isotropy H. The the orbit space is homeomorphic to either a closed interval or to a circle, and the following hold.*

i) If the orbit space of the action is an interval, then M is the union of two fiber bundles over the two singular orbits whose fibers are cones over spheres or the Poincaré homology sphere, that is, $M = G \times_{K^-} C(K^-/H) \cup_{G/H} G \times_{K^+} C(K^+/H)$.

ii) If the orbit space of the action is a circle, then M is equivariantly homeomorphic to a G/H-bundle over a circle with structure group $N(H)/H$.

The group diagram of the action is given by (G, H, K^-, K^+), where $K^{\pm}/H$ are spheres or the Poincaré homology sphere. Conversely, a group diagram (G, H, K^-, K^+), where $K^{\pm}/H$ are homeomorphic to a sphere or to the Poincaré homology sphere with $dim G/H \geq 4$, determines a cohomogeneity one topological manifold.

The following is an example of a cohomogeneity one topological manifolds with $K/H \approx \mathbb{P}^3$.

Example 3.2.2. *Let $S^3 \times SO(n+1), n \geq 1$, act on $\mathbb{P}^3 * S^n$ as the join action of the standard transitive action of $S^3 \cong SU(2)$ on $\mathbb{P}^3$ and $SO(n+1)$ on S^n. The orbit space is homeomorphic to $[-1, +1]$ and $K^+/H = \mathbb{P}^3$. By the Double Suspension Theorem [12], $Susp^2\mathbb{P}^3 \approx S^5$ and it follows that $\mathbb{P}^3 * S^n \approx Susp^{n+1}\mathbb{P}^3$ is homeomorphic to S^{n+4}.*

We will now review lemmas, theorems, and corollaries which we will used to prove 3.2.1. We begin with the case where $M/G = S^1$.

Corollary 3.2.3. *[5] Every Equivariant map, $G/H \to G/H$, is right translation by an element of $N(H)$ and is an equivalence of G-spaces. The map $a \mapsto {R_a}^{H,H}$ induces an isomorphism of $N(H)/H$ onto the group $Homeo^G(G/H)$. Under composition, of self-equivalences of the G-space G/H with the compact open topology.*

Proof. The right translation action $G/H \times N(H) \to G/H$ is continuous which implies the continuity of the map $N(H)/H \to Homeo^G(G/H)$ taking $a \mapsto {R_a}^{H,H}$. Therefore $N(H)/H \to Homeo^G(G/H)$ is continuous, one-one, and onto. Thus, it is a homeomorphism, since $N(H)/H$ is compact. □

Theorem 3.2.4. *[5] Suppose that $p : X \to B$ is a bundle with fiber F and structure group K. Suppose that F is also a left G-space and that the actions of G and K commute, i.e $(gf)k = g(fk)$. Then there is a unique G-action on X covering the trivial action on B and such that each chart $\phi : F \times U \to p^{-1}(U)$ is equivariant where G acts on $F \times U$ by $(g, (f, u)) \mapsto (gf, u)$.*

Proof. The action is to be defined by the equivariance of the charts and it suffices to prove that it is independent of the choice of a chart over U. In other words, it suffices to show that each $\phi^{-1}\psi : F \times U \to F \times U$ is equivariant. However,

$$g(\phi^{-1}\psi(f,u)) = g(f\cdot\theta(u),u) = (g(f\cdot\theta(u)),u) = (gf)\cdot\theta(u),u) = \phi^{-1}\psi(gf,u) = \phi^{-1}\psi(g(f,u)).$$

□

Theorem 3.2.5. *[5] Suppose M is a completely regular G-space, G compact Lie, and that all of the orbits have type G/H. Then the orbit map $M \to M/G$ is the projection in a fiber bundle with fiber G/H and structure group $N(H)/H$, acting by right translation on G/H. Conversely every such bundle comes from such an action.*

Proof. The converse follows from 3.2.4. For the first part we note that a tube in X is of the form $G \times_H A$. However, $G_{[e,a]} = H_a \subset H$ and the fact that $G_{[e,a]}$ is conjugate to H implies that $H_a = H$ for all $a \in A$. That is, A has trivial H-action. In this case we have $G \times_H A \approx (G/H) \times A$, equivariantly, and if we identify A with its homeomorphic image $A/H \approx (G \times_H A)/G$ in the orbit space, we get the following product representation

$$\begin{array}{ccc} \varphi_A : (G/H) \times A & \xrightarrow{\quad\approx\quad} & \pi^{-1}(A) \\ & \searrow \qquad \swarrow{\scriptstyle \pi} & \\ & A & \end{array}$$

If ϕ_A and ϕ_B are two such product representations, then

$$\begin{array}{ccc} \varphi_B^{-1}\varphi_A : (G/H) \times (A \cap B) & \xrightarrow{\qquad\qquad} & (G/H) \times (A \cap B) \\ & \searrow \qquad \swarrow & \\ & A \cap B & \end{array}$$

gives a map $\theta : A \cap B \to Homeo^G(G/H)$, where $\phi^{-1} - B\phi_A(gH, x) = (\theta(x)(gH), x)$ into the group of self equivalences of G/H in the compact-open topology and 3.1.2, $Homeo^G(G/H) \approx N(H)/H$.

□

This concludes the proof for the case where $M/G = S^1$. We now consider the case where $M/G = I$ and begin with the following result from [27].

Lemma 3.2.6. *[27] Let M be a cohomogeneity one G-manifold, with group diagram (G, H, K^-, K^+). Then, letting $N = K^{\pm}/H$ we have*

$$H_r(N) \approx H_r(S^n), 0 \leq r.$$

Recall the following theorem from [5] which allows us to completely classify integral homology spheres that are homogeneous spaces.

Theorem 3.2.7. *[5] Let G/H be an integral homology sphere. Then G/H is either simply connected, or a circle, or of the form $Sp(1)/I^* \approx SO(3)/I$, where I^* and I are the icosahedral subgroups of $Sp(1)$ and $SO(3)$ respectively.*

Corollary 3.2.8. *[5] Let G be a compact Lie group and H a closed subgroup of G. If G/H is a homology k-sphere, then G/H is homeomorphic to either $\mathbb{S}^k$ or to the Poincaré homology sphere, $\mathbb{P}^3$.*

3.2.1 Proof of the Structure Theorem from Galaz-Garica and Zarei

We are now ready to prove Theorem 3.2.1 in the case when $M/G = I$.

Proof. Let M^n be a closed topological n-manifold with an almost effective topological G action of cohomogeneity one with principal isotropy H. We have already proven, using 3.1.2, the orbit space is homeomorphic to either a closed interval or to a circle. Part (2) of the theorem follows from part (ii) of the original Mostert structure theorem [27]. Therefore we only need to prove part (1) of 3.2.1, where the orbit space M/G is homeomorphic to a closed interval. If we add $\mathbb{P}^3$ as a possibility for $K^{\pm}/H$ the 'if' statement in this case corresponds to part (iv) in [**?**].

Now we must prove the 'only if' statement.

Let (G, H, K^-, K^+) be a group diagram which satisfies the hypothesis of part (1) 3.2.1. We only need to consider the case where at least one of $K^{\pm}$ is the Poincaré homology sphere, since there other cases have already been examined by Mostert. In this case, $n \geq 5$.

Suppose, without loss of generality, that $K^+/H = \mathbb{P}^3$. Since $n \geq 5$, the singular orbit G/K^+ is at least one-dimensional. Now consider the following finite polyhedron,

$$X = G \times_K - C(K^-/H) \cup_{G/H} G \times_K^+ C(K^+/H).$$

Since the link of every point in the singular orbit G/K^+ is $\mathbb{S}^{n-5} * \mathbb{P}^3$, Theorem 2 in Edwards [12], implies that X is a topological manifold. □

CHAPTER 4

Classification of Cohomogeneity One Manifolds

This chapter will address the classification of simply connected cohomogeneity one manifolds. For a compact, connected cohomogeneity one manifold M and for the group G there exists an open, dense connected subset $M_0 \subset M$ which consists of codimension 1 orbits and by definition, the map $M_0 \to M_0/G$ is a locally trivial fiber bundle. Since M_0/G is one dimensional it must either be an interval or a circle [5]. In the case M_0/G is a circle $M = M_0$ and so M is fibered over a circle and must have an infinite fundamental group. Most of the interest lies in the cases where the manifolds are simply connected and therefore most attention will be paid to the case where $M_0/H \approx (-1, 1)$. Furthermore, M is compact so we have $M/G \approx [-1, 1]$. Therefore we will first, briefly, address the case where $M/G \approx S^1$ followed by the case where $M/G \approx [-1, 1]$, for each dimension up to dimension 7. Manifolds can be given in $dim \leq 5$ using classification theorems. Specifically, in dimension 4 from theorems of Freedmond [13], in dimension 5 from theorems of Smale [34]. However, we will omit the manifolds for dimensions 6 and 7, as they become much more difficult to find due to more invariants.

In this classification we will not consider torus actions beyond dimension 3 since these torus actions are *non-primitive*, in other words all the isotropy sub-groups, K^-,K^+, and H for some group diagram representation, are all contained in some proper subgroup L in G and these cases are well understood from the following work of Parker [31] and Pak [30].

Theorem 4.0.1. *If T^n acts effectively on a compact closed, orientable $(n + 1)$-manifold, M^{n+1}, then M^{n+1} must either be T^{n+1} or $M^3 \times T^{n-z}$, where M^3 is a T^2 cohomogeneity one manifold.*

Proof. There are two cases to consider. If $M/G \approx S^1$, then every point on S^1 must correspond to a principal orbit and the total space must be a T^n-bundle over S^1. However, these bundles are classified by

$$[S^1, K(\mathbb{Z},2) \times \cdots \times K(\mathbb{Z},2)] = H^2(S^1, \mathbb{Z} + \cdots + \mathbb{Z}) = 0.$$

so that bundle is trivial and $M = S^1 \times T^n = T^{n+1}$. If $M/G \approx [0,1]$, then there are only two circle subgroups of T^n corresponding to the stability groups at 0 and 1. Let T_0 be a subgroup which is generated by these two circle subgroups. Then, any $(n-2)$-dimensional subgroup T^{n-2} of T^n which is complimentary to T_0 must act freely on M. Then M/T^{n-2} is a 3 dimensional orientable manifold $\bar{M}$ and T_0 acts on so that $\bar{M}/T_0 \cong [0,1]$. However, T_0 actions on 3 manifolds whose orbit spaces are isomorphic to $[0,1]$ are classified as lens spaces which can be written as $M^3 \times T^{n-2}$. □

4.1 Classification in the Case when M/G is Homeomorphic to a Circle

In the case where the orbit space is homeomorphic to S^1, H is the only orbit type, therefore we have the following result.

Theorem 4.1.1. *[5] If M is a G-manifold, with principal isotropy H and $M/G \approx S^1$, then M fibers as*

$$G/H \to M \to S^1,$$

with structure group $N(H)/H$, where $N(H)$ is the normalizer in G of H.

Proof. The converse is a consequence of 2.2.10. For the first part we notice that a tube in X is of the form $G \times_H A$. However, $G_{[e,a]} = H_a \subset H$ and $G_{[e,a]}$ is conjugate to H, which combined implies that $H_a = H$ for all $a \in A$. In other words, A has trivial H-action. In this case we have $G \times_H A \approx (G/H) \times A$. If we identify A with its homeomorphic image $A/H \approx (G \times_H A)/G$ in the orbit space we get the following product representation

$$
\begin{array}{ccc}
\phi_A : (G/H) \times A & \xrightarrow{\approx} & \pi^{-1}(A) \\
& \searrow \quad \swarrow_{\pi} & \\
& A &
\end{array}
$$

If ϕ_A and ϕ_B are both such product representations, then

$$
\begin{array}{ccc}
\phi_B^{-1}\phi_A : (G/H) \times (A \cap B) & \longrightarrow & (G/H) \times (A \cap B) \\
& \searrow \quad \swarrow & \\
& A \cap B &
\end{array}
$$

gives a map $\theta : A \cap B \to Homeo^G(G/H)$ into the group of self-equivalences of G/H in the compact open topology. Therefore, by 3.2.3 we can say $Homeo^G(G/H) \approx N(H)/H$. □

Therefore, every such bundle results in a cohomogeneity one manifold which can be classified by the components of $N(H)/H$.

We will now focus on the more interesting case, where $M/G \approx I$.

4.2 Classification in Dimension 2

Closed, smooth manifold of cohomogeneity one in dimension 2 have been classified equivariantly by Mostert and Neumann.

The following proposition gives the possible dimensions G can have if it acts by cohomogeneity one.

Proposition 4.2.1. *[19] If a Lie group G acts almost effectively and by cohomogeneity one on the manifold M^n then, $n - 1 \leq dim(G) \leq n(n-1)/2$.*

Proof. Recall that for a principal orbit $G \cdot x \approx G/H$, $\dim G/H = n-1$, so the first inequality is trivial. Now we claim that G also acts almost effectively on a principal orbit $G \cdot x \approx G?H$. To see this, suppose an element $g \in G$ fixes $G \cdot x$ point-wise. Then in particular $g \in H$. We saw above that H fixes the geodesic c point-wise as well. Therefore g fixes all of M point-wise and hence G acts almost effectively on G/H. Now equip G/H with a G invariant metric. It then follows that G maps into $IsomG/H$ with finite kernel. Since $\dim G/H = n-1$, we know $\dim(IsomG/H) \leq n(n-1)/2$ and this proves the second inequality. $\square$

The next lemma shows how to give strong but simple conditions on the group diagrams which result in simply connected manifolds.

Lemma 4.2.2. *[17] Assume that G acts on M by cohomogeneity one with M simply connected and G connected, Then*

1. *There are no exceptional orbits, i.e., $l_\pm \geq 1$.*

2. *If both $l_\pm \geq 2$, then $K^\pm$ and H are all connected.*

3. *If one of $l_\pm$, say $l_- = 1$, and $l_+ \geq 2$, then $K^- = H \cdot S^1 = H_0 \cdot S^1$, $H = H_0 \cdot \mathbb{Z}_k$.*

Suppose $\dim(M) = 2$. From 2.2.7 we know that $\dim(G) = 1$ and hence $G = S^1$. For the action to be effective, H must be trivial. From 2.2.8 part (a), $K^\pm$ must both be S^1 and so the group diagram must be:

Cohomogeneity one Manifolds in Dimension 2				
G	H	K^-	K^+	M
S^1	1	S^1	S^1	S^2

This is the action of S^1 on S^2 by rotation about a fixed axis.

4.3 Classification in Dimension 3

The following proposition acts as a Van Kampen theorem for cohomogeneity one manifolds. It can be used to compute the fundamental group using only the group diagram. It will also be useful in the three dimensional classification of cohomogeneity one manifolds, specifically, if the dimension of $M = 3$, then the dimension of G can wither be 2 or 3. If $G = 2$ then $G = T^2$ and therefore H must be discrete. For the action to be effective H must be trivial. Thus $K^{\pm}$ must be circle groups.

Proposition 4.3.1. *[19] Let M be the cohomogeneity one manifold given by the group diagram (G, H, K^-, K^+) with $K^{\pm}/H = S^{l_{\pm}}$ and assume $l_{\pm} \geq 1$. Then $\pi_1(M) \approx \pi_1(G/H)/N_-N_+$ where*
$N_{\pm} = ker\{\pi_1(G/H) \rightarrow \pi_1(G/K^{\pm}\} = Im\{\pi_1(K^{\pm}/H) \rightarrow \pi_1(G/H)\}.$

In particular, M is simply connected iff the images of $K^{\pm}/H = S^{l_{\pm}}$ generate $\pi_1(G/H)$ under natural inclusions.

Proof. We will compute the fundamental group of M using Van Kampen's theorem. M can be decomposed as $\pi^{=1}([-1,0]) \cup \pi^{-1}([0,1])$ where $\pi([-1,0]) \cap \pi^{-1}([0,1]) = G \cdot x_0$. Here we know $\pi^{-1}([0,\pm 1])$ deformation retracts to $\pi^{-1}(\pm 1) = G \cdot x_{\pm} \approx G/K^{\pm}$. So we have a homotopy equivalence $\pi^{-1}([0,\pm 1]) \rightarrow G/K^{\pm} : g \cdot c(t) \mapsto gK^{\pm}$. Therefore we have the following commutative diagram of pairs:

$$\begin{array}{ccc} (G \cdot x_0, x_0) & \longrightarrow & (\pi^{-1}([0,\pm 1]), x_0) \\ \sim\downarrow & & \downarrow\sim \\ (G/H, H) & \longrightarrow & (G/K^{\pm}, k^{\pm}) \end{array}$$

Where the vertical maps are both homotopy equivalences, the top map is an inclusion and the bottom map is the natural quotient. Therefore, we get the corresponding diagram of fundamental groups:

$$\begin{array}{ccc} \pi_1(G \cdot x_0, x_0) & \longrightarrow & \pi_1(\pi^{-1}([0, \pm 1]), x_0) \\ \sim\downarrow & & \downarrow\sim \\ \pi_1(G/H, H) & \longrightarrow & \pi_1(G/K^{\pm}, k^{\pm}) \end{array}$$

Therefore we can freely use $\pi_1(G/H) \to \pi_1(G/K^{\pm}$ in place of $\pi_1(G \cdot x_0, x_0) \to \pi_1(\pi^{-1}([0, \pm 1]), x_0)$ for Van Kampen's theorem. Now consider the following fiber bundle,

$$K^{\pm} \to G/H \to G/K^{\pm} where K^{\pm}/H \approx S^{l_{\pm}}.$$

This results in a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_i(S^{l_{\pm}}) \longrightarrow \pi_i(G/H) \longrightarrow \pi_i(G/K^{\pm}) \longrightarrow \pi_{i-1}(S^{l_{\pm}}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(S^{l_{\pm}}) \longrightarrow \pi_1(G/H) \longrightarrow \pi_1(G/K^{\pm}) \longrightarrow \pi_0(S^{l_{\pm}})$$

Notice that this implies $\rho_*^{\pm} : \pi_1(G/H) \to \pi_1(G/K^{\pm})$ is onto, since $l_{\pm} > 0$.

By Van Kampen's theorem, $\pi_1(M) \approx \pi_1(G/H)/N_- N_+$ where $N_{\pm} = ker(\rho_*^{\pm})$. Finally, from out long exact sequences we find $N_{\pm} = ker(\rho_*^{\pm}) = Im(\imath_*^{\pm})$, and so the proof is complete.

□

The following lemma tells us that since $K_0^{\pm} \cap H = H_0$, it must be connected. This forces $K^{\pm}$ to also be connected.

Lemma 4.3.2. *[19] Let M be the cohomogeneity one manifold given by the group diagram (G, H, K^-, K^+) with $K^{\pm}/H = S^{l_{\pm}}$ and denote $H_{\pm} = H \cap K_0^{\pm}$. If M is simply connected then H is generated as a subgroup by H_- and H_+.*

This lemma is originally by Grove, Wilking and Ziller, lemma 1.7, but Hoelscher offers a more accessible proof.

Proof. Let $\alpha_{\pm} : [0, 1] \to K_0^{\pm})$ be paths with $\alpha_{\pm}(0) = 1$ which generate $\pi_1(K^{\pm}/H)$. In the case that either $dim(K^{\pm}/H > 1$ we can take $\alpha \equiv 1$ for corresponding $\pm$. Since M is simply connected by 4.3.1, $\pi_1(G/H) = \langle\alpha_-\rangle \cdot \langle\alpha_+\rangle$, where $\alpha_\pm$ are considered loops in G/H. Since $\alpha_\pm$ are loops in $K^\pm/H$ it follows that $a_\pm := \alpha_\pm(1) \in K_0^\pm \cap H$. We claim that $a_\pm$ and H_0 generate H as a group. Since $H_0 \subset K_0^\pm$, this will prove the lemma.

Chose an arbitrary component, hH_0, of H. We claim that some product of a_- and a_+ will lie in hH_0. For this, let $\gamma : [0,1] \to G$ be an arbitrary path with $\gamma(0) = 1$ and $\gamma(1) \in hH_0$. Then γ represents a loop in G/H and since $\pi_1(G/H) = \langle\alpha_-\rangle \cdot \langle\alpha_+\rangle$ we must have that $[\gamma] = [\alpha_-]^n \cdot [\alpha_+]^m$, where $[\cdot]$ denotes the corresponding class in $\pi_1(G/H)$.

In general, for compact Lie groups $J \subset L$, take paths $\beta_\pm : [0,1] \to L$, with $\beta_\pm(0) = 1$ and $\beta_\pm(1) \in J$. Then we see that $(\beta_- \cdot \beta_+(1)) \circ \beta_+$ is fixed endpoint homotopic to $\beta_- \cdot \beta_+$ in L, where $\beta_- \cdot \beta_+(1)$ is the path $t \mapsto \beta_-(t) \cdot \beta_+(1)$; $\circ$ denotes path composition; and $\beta_- \cdot \beta_+$ is the path $t \mapsto \beta_-(t) \cdot \beta_+(t)$. Therefore $[\beta_-] \cdot [\beta_+] = [\beta_- \cdot \beta_+]$ as classes in $\pi_1(J/L)$.
In our case, this implies

$$[\gamma] = [\alpha_-]^n \cdot [\alpha_+]^m = [\alpha_-^n \cdot \alpha_+^m]$$

in $\pi_1(G/H)$. Now look at the cover $G/H_0 \to G/H$. Since the paths γ and $\alpha_-^n \cdot \alpha_+^m$ both start at $1 \in G$, it follows that γ and $\alpha_-^n \cdot \alpha_+^m$ both end in the same component of H.

Hence, $(\alpha_-^n \cdot \alpha_+^m)(1) = \alpha_-(1)^n \cdot \alpha_+(1)^m = a_-^n \cdot a_+^m \in hH_0$.
Therefore, a_-, a_+ and H_0 generate H and the lemma is proved. □

The following proposition gives further restriction on the groups which can act by cohomogeneity one on simply connected manifolds, specifically when the group has abelian factor.

Proposition 4.3.3. *[19] Let M be the cohomogeneity one manifold given by the group diagram (G, H, K^-, K^+) where $G = G_1 \times T^m$ acts almost effectively and nonreducibly and G_1 is semisimple. Then we know $H_0 = H_1 \times 1 \subset G_1 \times 1$. Also, if M is simply connected then $m \leq 2$ and*

a) If $m = 1$ then at least on of $proj_2(K_0^\pm) = S^1$. Then $K^-/H \approx S^1$ and $K^- = S^1_- \cdot H$ for a circle group S^1_-, with $proj_2(S^1_-) = S^1$. Also, if $rk(H) = rk(G_1)$ or if H_1 is maximal-connected in G_1, then H,K^- and K^+ are all connected; $K^- = H_1 \times S^1$; and K^+ is either

$H_1 \times S^1$ or has the form $K_1 \times 1$, for $K_1/H_1 \approx S^{l_+}$.

b) If $m = 2$ then both $K^{\pm}/H$ are circles and $K^{\pm} = S^l_{\pm}$. H for circle groups $S^l_{\pm}$, with $proj_2(S^1_-) \cdot proj_2(S^l_-) = T^2$. Also, if $rk(H) = rk(G_1)$ then the G action is equivalent to the product action of $G_1 \times T^2$ on $(G_1/H_1) \times S^3$, where T^2 acts on $S^3 \subset \mathbb{C}^2$ by component-wise multiplication.

Proof. In all cases $proj_2(K_0^{\pm})$ is a compact connected subgroup of T^m. Now say $proj_2(K_0^-)$ is nontrivial. It must be a torus, $T^n \subset T^m$. Then we have $proj_2 : K_0^- \twoheadrightarrow T^n$ with kernel $K_0^- \cap (G_1 \times 1)$. Therefore we have the following fiber bundle

$$(K_0^- \cap (G_1 \times 1))?H_1 \times 1) \to K_0^-/(H_1 \times 1) \to K_0^-/(K_0^- \cap (G_1 \times 1)) \approx T^n$$

Which gives the following long exact sequence,

$$\pi_1(K_0^-/H_1 \times 1)) \to \pi_1(T^n \to \pi_0)(H_0^- \cap (G_1 \times 1))/(H_1 \times 1)).$$

The last group in this sequence is finite and the middle group is infinite. This means that K_0^-/H_0 has infinite fundamental group. Given that this space is a sphere, it follows that $K^-/H \approx S^1$. Therefore $K_0^- = H_0 \cdot S^1_-$, for some circle group S^1_- with $proj_2(S^1_-) = S^1 \subset T^m$. Similarly, if $proj_2(K_0^+)$ is nontrivial then $K_0^+ = H_0 \cdot S^1_+$, for S^l_+ with $proj_2(S^1_+) = S^1 \subset T^m$.

We know that $proj_2(K_0^-)$ and $proj_2(K_0^+)$ generate a torus $T^n \in T^m$, with $n \leq 2$. If $m > n$ then K^-/H and K^+/H will not generate $\pi_1(G/H)$ and so M will not be simply connected by 4.3.1. Therefore, $m \leq 2$. If $m = 1$ then one of $K^{\pm}$ must be a circle and if $m = 2$ then both $K^{\pm}$ have to be circles. Therefore the first part of the proposition has been proved.

For the second part, if $rkH_1 = rkG_1$ or if H_1 is maximal connected in G_1 we claim that $proj_1(K_0^-) - H_1$ if $K^-/H \approx S^1$.. If H_1 is maximal in G_1 our claim is true because if $proj_1(K_0^-$ is larger than H_1 it would have to be all of G_1. However, there is no compact semisimple group G_1 with subgroup H_1 where $G_1/H_1 \approx S^1$. When $rk(H_1 = rk(G_1)$, recall that for a general compact Lie group, the rank and the dimension have the same parity modulo 2. Since$K^- =$

$S^1_- \cdot H$, $proj_1(K_0^-)$ is at most one dimension larger than H_1. However, if $proj_1(K_0^-)$ is of one dimension higher than H_1 it would follow that $rk(prof_1(K_0^-)) = rk(H_1 + 1 = rk(G_1) + 1$. This is a contradiction because $proj_1(K_0^-) \subset G_1$. Therefore $proj_1(K_0^-) = H_1$ in either case. Then since $K^- = S^1_- \cdot H$ it follows that $K_0^- = H_1 \times S^1_- \subset G_1 \times T^m$. Similarly if $K^+/H \approx S^1$ then $K_0^+ = H_1 \times S^1_+$.

We can see that all of the groups in this case are connected by noticing that if $K_0^- \cap H$ is not H_0 then $H \cap 1 \times S^1$ is nontrivial and there is a more effective action for the same groups with $H \cap 1 \times S^1 = 1$. Therefore, we can assume that $K_0^- \cap H = H_0$. If also, $K^+/H \approx S^1$ then by the same argument $K_0^+ \cap H = H_0$ as well. If $dim(K^+/H) > 1$ then $K_0^+ \cap H = H_0$ already because $K_0^+/(H \cap K_0^+)$ would be a simply connected sphere. In any case we know that $K_0^{\pm} \cap H = H_0$. Then, from 4.3.2, H must be connected, which forces $K^{\pm}$ to both be connected as well.

Now we will prove the last statement of (c). In this case we already know $K^- = H_1 \times S^1_-$ and $K^+ = H_1 \times S^1_+$. $K^{\pm}/H$ must generate $\pi_1(G/H) \approx \pi_1((G_1/H_1) \times T^2)$ iff $S^1_{\pm}$ generate $\pi_1(T^2)$. This happens when there is an automorphism of T^2 taking S^1_- to $S^1 \times 1$ and S^1_+ to $1 \times S^1$. From 4.3.1 we can assume this automorphism exists. After this automorphism the group diagram has the form,

$$(G_1 \times S^1 \times S^1, H_1 \times 1 \times 1, H_1 \times S^1 \times 1, H_1 \times 1 \times S^1)$$

This is the action described in the proposition and so the proof is complete. □

We found that 4.2.1 gave the possible dimensions that the group, G, can have if it acts by cohomogeneity one. The following proposition is for the special case where G has the largest possible dimension.

Proposition 4.3.4. *[19] Let G be a compact Lie groups that acts almost effectively and by cohomogeneity one on the manifold M^n, $n > 2$, with the group diagram (G, H, K^-, K^+) which also has no exceptional orbits. If $dim(G) = n(n-1)/2$ and G is simply connected then G is isomorphic to $Spin(n)$ and the action is equivalent to the $Spin(n)$ action on $S^n \subset \mathbb{R}^n \times \mathbb{R}$ where $Spin(n)$ acts on $\mathbb{R}^n$ via $SO(n)$, leaving $\mathbb{R}$ point-wise fixed.*

Proof. We know that since G acts on M almost effectively we know that G also acts almost effectively on the principal orbits which are equivariantly diffeomorphic to G/H. Now endow G/H with a metric induced from a bi-variant metric on G, so that G acts by isometry. Therefore we have a Lie group homomorphism $G \to IsomG/H$ with a finite kernel. Since $dimG = n(n-1)/2$ and $dimG/H = n-1$ it follows that G/H must be a space form. This is further explained in Peterson's Riemannian Geometry. Since G is simply connected it follows that G/H_0 is a compact simply connected space form. Therefore G/H_0 is isometric to S^{n-1} and G still acts almost effectively and by isometry on S^{n-1}. So $G \to IsomS^{n-1} = SO(n)$ as a Lie group homomorphism with finite kernel. G must be isomorphic to $Spin(n)$ because $dimG - dimSO(n)$. We also know that the only way $Spin(n)$ can act transitively on an $(n-1)$-sphere is with $Spin(n-1)$ isotropy. Therefore by work of Ziller there is an isomorphism $G \to Spin(n)$ taking H_0 to $Spin(n-1)$.

We also see that $Spin(n-1)$ is maximal among connected subgroups of $Spin(n)$. Therefore $K^{\pm}$ must both be $Spin(n)$ and hence H is connected because $n > 2$. So the resulting group diagram for this action is, $(Spin(n), Spin(n-1), Spin(n), Spin(n))$. The $Spin(n)$ action on S^n described by the proposition gives the same group diagram so we can say the two actions are equivalent. □

Now, Suppose $dim(M) = 3$. The $dim(G)$ must either be 2 or 3. If $dim(G) = 2$ then $G = T^2$ and H is discrete. For the action to be effective, H must be trivial. Therefore, both $K^{\pm}$ are circle groups in T^2. From 4.3.1 we know M will be simply connected if and only if $K^{\pm}$ generate $\pi_1(T^2)$. This happens where there is an automorphism of T^2 taking K^- to

$S^1 \times 1$ and K^+ to $1 \times S^1$. Therefore our group diagram is $(T^2, 1, S^1 \times 1, 1 \times S^1)$, up to automorphism. This is the action of T^2 on $S^3 \subset \mathbb{C}^2$ by $(z, w) \star (x, y) = (zx, wy)$.

Next, if $dimG = 3$ we know from 4.3.3, that $G = S^3$ up to cover and 4.3.4 tells us this action is a two-fixed-point action on a sphere.

Cohomogeneity one Manifolds in Dimension 3				
G	H	K^-	K^+	M
T^2	1	$S^1 \times 1$	$1 \times S^1$	S^3
SU^2	S^1	SU^2	SU^2	S^3

4.4 Classification in Dimension 4

Parker originally obtained the equivariant and topological classifications of cohomogeneity one manifolds in dimension 4 and Hoelscher latter addressed and omission in Parker's classification and classified dimension 4 assuming simply connectedness.

Proposition 4.4.1. *[19] Every cohomogeneity one action on a compact simply connected manifold with a fixed point is equivalent to on of the isometric actions on a compact rank one symmetric space described below.*

Table 4.1: 2.2.5

	G	H	K^-	K^+
1	$SU(n)$	$SU(n-1)$	$SU(n)$	$S(U(n-1)U(1))$
2	$U(n)$	$U(n-1)$	$U(n)$	$U(n-1)U(1)$
3	$Sp(n)$	$Sp(n-1)$	$Sp(n)$	$Sp(n-1)Sp(1)$
4	$Sp(n)$	$Sp(n-1)$	$Sp(n)$	$Sp(n-1)U(1)$
5	$Sp(n)\times Sp(1)$	$Sp(n-1)\Delta Sp(1)$	$Sp(n)\times Sp(1)$	$Sp(n-1)Sp(1)\times Sp(1)$
6	$Sp(n)\times U(1)$	$Sp(n-1)\Delta U(1)$	$Sp(n)\times U(1)$	$Sp(n-1)Sp(1)\times U(1)$
7	$Sp(n)\times U(1)$	$Sp(n-1)\Delta U(1)$	$Sp(n)\times U(1)$	$Sp(n-1)U(1)\times U(1)$

Corollary 4.4.2. *[19] M is simply connected if and only if $\alpha_{\pm}(1)$, where $\alpha_{\pm}$ are loops in $K^{\pm}/H$, and H_0 generates H as a group, and $\alpha_{\pm}$ generate $\pi_1(G/H_0)$*

Proof. We know from 4.3.1, that M is simply connected iff $\alpha_{\pm}$ generate $\pi_1(G/H)$ when considered as loops in G/H. Also, the map $G/H_0 \to G/H$ is a covering map. So $\pi_1(G/H)$ is generated by $\pi_1(G/H_0)$ and a collection of curves in G/H_0 which go from H_0 to each component of H. Saying $\alpha_{\pm}(1)$ an H_0 generate H is equivalent to saying that combinations of $\alpha_{\pm}$ can reach any component of H, when considered as paths in G/H_0. □

We previously gave a definition of equivalent group diagrams, however, when $G_1 = G_2$ the resulting equivalence is stronger than the previous type of equivalence.

Definition 4.4.3. *G-Equivariant A map f $M_1 \to M_2$ between two G-manifolds is G-equivariant if*

$$f(g \cdot x) = g \cdot f(x) \forall x \in M, g \in G.$$

The following proposition from Grove, Wilking, and Ziller gives the conditions for this type of equivarience.

Proposition 4.4.4. *[19] Let a cohomogeneity one action of G on M be given by the group diagram (G, H, K^-, K^+). Any of the following operations on the group diagram will result in an equivariantly diffeomorphic manifold.*

i) Switching K^- and K^+,

ii) Conjugating each group in the diagram by the same element of G,

iii) Replacing K^- with aK^-a^{-1} for $a \in N(h)_0$.

Conversely, the group diagrams for two equivariantly diffeomorphic manifolds must be taken to each other by some combination of these three operations.

Suppose $dimM = 4$, so that $3 \leq dimG \leq 6$. Up to cover, 4.2.1 says the the only options for G are the following groups: $S^3, S^3 \times S^1, S^3 \times T^2$,or$S^3 \times S^3$. If $G = S^3 \times S^3$ then 4.3.4 implies that the action is a two-fixed-point action on S^4. If $G = S^3 \times T^2$ then H_0 would have to be a two dimensional subgroup of $S^3 \times 1$ for the action to be nonreducible, which is impossible. Next, if $G = S^3 \times S^1$, then H_0 would be a one dimensional subgroup of $S^3 \times 1$, say $H_0 = S^1 \times 1$ then, by 4.3.3, $K^\pm$ and H are all connected and we can assume $K^- = S^1 \times S^1$. The same proposition also says K^+ is either $S^3 \times 1$ or $S^1 \times S^1$. Therefore there are only two possibilities for our group diagram.

	G	H	K^-	K^+
1	$S^3 \times S^1$	$S^1 \times S^1$	$S^1 \times S^1$	$S^1 \times 1$
2	$S^3 \times S^1$	$S^1 \times S^1$	$S^3 \times S^1$	$S^1 \times 1$

The first is a product action on $S^2 \times S^2$ and the second is a sum action on S^4.

The only other case is when $G = S^3$, where H is discrete. 4.4.1 tells us that we may assume that $K^\pm$ are both proper subgroups of G, and so they must both be circle groups. After conjugation we may assume $K_0^- = \{e^{\imath\theta}\}$ and $K_0^+ = \{e^{x\theta}\}$ for some $x \in Im\mathbb{H} \cap S^3$. If $x = \pm\imath$ then 4.4.2 implies that H must be a cyclic subgroup of $K^- = K^+$. In this case we get the following group diagram.

G	H	K^-	K^+
S^3	$\mathbb{Z}_n$	S^1	S^1

The definition of Action 50 in Parker's paper, "4-Dimensional g-Manifolds with 3-Dimensional Orbits tells us that if n is even this is an action on $S^2 \times S^2$. If n is odd then this is an action on $\mathbb{CP}^2 \# - \mathbb{CP}^2$.

Now suppose $x \neq \pm\imath$. For an arbitrary closed subgroup, $L \subset N(l_0)$. In particular $K^- \subset (\{e^{\imath\theta}\} \cup \{je^{\imath\theta}\})$ and $K^+ \subset (\{e^{x\theta}\} \cup \{ye^{x\theta}\})$ for some $y \in x^\perp \cap Im\mathbb{H} \cap S^3$. Thus H must be a subgroup of the intersection of these two sets. If $x \neq \imath^{bot}$ then these sets intersect in $\{\pm 1\} \cup (\{je^{\imath\theta}\} \cap \{ye^{x\theta}\})$ but 4.3.2 tells us that H must be generated by its intersections with K_0^- and K_0^+. Therefore, in this case, $H \subset \{\pm 1\}$ and so $N(H) = G$. Then 4.4.4 allows us to conjugate K^+ by an element of G to make $K_0^+ = K_0^-$. This is the case we already considered.

Now we may assume that $\imath \perp x$. Recall that conjugation of S^3 by the element $e^{\imath\theta_0}$ rotates the jk-plane by the angle $2\theta_0$ and fixes the $1\imath$-plane. After the conjugation we can assume that $K_0^+ = \{e^{j\theta}\}$ and $K_0^- = \{e^{\imath\theta}\}$.

So,

$$H \subset N(K_0^-) \cap N(K_0^+) = \{\pm 1, \pm \imath, \pm j, \pm k\} =: Q$$

By the argument above we can also assume that $H \not\subseteq \{\pm 1\}$. So H contains some element of $Q/\{\pm 1\}$ and since H is generates by its intersection with K_0^- and K_0^+ we can assume $\imath \in H$. If $H = \langle \imath \rangle$ then we get the following group diagram.

G	H	K^-	K^+
S^3	$\langle \imath \rangle$	$\{e^{\imath\theta}\}$	$\{e^{j\theta}\} \cup \{\imath e^{j\theta}\}$

This is an action which was missing from Parker's classification.

If H contains any other element of Q, along with $\langle \imath \rangle$, then $H = Q$. This gives us the following diagram.

G	H	K^-	K^+
S^3	$\{\pm 1, \pm \imath, \pm j, \pm k\}$	$\{e^{\imath\theta}\} \cup \{j e^{j\theta}\}$	$\{e^{j\theta}\} \cup \{\imath e^{j\theta}\}$

The omissions combined with the original work of Parker gives the current classification of cohomogeneity one manifolds in dimension 4 found in the follow table.

Cohomogeneity One Manifolds in Dimension 4					
	G	H	K^-	K^+	M
1	$S^3 \times S^1$	$S \times 1$	$S^1 \times S^1$	$S^1 \times S^1$	$S^2 \times S^2$
2	$S^3 \times S^1$	$S \times 1$	$S^1 \times S^1$	$S^3 \times S^1$	S^4
3	S^3	$\mathbb{Z}_n$	S^1	S^1	$S^2 \times S^2$ [1] or $C\mathbb{P}^2 \# - C\mathbb{P}^2$ [2]
4	S^3	$\langle \imath \rangle$	$\{e^{\imath\theta}\}$	$\{e^{j\theta}\} \cup \{\imath e^{j\theta}\}$	$C\mathbb{P}^2$
5	S^3	$\{\pm 1, \pm \imath, \pm j, \pm k\}$	$\{e^{\imath\theta}\} \cup \{j e^{j\theta}\}$	$\{e^{j\theta}\} \cup \{\imath e^{j\theta}\}$	S^4
6	$SU(2)$	1	$SU(2)$	$SU(2)$	S^4
7	$SO(4)$	$SO(3)$	$SO(4)$	$SO(4)$	S^4
8	$SO(3) \times T^1$	$SO(2)$	T^2	T^2	$S^2 \times S^2$
9	$SO(3) \times T^1$	$SO(2)$	T^2	$SO(3)$	S^4
10	$SU(2)$	1	T^1	T^1	$C\mathbb{P}^2 \# - C\mathbb{P}^2$
11	$SU(2)$	1	T^1	$SU(2)$	$C\mathbb{P}^2$

4.5 Classification in Dimension 5

Hoelscher gave the smooth classification for dimension 5 assuming simply connectedness. Throughout the classification of dimension 5, M will denotes a 5-dimensional compact simply connected cohomogeneity one manifold for a compact connected group G which will act almost effectively and nonreducibly. We will continue to use the group diagram, (G, H, K^-, K^+) where $K^\pm/H \approx S^{l_\pm}$.

To find all diagrams which give simply connected manifolds in dimension 5 we will first find all possible options for G therefore simplifying our problem. We allow the action to have finite ineffective kernel and after lifting the action to a cover of G, we assume G is a product of groups. We will also need to know the subgroups of those groups, for certain

[1] for n even

[2] for n odd

dimensions. These subgroups are well know by work of Dynkin [11]. From this point we can now compile a list of possibilities for G and H_0

Proposition 4.5.1. *[19] G and H_0 must be one of the following pairs, up to equivalence.*

G	H_0
$S^3 \times S^1$	$\{1\}$
$S^3 \times T^2$	$S^1 \times 1$
$S^3 \times S^3$	T^2
$SU(3)$	$U(2)$ [3]
$Spin(5)$	$Spin(4)$

All of these cases are examined in Hoelscher's Cohomogeneity One Manifolds in Lower Dimensions. The results are compiled in the table of possible group diagrams for cohomogeneity one manifolds in dimension 5 found below.

In general describing the full topology of all of the manifolds in the classification is a difficult problem. However, in dimension 5, work of Hoelscher [19] and Galaz-García and Zarei [14] allow us to explicitly state the resulting manifold.

Theorem 4.5.2. *[19] Every compact simply connected cohomogeneity one manifold of dimension 5 must be diffeomorphic to one of the following*

i) S^5

ii) $SU(3)/SO(3)$

iii) one of the two S^3 bundles over S^2

The following theorem, by work of Galaz-García and Zarei, gives us the diagrams of the cases missing as a result of the second omission in the original structure theorem. These are the possible cases where $K/H = \mathbb{P}^3$.

Theorem 4.5.3. *[14] Let M be a closed, simply connected topological n-manifold, $n \leq 7$, with an (almost) effective cohomogeneity one action of a compact connected Lie group, G. If the action is non-smoothable then it is given by one of the diagrams from the table below.*

Non-Smoothable Cohomogeneity One Actions in Dimensions 5,6, and 7				
Dimension	G	H	K^-	K^+
5	$S^3 \times S^3$	$I^* \times \mathbb{Z}_k$	$I^* \times S^1$	$S^3 \times \mathbb{Z}_K$
6	$S^3 \times S^3$	$I^* \times S^1$	$S^3 \times S^1$	$S^3 \times S^1$
6	$S^3 \times S^3$	$S^1 \times I^*$	$S^3 \times I^*$	$S^1 \times S^3$
7	$S^3 \times S^3$	$I^* \times 1$	$I^* \times S^3$	$S^3 \times 1$
7	$S^3 \times S^3$	ΔI^*	$I^* \times S^3$	ΔS^3
7	$S^3 \times S^3$	$I^* \times \mathbb{Z}_k$	$I^* \times S^1$	$S^3 \times \mathbb{Z}_k$
7	$S^3 \times S^3$	$I^* \times I^*$	$S^3 \times I^*$	$I^* \times S^3$
7	$S^3 \times S^3$	$I^* \times 1$	$S^3 \times 1$	$S^3 \times 1$
7	$S^3 \times S^3$	ΔI^*	ΔS^3	ΔS^3
7	$S^3 \times S^3 \times S^1$	$I^* \times S^1 \times \mathbb{Z}_k$	$I^* \times T^2$	$S^3 \times S^1 \times \mathbb{Z}_k$

From the two classification theorems given we can now compile all of the possible group diagrams for cohomogeneity one manifolds in dimension 5 and explicitly state the resulting manifold.

Cohomogeneity One Manifolds in Dimension 5					
	G	H	K^-	K^+	M
1	$S^3 \times S^1$	$\mathbb{Z}_n$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	S^5
2	$S^3 \times S^3$	$S^1 \times S^1$	$S^3 \times S^1$	$S^3 \times S^1$	S^5
3	$S^3 \times S^3$	$S^1 \times S^1$	$S^3 \times S^1$	$S^1 \times S^3$	$S^3 \times S^2$
4	$S^3 \times S^1$	$\langle (j, i) \rangle$	$\{(e^{i\theta}, 1)\} \cdot H$	$\{(e^{jp\theta}, e^{i\theta})\}$ Where $p \equiv 1 \bmod 4$	$\frac{SU(3)}{SO(3)}$
5	$S^3 \times S^1$	$H_- \cdot H_+$	$\{(e^{ip_-\theta}, e^{iq_-)\theta}\} \cdot H$	$\{(e^{ip_+\theta}, e^{iq_+\theta})\} \cdot H$ [4]	$S^3 \times S^2$
6	$S^3 \times S^1$	$\langle (j, -1) \rangle$	$\{(e^{i\theta}, 1)\} \cdot H$	$\{(e^{jp\theta}, e^{i\theta})\}$ Where $p \equiv 1 \bmod 4$	S^5
7	$S^3 \times S^1$	$\langle (j, -1) \rangle$	$\{(e^{i\theta}, 1)\} \cdot H$	$\{(e^{jp\theta}, e^{2i\theta})\}$ [5]	S^5
8	$S^3 \times S^3$	$I^* \times \mathbb{Z}_k$	$I^* \times S^1$	$S^3 \times \mathbb{Z}_K$	$\mathbb{P}^3 * S^1 \cong S^5$

[4]Where $K^- \neq K^+$, $(q_-, q_+) \neq 0$, $gcd(q_-, q_+, d) = 1$ where $d = \#(K_0^- \cap K_0^+)/\#(H \cap K_0^- \cap K_0^+)$.
[5]Where $p > 0$ is odd

4.6 Classification in Dimension 6

Hoelscher gave the classification for dimension 6 assuming simply connectedness. Throughout the classification of dimension 6, M will denotes a 6-dimensional compact simply connected cohomogeneity one manifold for a compact connected group G which will act almost effectively and nonreducibly. We will continue to use the group diagram, (G, H, K^-, K^+) where $K^\pm/H \approx S^{l_\pm}$. Just as in the previous dimension the first step to finding all desired group diagrams is to find all possible groups which is addressed in the following proposition.

Proposition 4.6.1. *[19] G and H_0 must be one of the following pairs, up to equivalence.*

G	H_0
$S^3 \times T^2$	$\{1\}$
$S^3 \times S^3$	$\{(e^{\imath p\theta}, e^{\imath q\theta})\}$
$S^3 \times S^3 \times S^1$	$T^2 \times 1$
$SU(3)$	$SU(2)$, $SO(3)$
$SU(3) \times S^1$	$U(2) \times 1$
$Sp(2) \times S^1$	$Sp(1)Sp(1) \times 1$
$Spin(6)$	$Spin(5)$

All of these cases are examined in Hoelscher's Cohomogeneity One Manifolds in Lower Dimensions except for those case which were covered by 4.5.3. The results are compiled in the following table of possible group diagrams for cohomogeneity one manifolds in dimension 6.

Cohomogeneity One Manifolds in Dimension 6				
	G	H	K^-	K^+
1	$S^3 \times S^3$	$\Delta S^1 \cdot \mathbb{Z}_n$	T^2	$\Delta S^3 \cdot \mathbb{Z}_n$ [6]
2	$S^3 \times S^3$	ΔS^1	ΔS^3	ΔS^3
3	$S^3 \times S^3$	ΔS^1	ΔS^3	$S^3 \times S^1$
4	$S^3 \times S^3$	ΔS^1	$S^3 \times S^1$	$S^3 \times S^1$
5	$S^3 \times S^3 \times S^1$	$S^1 \times S^1 \times S^1$	$S^1 \times S^1 \times S^1$	$S^1 \times S^1 \times S^1$
6	$S^3 \times S^3 \times S^1$	$S^1 \times S^1 \times S^1$	$S^1 \times S^1 \times S^1$	$S^3 \times S^1 \times S^1$
7	$SU(3)$	$\{diag(A, 1)\} \cdot \mathbb{Z}_n$	$\{diag(A, det(\bar{A}))\}$	$\{diag(A, det(\bar{A}))\}$
8	$SU(3) \times S^1$	$U(2) \times 1$	$U(2) \times S^1$	$U(2) \times S^1$
9	$Sp(2) \times S^1$	$Sp()Sp(1) \times 1$	$Sp(1)Sp(1) \times S^1$	$Sp(2) \times 1$
10	$Sp(2) \times S^1$	$Sp()Sp(1) \times 1$	$Sp(1)Sp(1) \times S^1$	$Sp(1)Sp(1) \times S^1$
11	$S^3 \times T^2$	H	$\{(e^{\imath a_- \theta}, e^{\imath b_- \theta}, e^{\imath c_- \theta})\} \cdot H$	$\{(e^{\imath a_+ \theta}, e^{\imath b_+ \theta}, e^{\imath c_+ \theta})\} \cdot H$ [7]
12	$S^3 \times S^3$	$\{(e^{\imath p\theta}, e^{\imath q\theta})\} \cdot \mathbb{Z}_n$	$\{(e^{\imath\theta}, e^{\imath\phi})\}$	$\{(e^{\imath\theta}, e^{\imath\phi})\}$
13	$S^3 \times S^3$	$S^1 \times \mathbb{Z}_n$	T^2	$S^3 \times \mathbb{Z}_n$
14	$S^3 \times S^3$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	T^2	$S^3 \times S^1$
15	$S^3 \times S^3$	$S^1 \times 1$	$S^3 \times 1$	$S^1 \times S^3$
16	$S^3 \times S^3$	$S^1 \times 1$	$S^3 \times 1$	$S^3 \times 1$
17	$S^3 \times S^3$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	$S^3 \times S^1$	$S^3 \times S^1$
18	$S^3 \times S^3$	$I^* \times S^1$	$S^3 \times S^1$	$S^3 \times S^1$
19	$S^3 \times S^3$	$S^1 \times I^*$	$S^3 \times I^*$	$S^1 \times S^3$

[6] Where $n = 1$ or 2

[7] Where $K^- \neq K^+$, $H = H_- \cdot H_+$, $gcd(b_\pm, c_\pm = 1$, $a_\pm = rb_\pm + sc_\pm$, and $K_0^- \cap K_0^+ \subset H$

4.7 Classification in Dimension 7

Hoelscher gave the classification for dimension 7 assuming simply connectedness and again we will also include the cases address in **??**. We will continue to use the group diagram, (G, H, K^-, K^+) where $K^\pm/H \approx S^{l_\pm}$. Just as in previous sections, we will examine possibilities for our groups to make discovering group diagrams for the desired cohomogeneity one manifolds in dimension 7.

Proposition 4.7.1. *[19] G and H_0 must be one of the following pairs, up to equivalence.*

G	H_0
$S^3 \times S^3$	$\{1\}$
$S^3 \times S^3 \times S^1$	$\{(e^{\imath p\theta}, e^{\imath q\theta})\} \times S^1$
$S^3 \times S^3 \times T^2$	$T^2 \times 1$
$SU(3)$	T^2
$S^3 \times S^3 \times S^3$	$T^3 \times 1$
$SU(3) \times S^1$	$SU(2) \times 1$, $SO(3) \times 1$
$SU(3) \times T^2$	$U(2) \times 1$
$Sp(2)$	$U(2)_{max}$, $Sp(1)SO(2)$
$SU(3) \times S^3$	$U(2) \times S^1$
$Sp(2) \times T^2$	$Sp(1)Sp(1) \times 1$
$Sp(2) \times S^3$	$Sp(1)Sp(1) \times S^1$
G_2	$SU(3)$
$SU(4)$	$U(3)$
$SU(4) \times S^1$	$Sp(2) \times 1$
$Spin(7)$	$Spin(6)$

All of these cases are examined in Hoelscher's Cohomogeneity One Manifolds in Lower Dimensions and the cases where $K/H = \mathbb{P}^3$ is given by 4.5.3. The results are compiled in the following table of possible group diagrams for cohomogeneity one manifolds in dimension 7.

Cohomogeneity One Manifolds in Dimension 7				
	G	H	K^-	K^+
1	$S^3 \times S^1$	$\mathbb{Z}_n$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	$S^3 \times \mathbb{Z}_n$
2	$G^1 \times S^1$	H	$H_+ \cdot \{(\beta(m_-\theta), \delta(n_-\theta))\}$	$H_- \cdot \{(\beta(m_+\theta), \delta(n_+\theta))\}$
3	$G^1 \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(\beta(m\theta), \delta\theta))\} \cdot H_0$	$\{(\beta(m\theta), \delta\theta))\} \cdot H_0$
4	$S^3 \times S^3 \times S^1$	$S^1 \times 1 \times 1 \cdot \mathbb{Z}_n$	$\{(e^{\imath\phi}, e^{\imath b\theta}, e^{\imath\theta})\}$	$S^3 \times 1 \times 1 \cdot \mathbb{Z}_n$
5	$S^3 \times S^3 \times S^1$	$\Delta S^1 \times 1 \times 1 \cdot \mathbb{Z}_n$	$\{(e^{\imath\phi}, e^{\imath\phi}e^{\imath b\theta}, e^{\imath\theta})\}$	$\Delta S^3 \times 1 \times 1 \cdot \mathbb{Z}_n$
6	$SU(3) \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(\beta(m\theta), e^{\imath\theta})\} \cdot H_0$	$SU(3) \times \mathbb{Z}_n$
7	$SU(3) \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(\beta(m\theta), e^{\imath\theta})\} \cdot H_0$	$SU(3) \times \mathbb{Z}_n$
8	$SU(3) \times S^1$	$SO(3) \times 1$	$SO(3) \times S^1$	$SO(3) \times S^1$
9	$S^3 \times S^3$	$\langle(\imath, \imath,)\rangle$	$\{(e^{\imath p_-\theta}, e^{\imath q_-\theta})\} \cdot H$	$\{(e^{\imath p_+\theta}, e^{\imath q_+\theta})\} \cdot H$ [8]
10	$SO(4) \times S^1$	$Sp(2) \times 1$	$Sp(2) \times S^1$	$Sp(2) \times S^1$
11	$S^3 \times S^3$	ΔQ	$\{(e^{\imath p_-\theta}, e^{\imath q_-\theta})\} \cdot H$	$\{(e^{jp_+\theta}, e^{jq_+\theta})\} \cdot H$ [9]
12	$S^3 \times S^3$	$H_- \cdot H_+$	$\{(e^{\imath p_-\theta}, e^{\imath q_-\theta})\} \cdot H_+$	$\{(e^{\imath p_+\theta}, e^{\imath q_+\theta})\} \cdot H_-$
13	$S^3 \times S^3$	$H_- \cdot H_+$	$\{(e^{\imath p\theta}, e^{\imath q\theta})\} \cdot H_+$	$\{(e^{j\theta}, 1)\} \cdot H_-$ [10]
14	$SU(3) \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(\beta(m\theta), e^{\imath\theta})\} \cdot H_0$	$\{(\beta(m\theta), e^{\imath\theta})\} \cdot H_0$
15	$S^3 \times S^3 \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(z^p w^{\lambda m}, z^q w^{\mu m}, w)\}$	$\{(z^p w^{\lambda m}, z^q w^{\mu m}, w)\}$ [11]
16	$S^3 \times S^3 \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(z^p w^{\lambda m_-}, z^q w^{\mu m_-}, w^{n_-})\}$	$\{(z^p w^{\lambda m_+}, z^q w^{\mu m_+}, w^{n_+})\}$ [12]
17	$S^3 \times S^3 \times S^1$	$\{(e^{\imath p\phi}, e^{\imath\phi}, 1)\} \cdot \mathbb{Z}_n$	$\{(e^{\imath p\phi}, e^{\imath a\theta}, e^{\imath\phi}, e^{\imath\theta})\}$	$S^3 \times S^1 \times \mathbb{Z}_n$
18	$SU(3)$	T^2	$S(U(1)U(2))$	$S(U(1)U(2))$
19	$SU(3)$	T^2	$S(U(1)U(2))$	$S(U(2)U(1))$

[8] Where $p_-, q_- \equiv 1 \mod 4$

[9] Where $p_\pm, q_\pm \equiv 1 \mod 4$

[10] Where $H_\pm = \mathbb{Z}_{n\pm} \subset K_0^\pm, n_+ \leq 2, 4 \mid n_-$ and $p_- \equiv \pm\frac{n_-}{4} \mod n_-$

[11] Where $H_0 = \{(z^p, z^q, 1)\}, p\mu - q\lambda = 1$ and $\mathbb{Z}_n \subset \{(w^{\lambda m}, w^{\mu m}, w)\}$

[12] Where $H = H_- \cdot H_+$, $H_0 = \{(z^p, z^q, 1)\}$, $K^- \neq K^+$, $p\mu - q\lambda = 1$, $gcd(n_-, n_+, d) = 1$ where d is the index of $H \cap K_0^- \cap K_0^+$ in $K_0^- \cap K_0^+$

20	$SU(3) \times S^1$	H	$\{(\beta(m_{-}\theta), e^{\imath n_{-}\theta})\} \cdot H$	$\{(\beta(m_{+}\theta).e^{\imath n_{+}\theta})\} \cdot H$ [13]
21	$SU(4) \times S^1$	$Sp(2) \times 1$	$Sp(2) \times S^1$	$SU(4) \times 1$
22	$SU(3) \times S^3$	$U(2) \times S^1$	$U(2) \times S^3$	$U(2) \times S^3$
23	$Sp(2) \times S^3$	$Sp(1)Sp(1) \times S^1$	$Sp(1)Sp(1) \times S^3$	$Sp(2) \times S^1$
24	$Sp(2) \times S^3$	$Sp(1)Sp(1) \times S^1$	$Sp(1)Sp(1) \times S^3$	$Sp(1)Sp(1) \times S^3$
25	$Sp(2) \times S^3$	$Sp(1)Sp(1) \times S^1$	$Sp(2) \times S^1$	$Sp(2) \times S^1$
26	$S^3 \times S^3$	1	$S^3 \times 1$	$S^3 \times 1$
27	$S^3 \times S^3$	1	$S^3 \times 1$	$1 \times S^3$
28	$S^3 \times S^3$	1	$S^3 \times 1$	ΔS^3
29	$S^3 \times S^3$	1	ΔS^3	ΔS^3
30	$S^3 \times S^3 \times S^3$	$S^1 \times S^1 \times S^1$	$S^3 \times S^1 \times S^1$	$S^3 \times S^1 \times S^1$
31	$S^3 \times S^3 \times S^3$	$S^1 \times S^1 \times S^1$	$S^3 \times S^1 \times S^1$	$S^1 \times S^3 \times S^1$
32	$Sp(2)$	$Sp(1)SO(2)$	$Sp(1)Sp(1)$	$Sp(1)Sp(1)$
33	$S^3 \times S^3$	$I^* \times 1$	$I^* \times S^3$	$S^3 \times 1$
34	$S^3 \times S^3$	ΔI^*	$I^* \times S^3$	ΔS^3
35	$S^3 \times S^3$	$I^* \times \mathbb{Z}_k$	$I^* \times S^1$	$S^3 \times \mathbb{Z}_k$
36	$S^3 \times S^3$	$I^* \times I^*$	$S^3 \times I^*$	$I^* \times S^3$
37	$S^3 \times S^3$	$I^* \times 1$	$S^3 \times 1$	$S^3 \times 1$
38	$S^3 \times S^3$	ΔI^*	ΔS^3	ΔS^3
39	$S^3 \times S^3 \times S^1$	$I^* \times S^1 \times \mathbb{Z}_k$	$I^* \times T^2$	$S^3 \times S^1 \times \mathbb{Z}_k$

[13] Where $H_0 = SU(1)SU(2) \times 1$, $H = H_{-} \cdot H_{+}$, $K^{-} \neq K^{+}$, $\beta(\theta) = diag(e^{\imath\theta}, e^{\imath\theta}, 1)$, $gnd(n_{-}, n_{+}, d) = 1$ where d is the index of $H \cap K_0^{-} \cap K_0^{+}$ in $K_0^{-} \cap K_0^{+}$

CHAPTER 5

Conclusion

From the work of Mostert [27], Neumann [28], Galaz-Garcia and Zarei [14], we now have a complete structure theorem for closed cohomogeneity one manifolds. For dimensions less than or equal to 7, by work of Mostert [27], Neumann [28], Parker [31], Pak [30], Hoelscher [19], and Galaz-Garcia and Zarei [14] we can find every such group diagram. In particular, closed, smooth cohomogeneity one manifolds have been classified by Mostert [27] and Neumann [28] in dimensions 2 and 3, noting that Neumann corrected an omission in the original structure theorem by Mostert. Such manifolds were classified by Parker [31] for dimension 4. If you assume simple connectivity, Hoelscher classified such manifolds for dimensions 5,6, and 7. Hoelscher also corrected omissions from earlier classifications.

However, while the classification of closed, connected cohomogeneity one manifolds was complete in dimensions up to 4, Hoelscher's results in dimension 5,6, and 7 gave a classification for smooth manifolds. Due to another omission in Moster's structure theorem, discovered by Galaz-Garcia and Zarei, classification for topological manifolds was not complete. Work of Galaz-Garcia and Zarei [14] completes the topological classification in dimensions less than or equal to 7.

The importance of this work is far reaching. A fruitful approach to understanding manifolds with lower curvature bounds has been through researching symmetries. Homogeneous spaces admit metrics of non-negative sectional curvature and those with positive sectional curvature were classified by [36], [4], and [3]. A natural next step is to understand those with cohomogeneity one metrics.

Recently, Grove-Ziller constructed a class of non-negatively curved metrics on certain cohomogeneity one manifolds. They also showed that every compact cohomogeneity one manifold admits a metric of non-negative Ricci curvature and also admits a metric of positive Ricci curvature if and only if its fundamental group is finite. Galaz-Garcia and Searle generalized the structure of closed, cohomogeneity one manifolds for Alexandrov spaces and classified cohomogeneity one, closed Alexandrov spaces in dimensions 3 and 4 [15], extending the previous classification. In the future the hope is that cohomogeneity one manifolds will help in finding more examples of manifolds with non-negative and positive curvature.

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APPENDIX

APPENDIX A

Tables of Cohomogeneity One Manifolds

Cohomogeneity One Manifolds in Dimension 4				
	G	H	K^-	K^+
1	$S^3 \times S^1$	$S \times 1$	$S^1 \times S^1$	$S^1 \times S^1$
2	$S^3 \times S^1$	$S \times 1$	$S^1 \times S^1$	$S^3 \times S^1$
3	S^3	$\mathbb{Z}_n$	S^1	S^1
4	S^3	$\langle \imath \rangle$	$\{e^{\imath\theta}\}$	$\{e^{j\theta}\} \cup \{\imath e^{j\theta}\}$
5	S^3	$\{\pm 1, \pm \imath, \pm j, \pm k\}$	$\{e^{\imath\theta}\} \cup \{je^{j\theta}\}$	$\{e^{j\theta}\} \cup \{\imath e^{j\theta}\}$
6	$SU(2)$	1	$SU(2)$	$SU(2)$
7	$SO(4)$	$SO(3)$	$SO(4)$	$SO(4)$
8	$SO(3) \times T^1$	$SO(2)$	T^2	T^2
9	$SO(3) \times T^1$	$SO(2)$	T^2	$SO(3)$
10	$SU(2)$	1	T^1	T^1
11	$SU(2)$	1	T^1	$SU(2)$
12	$SU(2)$	1	$SU(2)$	$SU(2)$

APPENDIX A (continued)

Cohomogeneity One Manifolds in Dimension 5					
	G	H	K^-	K^+	M
1	$S^3 \times S^1$	$\mathbb{Z}_n$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	S^5
2	$S^3 \times S^3$	$S^1 \times S^1$	$S^3 \times S^1$	$S^3 \times S^1$	S^5
3	$S^3 \times S^3$	$S^1 \times S^1$	$S^3 \times S^1$	$S^1 \times S^3$	$S^3 \times S^2$
4	$S^3 \times S^1$	$\langle (j, i) \rangle$	$\{(e^{i\theta}, 1)\} \cdot H$	$\{(e^{jp\theta}, e^{i\theta})\}$ Where $p \equiv 1 \bmod 4$	$\frac{SU(3)}{SO(3)}$
5	$S^3 \times S^1$	$H_- \cdot H_+$	$\{(e^{ip_-\theta}, e^{iq_-)\theta}\} \cdot H$	$\{(e^{ip_+\theta}, e^{iq_+\theta})\} \cdot H$ [1]	$S^3 \times S^2$
6	$S^3 \times S^1$	$\langle (j, -1) \rangle$	$\{(e^{i\theta}, 1)\} \cdot H$	$\{(e^{jp\theta}, e^{i\theta})\}$ Where $p \equiv 1 \bmod 4$	S^5
7	$S^3 \times S^1$	$\langle (j, -1) \rangle$	$\{(e^{i\theta}, 1)\} \cdot H$	$\{(e^{jp\theta}, e^{2i\theta})\}$	S^5
8	$S^3 \times S^3$	$I^* \times \mathbb{Z}_k$	$I^* \times S^1$	$S^3 \times \mathbb{Z}_K$	$\mathbb{P}^3 * S^1 \cong S^5$

[1]Where $K^- \neq K^+$, $(q_-, q_+) \neq 0$, $gcd(q_-, q_+, d) = 1$ where $d = \#(K_0^- \cap K_0^+)/\#(H \cap K_0^- \cap K_0^+)$.

Cohomogeneity One Manifolds in Dimension 6				
	G	H	K^-	K^+
1	$S^3 \times S^3$	$\Delta S^1 \cdot \mathbb{Z}_n$	T^2	$\Delta S^3 \cdot \mathbb{Z}_n$ [2]
2	$S^3 \times S^3$	ΔS^1	ΔS^3	ΔS^3
3	$S^3 \times S^3$	ΔS^1	ΔS^3	$S^3 \times S^1$
4	$S^3 \times S^3$	ΔS^1	$S^3 \times S^1$	$S^3 \times S^1$
5	$S^3 \times S^3 \times S^1$	$S^1 \times S^1 \times S^1$	$S^1 \times S^1 \times S^1$	$S^1 \times S^1 \times S^1$
6	$S^3 \times S^3 \times S^1$	$S^1 \times S^1 \times S^1$	$S^1 \times S^1 \times S^1$	$S^3 \times S^1 \times S^1$
7	$SU(3)$	$\{diag(A,1)\} \cdot \mathbb{Z}_n$	$\{diag(A, det(\bar{A}))\}$	$\{diag(A, det(\bar{A}))\}$
8	$SU(3) \times S^1$	$U(2) \times 1$	$U(2) \times S^1$	$U(2) \times S^1$
9	$Sp(2) \times S^1$	$Sp()Sp(1) \times 1$	$Sp(1)Sp(1) \times S^1$	$Sp(2) \times 1$
10	$Sp(2) \times S^1$	$Sp()Sp(1) \times 1$	$Sp(1)Sp(1) \times S^1$	$Sp(1)Sp(1) \times S^1$
11	$S^3 \times T^2$	H	$\{(e^{\imath a_- \theta}, e^{\imath b_- \theta}, e^{\imath c_- \theta})\} \cdot H$	$\{(e^{\imath a_+ \theta}, e^{\imath b_+ \theta}, e^{\imath c_+ \theta})\} \cdot H$ [3]
12	$S^3 \times S^3$	$\{(e^{\imath p\theta}, e^{\imath q\theta})\} \cdot \mathbb{Z}_n$	$\{(e^{\imath\theta}, e^{\imath\phi})\}$	$\{(e^{\imath\theta}, e^{\imath\phi})\}$
13	$S^3 \times S^3$	$S^1 \times \mathbb{Z}_n$	T^2	$S^3 \times \mathbb{Z}_n$
14	$S^3 \times S^3$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	T^2	$S^3 \times S^1$
15	$S^3 \times S^3$	$S^1 \times 1$	$S^3 \times 1$	$S^1 \times S^3$
16	$S^3 \times S^3$	$S^1 \times 1$	$S^3 \times 1$	$S^3 \times 1$
17	$S^3 \times S^3$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	$S^3 \times S^1$	$S^3 \times S^1$

[2] Where $n = 1$ or 2

[3] Where $K^- \neq K^+$, $H = H_- \cdot H_+$, $gcd(b_\pm, c_\pm) = 1$, $a_\pm = rb_\pm + sc_\pm$, and $K_0^- \cap K_0^+ \subset H$

APPENDIX A (continued)

Cohomogeneity One Manifolds in Dimension 7				
	G	H	K^-	K^+
1	$S^3 \times S^1$	$\mathbb{Z}_n$	$\{(e^{\imath p\theta}, e^{\imath\theta})\}$	$S^3 \times \mathbb{Z}_n$
2	$G^1 \times S^1$	H	$H_+ \cdot \{(\beta(m_-\theta), \delta(n_-\theta))\}$	$H_- \cdot \{(\beta(m_+\theta), \delta(n_+\theta))\}$
3	$G^1 \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(\beta(m\theta), \delta\theta))\} \cdot H_0$	$\{(\beta(m\theta), \delta\theta))\} \cdot H_0$
4	$S^3 \times S^3 \times S^1$	$S^1 \times 1 \times 1 \cdot \mathbb{Z}_n$	$\{(e^{\imath\phi}, e^{\imath b\theta}, e^{\imath\theta})\}$	$S^3 \times 1 \times 1 \cdot \mathbb{Z}_n$
5	$S^3 \times S^3 \times S^1$	$\Delta S^1 \times 1 \times 1 \cdot \mathbb{Z}_n$	$\{(e^{\imath\phi}, e^{\imath\phi}e^{\imath b\theta}, e^{\imath\theta})\}$	$\Delta S^3 \times 1 \times 1 \cdot \mathbb{Z}_n$
6	$SU(3) \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(\beta(m\theta), e^{\imath\theta})\} \cdot H_0$	$SU(3) \times \mathbb{Z}_n$
7	$SU(3) \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(\beta(m\theta), e^{\imath\theta})\} \cdot H_0$	$SU(3) \times \mathbb{Z}_n$
8	$SU(3) \times S^1$	$SO(3) \times 1$	$SO(3) \times S^1$	$SO(3) \times S^1$
9	$S^3 \times S^3$	$\langle(\imath, \imath,)\rangle$	$\{(e^{\imath p_-\theta}, e^{\imath q_-\theta})\} \cdot H$	$\{(e^{\imath p_+\theta}, e^{\imath q_+\theta})\} \cdot H$ [4]
10	$SO(4) \times S^1$	$Sp(2) \times 1$	$Sp(2) \times S^1$	$Sp(2) \times S^1$
11	$S^3 \times S^3$	ΔQ	$\{(e^{\imath p_-\theta}, e^{\imath q_-\theta})\} \cdot H$	$\{(e^{j p_+\theta}, e^{j q_+\theta})\} \cdot H$ [5]
12	$S^3 \times S^3$	$H_- \cdot H_+$	$\{(e^{\imath p_-\theta}, e^{\imath q_-\theta})\} \cdot H_+$	$\{(e^{\imath p_+\theta}, e^{\imath q_+\theta})\} \cdot H_-$
13	$S^3 \times S^3$	$H_- \cdot H_+$	$\{(e^{\imath p\theta}, e^{\imath q\theta})\} \cdot H_+$	$\{(e^{j\theta}, 1)\} \cdot H_-$ [6]
14	$SU(3) \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(\beta(m\theta), e^{\imath\theta})\} \cdot H_0$	$\{(\beta(m\theta), e^{\imath\theta})\} \cdot H_0$
15	$S^3 \times S^3 \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(z^p w^{\lambda m}, z^q w^{\mu m}, w)\}$	$\{(z^p w^{\lambda m}, z^q w^{\mu m}, w)\}$ [7]
16	$S^3 \times S^3 \times S^1$	$H_0 \cdot \mathbb{Z}_n$	$\{(z^p w^{\lambda m_-}, z^q w^{\mu m_-}, w^{n_-})\}$	$\{(z^p w^{\lambda m_+}, z^q w^{\mu m_+}, w^{n_+})\}$ [8]
17	$S^3 \times S^3 \times S^1$	$\{(e^{\imath p\phi}, e^{\imath\phi}, 1)\} \cdot \mathbb{Z}_n$	$\{(e^{\imath p\phi}, e^{\imath a\theta}, e^{\imath\phi}, e^{\imath\theta})\}$	$S^3 \times S^1 \times \mathbb{Z}_n$
18	$SU(3)$	T^2	$S(U(1)U(2))$	$S(U(1)U(2))$
19	$SU(3)$	T^2	$S(U(1)U(2))$	$S(U(2)U(1))$
20	$SU(3) \times S^1$	H	$\{(\beta(m_-\theta), e^{\imath n_-\theta})\} \cdot H$	$\{(\beta(m_+\theta).e^{\imath n_+\theta})\} \cdot H$ [9]

[4]Where $p_-, q_- \equiv 1 \bmod 4$

[5]Where $p_\pm, q_\pm \equiv 1 \bmod 4$

[6]Where $H_\pm = \mathbb{Z}_{n\pm} \subset K_0^\pm, n_+ \leq 2, 4 \mid n_-$ and $p_- \equiv \pm\frac{n_-}{4} \bmod n_-$

[7]Where $H_0 = \{(z^p, z^q, 1)\}, p\mu - q\lambda = 1$ and $\mathbb{Z}_n \subset \{(w^{\lambda m}, w^{\mu m}, w)\}$

[8]Where $H = H_- \cdot H_+$, $H_0 = \{(z^p, z^q, 1)\}$, $K^- \neq K^+$, $p\mu - q\lambda = 1$, $gcd(n_-, n_+, d) = 1$ where d is the index of $H \cap K_0^- \cap K_0^+$ in $K_0^- \cap K_0^+$

[9]Where $H_0 = SU(1)SU(2) \times 1$, $H = H_- \cdot H_+$, $K^- \neq K^+$, $\beta(\theta) = diag(e^{\imath\theta}, e^{\imath\theta}, 1)$, $gnd(n_-, n_+, d) = 1$ where d is the index of $H \cap K_0^- \cap K_0^+$ in $K_0^- \cap K_0^+$

APPENDIX A (continued)

21	$SU(4) \times S^1$	$Sp(2) \times 1$	$Sp(2) \times S^1$	$SU(4) \times 1$
22	$SU(3) \times S^3$	$U(2) \times S^1$	$U(2) \times S^3$	$U(2) \times S^3$
23	$Sp(2) \times S^3$	$Sp(1)Sp(1) \times S^1$	$Sp(1)Sp(1) \times S^3$	$Sp(2) \times S^1$
24	$Sp(2) \times S^3$	$Sp(1)Sp(1) \times S^1$	$Sp(1)Sp(1) \times S^3$	$Sp(1)Sp(1) \times S^3$
25	$Sp(2) \times S^3$	$Sp(1)Sp(1) \times S^1$	$Sp(2) \times S^1$	$Sp(2) \times S^1$
26	$S^3 \times S^3$	1	$S^3 \times 1$	$S^3 \times 1$
27	$S^3 \times S^3$	1	$S^3 \times 1$	$1 \times S^3$
28	$S^3 \times S^3$	1	$S^3 \times 1$	ΔS^3
29	$S^3 \times S^3$	1	ΔS^3	ΔS^3
30	$S^3 \times S^3 \times S^3$	$S^1 \times S^1 \times S^1$	$S^3 \times S^1 \times S^1$	$S^3 \times S^1 \times S^1$
31	$S^3 \times S^3 \times S^3$	$S^1 \times S^1 \times S^1$	$S^3 \times S^1 \times S^1$	$S^1 \times S^3 \times S^1$
32	$Sp(2)$	$Sp(1)SO(2)$	$Sp(1)Sp(1)$	$Sp(1)Sp(1)$