TORUS ACTIONS ON SIMPLY CONNECTED 4-MANIFOLDS

A Thesis by

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

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DEDICATION

To my nieces, Londyn and Alani.
Never doubt that you are valuable and powerful and deserving of every chance and opportunity in the world to pursue and achieve your own dreams. -H.R.C

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ABSTRACT

In this paper, we study smooth effective actions of the 2-dimensional torus group $T^2 \cong SO(2)\times SO(2)$ on simply connected, closed 4-dimensional manifolds. Using the conical orbit space of the quotient, a cross-sectioning theorem for the orbit map $\pi : M \to M/G$ is achieved. An equivariant classification theorem is obtained as an application of the cross-sectioning theorem; it is shown that such a manifold is S^4, $S^2\times S^2$, $\mathbb{C}P^2$, or $\overline{\mathbb{C}P^2}$, that is,$\mathbb{C}P^2$ with the reverse orientation, or an equivariant connected sum of $S^2 \times S^2$, $\mathbb{C}P^2$, or $\overline{\mathbb{C}P^2}$ up to equivariant diffeomorphism. The decompositions are not unique.

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CHAPTER 1

Introduction

Suppose M is a closed, orientable, smooth manifold of dimension $(n + 2)$ with a smooth and effective T^n-action. A lot of work on the classification of these manifolds has been done and expanded throughout the years. Orlik and Raymond in [32] and Raymond in [40] have obtained result in the case $n = 1$. For the $n = 2$ case, work has been completed by Melvin in [27], Orlik and Raymond in [33], and Pao in [35]. Oh in [31] completed the case for $n = 3$. Various results have been obtained by Kim, McGavran and Pak [19] for $n \geq 1$ and McGavran in [25] for $n \geq 2$ on simply connected $(n + 2)$-manifolds. Fintushel in [6] looked at circle actions on 4-manifolds and in [7] for the simply connected case.

In this paper, we study smooth and effective actions of the 2-dimensional torus $T^2 \cong SO(2) \times SO(2)$ on closed simply connected 4-dimensional manifolds motivated by [33]. Our motivation for this study is that it allows one to construct a large number of nicely behaved 4-manifolds. For T^2 actions on simply connected 4-manifolds, we find that the orbit space is the 2-disk, D^2 without finite isotropy groups and with the presence of fixed points. These 4-manifolds can be described in terms of their orbit space and orbit structure. But in order to construct these 4-manifolds, we are presented with two classification problems: the equivariant problem and the topological problem. By establishing the existence of a conical orbit structure and thereby generating a cross-sectioning theorem, we will see the crucial role it plays in solving these two problems.

For the equivariant problem, our goal is to extract enough information about the orbit structure so that when two sets of data arise from two actions, we can deduce if there is a homeomorphism between the two manifolds which is equivariant with respect to both actions. That is, a simply connected 4-manifold can be determined, up to equivariant homeomorphism, by its weighted orbit space. We obtain the equivariant classification theorem using the cross-sectioning theorem as a tool.

The topological classification arises from the information obtained during the equivariant classification. We will look at the correlation between the fixed points and the Euler characteristic of a 4-manifold M and geometrically interpret these results. We then find that for such manifolds, M, admitting a T^2-action, M must be one of these 4-manifolds: S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, or $\overline{\mathbb{C}P^2}$ ($\mathbb{C}P^2$ with reverse orientation), or M must be an equivariant connected sum of $S^2 \times S^2$, $\mathbb{C}P^2$, or $\overline{\mathbb{C}P^2}$. We note that these decompositions are not unique.

We will organize this thesis as follows. In Chapter 2, we will discuss the background knowledge needed to begin our study of T^2-actions on 4-manifolds. In Chapter 3, we describe the orbit structure of T^2-actions on 4-manifolds and prove some lemmas that are essential tools for our study. Here we show that the only possible isotropy groups are the identity, $\{e\}$, the circle group $G(m, n)$ and the whole group T^2. Shown in Chapter 4, in the presence of fixed points, we can construct a cross-section to an orbit map. With the existence of a cross-section, we prove in Chapter 5 the equivariant classification theorem. Chapter 6 is devoted to understanding intersection form on 4-dimensional manifolds needed for the topological classifications. The main result of the paper is presented in Chapter 7, where we will topologically classify the simply connected 4-manifolds which admit a T^2-action. Recall that M must be one of these 4-manifolds: S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, or $\overline{\mathbb{C}P^2}$ ($\mathbb{C}P^2$ with reverse orientation), or M must be an equivariant connected sum of $S^2 \times S^2$, $\mathbb{C}P^2$, or $\overline{\mathbb{C}P^2}$.

We assume throughout this paper that all manifolds are closed, compact, connected, orientable and 4-dimensional with orientation preserving T^2-actions. Their orbit spaces are 2-disks with boundary, consisting of isotropy groups which span T^2 and which have no exceptional orbits. Further, all actions will be assumed to be locally smooth and effective.

CHAPTER 2

Transformation Groups

In this chapter, we present useful background information on transformation groups and compact Lie Groups needed for use in later chapters. The definitions and notations follow from Bredon [2] and Lee [22].

2.1 Group Actions

We begin by recalling basic definitions and facts about group actions. A *topological group* is a Hausdorff space G together with a continuous multiplication map $m : G \times G \to G$ with $(g, h) \mapsto gh$ for all $g, h \in G$ which makes G into a group and such that the map $i : G \to G$, with $g \mapsto g^{-1}$, is continuous.

Definition 2.1.1. (Action) *Let G be a topological group, X a Hausdorff topological space and $\Theta : G \times X \to X$ a map such that*

(*i*) $\Theta(g, \Theta(h, x)) = \Theta(gh, x)$, *for all* $g, h \in G$ *and* $x \in X$,

(*ii*) $\Theta(e, x) = x$, *for all* $x \in X$, *where* $\{e\}$ *is the identity of* G.

The map Θ is called an action *of G on X.*

The space X together with a given action Θ of G is called a G-space. We will consider an action of this kind to be a *left group action* and X a left G-space. An analogous notion of *right group action* can be stated with X a right G-space. Letting $g(x)$ denote $\Theta(g, x)$, we can reformulate the conditions above as:

(*i*) $(g(hx)) = (gh)(x)$, for all $g, h \in G$, and for all $x \in X$,

(*ii*) $e(x) = x$, for all $x \in X$.

For C a subset of G and A a subset of X, let $C(A) = \{g(x) \mid g \in C,\ x \in A\}$. A set A is said to be *invariant* under G if $G(A) = A$. Topological invariants are properties that are preserved by homeomorphism and are used to prove two spaces, X and Y are not topologically equivalent, that is, X is not homeomorphic to Y.

For $g \in G$, there is a *left translation map* $\theta_g : X \to X$ defined by $\theta_g(x) = g(x) = \Theta(g, x)$. Then $\theta_g \theta_h = \theta_{gh}$ and $\theta_e = \mathbf{1}_X$, the identity map of X by (i) and (ii) of Definition 2.1.1. Therefore,

$$\theta_g \theta_{g^{-1}} = \theta_e = \mathbf{1}_X = \theta_{g^{-1}} \theta_g$$

and each θ_g is a homeomorphism of X. Then G acts on X if there is a continuous homomorphism

$$\theta : G \to \mathrm{Homeo}(X),$$

where g maps into θ_g and $\mathrm{Homeo}(X)$ denotes the group under composition of all homeomorphism of X onto itself. The kernel of this homomorphism θ will be called the *kernel of the action* Θ denoted by

$$\ker(\Theta) = \{g \in G \mid g(x) = x \text{ for all } x \in X\}.$$

This is a normal subgroup of G and is closed in G. The action Θ is called *effective* if $\ker(\Theta)$ is trivial and *ineffective* if not.

2.2 Equivariant Maps and Isotropy Groups

Definition 2.2.1. (Equivariant Map) *An* equivariant map $\varphi : X \to Y$ *between (smooth) G-spaces is a map which commutes with the group actions. That is,*

$$\varphi(g(x)) = g(\varphi(x)) \text{ for all } g \in G, \ x \in X.$$

If φ is a homeomorphism (diffeomorphism), then φ is an *equivalence* of G-spaces and we say that X and Y are equivariantly homeomorphic (diffeomorphic). A slightly weaker notion of equivalence is given by *weakly equivalent*. We say that two G-spaces X and Y are weakly equivalent if there is an automorphism α of G and a homeomorphism (diffeomorphism) where,

$$\varphi(g(x)) = \alpha(g)(\varphi(x)) \text{ for all } g \in G, \ x \in X.$$

Definition 2.2.2. (Isotropy Group) *Let X be a G-space and let $x \in X$. The set of elements of G leaving x fixed is a closed subgroup of G denoted by,*

$$G_x = \{g \in G \mid g(x) = x\}.$$

Then G_x is the isotropy group *of G at x.*

Notice that if $\varphi : X \to Y$ is an equivariant map, we have $G_x \subset G_{\varphi(x)}$. A point x in a G-space X is said to be a *fixed point* of the group action when $G_x = G$. We denote the subspace of fixed points of G on X by,

$$X^G = \{x \in X \mid g(x) = x \text{ for all } g \in G\}$$

and also by $\text{Fix}(G, X)$. When H is a subgroup of G, the set of fixed points under H is denoted by $X^H = \text{Fix}(X, H)$.

Definition 2.2.3. Normalizer *For a subgroup $H \subseteq G$, the* normalizer $N(H)$ *is the subgroup given by $N(H) = \{g \in G \mid Hg = gH\}$.*

Elements of the $N(H)$ act on the fixed point set M^H. That is, given $g \in N(H)$ and $p \in M^H$, the point $g(p) \in M^H$ since for all $h \in H$, there exists h' such that $hg = gh'$ and hence

$$h(g(p)) = (hg)p = (gh')p = g(h'p) = g(p).$$

This fact is useful for abelian group actions since the normalizer of any subgroup is the entire group. Note that the kernel of an action, Θ, is then given by $\ker(\Theta) = \cap_{x \in X} G_x$. An action of G on X is said to be *free* if G_x is trivial for each $x \in X$. That is, the action is free if each nontrivial element of G moves every point in X. We can also rephrase our earlier definition of an effective action. An action is *effective* if each $g \neq \{e\}$ in G moves at least one point.

2.3 Orbits

The notion of G-orbit allows us to structure our space X as follows:

Definition 2.3.1. (Orbit) *If X is a G-space and $x \in X$, then the subspace of x under G*

$$G(x) = \{g(x) \in X \mid g \in G\}$$

is called an orbit.

We can regard $G(x)$ as the set of all images of x under the action by elements of G. If $g(x) = h(y)$ for some $g, h \in G$ and $x, y \in X$, then for any $g' \in G$ we have,

$$g'(x) = g'g^{-1}g(x) = g'g^{-1}h(y) \in G(y)$$

such that $G(x) \subset G(y)$. Conversely, using the same method, we have that $G(y) \subset G(x)$. Hence the orbits $G(x)$ and $G(y)$ of any two points $x, y \in X$ are either equal or disjoint. In particular, the sets of obits partition X.

Notice that the fixed points in the same orbit are conjugate to each other. That is for all $g \in G$, $x \in X$,

$$gG_xg^{-1}(g(x)) = gG_x(x) = g(x)$$

and we have $gG_xg^{-1} \subset G_{g(x)}$. Conversely,

$$g^{-1}G_{g(x)}g \subset G_{g^{-1}g(x)} = G_x$$

so that $G_{g(x)} = gG_xg^{-1}$. Therefore the isotropy groups which occur at points in an orbit form a complete conjugacy class of subgroups of G. Thus for an orbit $G(x)$, let its orbit type, denoted by (H) be the conjugacy class of H in G where $G(x) \cong G/H$.

Definition 2.3.2. (Orbit Space) *Let $X/G = X^*$ denote the set whose elements are the orbits $G(x)$ of G on X. Let $\pi : X \to X/G$ denote the natural map taking x to its orbit x^*, that is, $\pi(x) = x^*$. Let X/G be endowed with the quotient topology, that is, $U \subset X/G$ is open if and only if $\pi^{-1}(U)$ is open in X. Then X/G is called the* orbit space *of X with respect to G.*

The natural map $\pi : X \to X/G$ is called the orbit map. Note that $G(x) \cong G/G_x$. Orbits will be either principal, exceptional, or singular, depending on the relative size of their isotropy subgroups. *Principal orbits* are those orbits with the smallest possible isotropy subgroup and each principal orbit has the same dimension. *Exceptional orbits* are orbits whose isotropy subgroup is a finite extension of the principal isotropy subgroup and will sometimes be referred to as E-orbits. *Singular orbits* are the orbits whose isotropy subgroup is of strictly larger dimension than that of the principal isotropy subgroup. The *cohomogeneity* of the action is the dimension of the orbit space.

2.4 Tubes and Slices

We now consider the structure of the topological G-space X in the neighborhood of an orbit of type $G(x)$ for $x \in X$. Assume G to be compact. We define a *cross-section* for the orbit map $\pi : X \to X/G$ to be a continuous map $\sigma : X/G \to X$ such that $\pi \circ \sigma$ is the identity on X/G. Cross-sections do not always exist but the following theorem from [2] gives a necessary and sufficient condition for their existence.

Theorem 2.4.1. [2] *Let X be a G-space with G compact. Let C be a closed subset of X touching each orbit in exactly one point. Then the map $\sigma : X/G \to X$ defined by $\sigma(x^*) = G(x) \cap C$ is a cross-section. Conversely, the image of a cross-section is closed in X.*

Proof. We will show that σ is continuous. Let $A \subset C$ be closed. Then $\sigma^{-1}(A) = \pi(A)$ is closed. For the converse, let $C = \sigma(X/G)$ and let $\{x_\alpha\}$ be a sequence in C converging to $x \in X$. Then $\lim \pi(x_\alpha) = \pi(x)$ such that $x = \lim x_\alpha = \lim \sigma\pi(x_\alpha) = \sigma\pi(x) \in C$ and hence C is closed. □

Definition 2.4.2. (Twisted Product) *Consider H, a compact, closed subgroup of G. Let G be a right H-space and A a left H-space. Then a left H-action on $G \times A$ is given by $(h, (g, a)) \mapsto (gh^{-1}, ha)$. Then the* twisted product *of G and A denoted by*

$$G \times_H A$$

is the orbit space of this H-action.

We can regard $G \times_H A$ as the quotient space of $G \times A$ under the equivalence relation which relates (gh, a) to (g, ha) for all $g \in G$, $a \in A$, and $h \in H$. The equivalence class (orbit) of (g, a) is denoted by $[g, a]$ where $[g, a] = [g', a']$ if and only if there is a $h \in H$ with $g' = gh^{-1}$ and $a' = ha$. Note that $[gh, a] = [g, ha]$.

Define a G-action on $G \times_H A$ by setting $g'[g, a] = [g'g, a]$. Let $i_e : A \to G \times_H A$ be a map such that $i_e(a) = [e, a]$. Then i_e is H-equivariant since $[e, ha] = [h, a] = h[e, a]$. Then i_e is continuous, one-to-one, and closed since it is the composition

$$A \to G \times A \to G \times_H A$$

of closed maps. Thus i_e is an embedding. We now define a tube about an orbit P.

Definition 2.4.3. (Tube) *Let X be a G-space with G compact and let $P \subset X$ be an orbit of type G/H. Then a* tube *about P is a G-equivariant embedding*

$$\varphi : G \times_H A \to X$$

onto an open neighborhood of P in X, where A is some space on which H acts.

Therefore every G-orbit in $G \times_H A$ passes through a point of the form $[e, a]$ and we can find the isotropy group at a point of the twisted product. Consider $g \in G_{[e,a]}$ if

$$[e, a] = g[e, a] = [g, a].$$

Then for some $h \in H$, such that $(h^{-1}, h(a)) = (g, a)$ where $gh^{-1} = e$ and $ha = a$, we have $h \in H_a$ implying $g = h$. Hence $G_{[e,a]} = H_a$ in $G \times_H A$. Thus for $a \in A$, such that $\varphi[e, a] \in P$ and setting $x = \varphi[e, a]$, then $P = G(x)$. Thus

$$G_x = G_{[e,a]} = H_a \subset H$$

and since G_x is conjugate to H, we have $G_x = H_a = H$. Therefore such a point $a \in A$ is fixed under H.

As before, since $i_e : G \times_H A$ is an H-embedding, the composition $\varphi \circ i_e : A \to X$ is also an H-embedding when φ is a tube. Therefore we assume that $A \subset X$ and we have the following definition:

Definition 2.4.4. (Slice) *Let $x \in X$, a G-space. Let $x \in S \subset X$ be such that $G_x(S) = S$. Then S is called a* slice *at x if the map $G \times_{G_x} S \to X$ taking $[g, s] \mapsto gs$ is a tube about $G(x)$.*

The following theorem from [2] presents the characteristics of a slice.

Theorem 2.4.5. *Let X be a G-space and let $x \in S \subset X$. Suppose that:*

(i) S is closed in $G(S)$.

(ii) $G(S)$ is an open neighborhood of $G(x)$.

(iii) $G_x(S) = S$.

(iv) $(gS) \cap S \neq \emptyset$ for $g(S)$ a slice at $g(x)$ implies $g \in G_x$.

Then S is a slice at x. Conversely, every slice satisfies these conditions.

We state the following important theorem that provides the existence of a slice from Montgomery and Yang [28] without proof:

Theorem 2.4.6. (Slice Theorem) *Let G be a compact transformation group acting on a space M. Then at any point x of M, there exists a slice.*

2.5 Lie Groups

The definitions above apply to topological groups and thus hold for Lie groups as they are topological groups with a smooth structure. In order to define a Lie group, we first need to define the notion of a smooth manifold.

Definition 2.5.1. (Topological Manifold) *Suppose M is a topological space. We say that M is a* topological manifold *of dimension n if it has he following properties:*

(i) M is a Hausdorff space.

(ii) M is second countable, that is, there exists a countable basis for the topology of M.

(iii) M is locally Euclidean of dimension n. Each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

A manifold is illustrated in Figure 2.1:

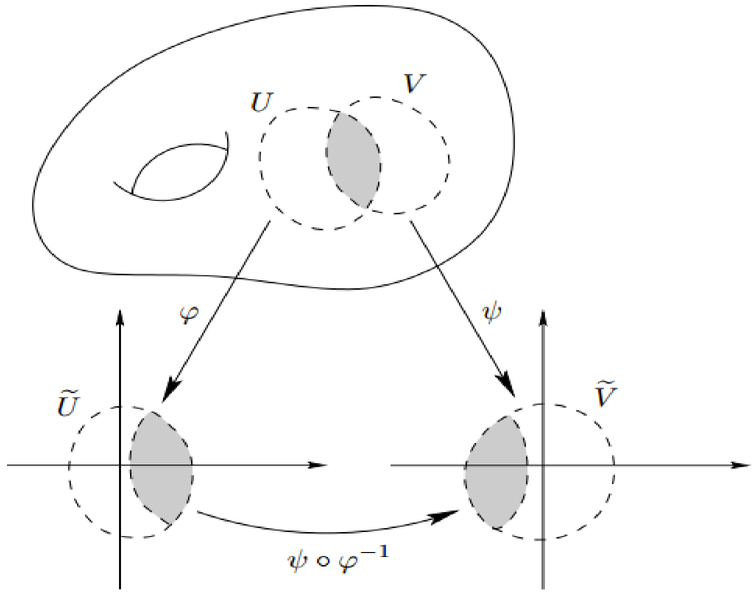


Figure 2.1: Manifold

A *chart* on M is a pair (U, φ) where U is an open subset of M and $\varphi : U \to V$ where U is a homeomorphism from U to an open subset $V = \varphi(U) \subseteq \mathbb{R}^n$. Each point $p \in M$ is contained in the domain of some chart (U, φ). An *atlas*, $\mathcal{A}$, for M is a collection of charts whose domains cover M. We can call $\mathcal{A}$ smooth if any two charts in $\mathcal{A}$ are smoothly compatible with each other and $\mathcal{A}$ is *maximal* if it is not properly contained in any larger smooth atlas. If M is a topological manifold, a smooth structure on M is a maximal smooth atlas. Hence, a *smooth manifold* is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Definition 2.5.2. (Lie Group) *A* Lie group *is therefore a smooth manifold G with a group structure such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ given by*

$$(g, h) \mapsto gh \qquad g \mapsto g^{-1}$$

are both smooth.

In particular, a Lie group is a topological group. They are useful tools in studying general manifolds because of the role they play as groups of symmetries of other manifolds. In addition, one of the most important application of Lie groups involves actions by Lie groups on manifolds. Some examples of Lie groups are:

Example 2.5.3. *The circle* $\boldsymbol{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ *is a smooth manifold and a group under complex multiplication. Multiplication and inversion have the smooth coordinate expressions* $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$ *and* $\theta \mapsto -\theta$. *Hence* $\boldsymbol{S}^1$ *is a Lie group, called the circle group.*

Example 2.5.4. *The n-torus* $\boldsymbol{T}^n = \boldsymbol{S}^1 \times \cdots \times \boldsymbol{S}^1$ *is a product of copies of the circle and hence an n-dimensional abelian Lie group.*

These are the Lie groups useful for the purpose of this paper. Another important application of Lie groups involve sub-manifolds. When given an action by a Lie group on a manifold, we find that we can consider orbits as embedded sub-manifolds of the manifold begin acted upon.

2.6 Fiber Bundles

Fiber bundles are essential tools for the topological classifications given in Chapter 7. We can think of fiber bundles as a parametrized family of objects, each isomorphic to the fiber F, where the family is parametrized by points in the base space B. Fiber bundles encode topological and geometric information about the spaces over which they are defined. The definition follows from Davis and Kirk [3]:

Definition 2.6.1. (Fiber Bundle) *Let* G *be a topological group acting effectively on a space* F. *A* fiber bundle E *over* B *with fiber* F *and structure group* G *is a map* $p : E \to B$ *together with a collection of homeomorphism* $\varphi : U \times F \to p^{-1}(U)$ *for open sets* U *in* B *(*φ *is a chart over* U*) such that:*

(i) The diagram commutes for each chart φ *over* U.

$$\begin{array}{ccc} U \times F & \xrightarrow{\varphi} & p^{-1}(U) \\ & {\scriptstyle proj} \searrow \quad \swarrow {\scriptstyle p} & \\ & U & \end{array}$$

(ii) Each point of B has neighborhood over which there is a chart.

(iii) If φ is a chart over U and $V \subset U$ is open, the the restriction of φ to V is a chart over V.

(iv) For any charts φ, φ' over U, there is a continuous map $\theta_{\varphi,\varphi'} : U \to G$ so that

$$\varphi'(u, f) = \varphi(u, \theta_{\varphi,\varphi'}(u)(f))$$

for all $u \in U$ and all $f \in F$. The map $\theta_{\varphi,\varphi'}$ is called the transition function for φ, φ'.

(v) The collection of charts is maximal among collections satisfying the previous conditions.

The standard terminology is to call B the base space, F the fiber, and E the total space. We will use the notation $F \hookrightarrow E \xrightarrow{p} B$ or

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow {\scriptstyle p} \\ & & B \end{array}$$

to indicate a fiber bundle $p : E \to B$ with fiber F.

A map $p : E \to B$ is called a *locally trivial bundle* if it meets the first three requirements of Definition 2.6.1. Local trivialization is the important distinction between a fiber bundle and an arbitrary map. An example of a locally trivial bundle is the *trivial bundle*. It is the projection $p_B : B \times G \to B$.

Example 2.6.2. Bundles over $\mathbf{S}^2$

For integers $n \geq 0$, we can construct an S^1 bundle over S^2 with structure group $SO(2)$. We call n the Euler number of the bundle. For $n = 0$ we have the product bundle $p : S^2 \times S^1 \to S^2$. When $n = 1$, we obtain the Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$.

An important case of fiber bundles are principal bundles. Principal bundles are parametrized families of topological groups, and often Lie groups. They are more simple in the sense that the fiber F is the group G with a canonical action.

Definition 2.6.3. (Principal G-Bundle) *Let G be a topological group. It acts on itself by left translation. That is*

$$G \to Homeo(G), \qquad g \mapsto (x \mapsto gx).$$

A principal G-bundle *over B is a fiber bundle $p : P \to B$ with fiber $F = G$ and structure group G acting by left translations.*

Since the group action preserves the fibers of p it follows that the orbits of the G-action are these fibers and the orbit space E/G is homeomorphic to the base space B. We have the following lemma from [3]:

Lemma 2.6.4. [3] *If $p : P \to B$ is a principal G-bundle, then G acts freely on P on the right with orbit space B.*

Proof. Consider G acting on the local trivializations on the right

$$(U \times G) \times G \to U \times G$$

$$(u, g)g' = (u, gg').$$

This commutes with the action of G on itself by left translations, such that there exists a well-defined right action of G on E using the identification provided by the chart

$$U \times G \xrightarrow{\varphi} p^{-1}(U).$$

Define $\varphi(u, g)g' = \varphi(u, gg')$. If φ' is another chart over U, then

$$\varphi(u, g) = \varphi'(u, \theta_{\varphi,\varphi'}(u)g),$$

and $\varphi(u, gg') = \varphi'((u, \theta_{\varphi,\varphi'}(u)g)g')$, so the action is independent of the choice of chart. The action is free, since the local action $(U \times G) \times G \to U \times G$ is free. Since $U \times G/G = U$, it follows that $E/G = B$. □

A principal G-bundle is *trivial* if it is isomorphic to the product principal bundle $B \times G \to B$ which is locally trivially by construction.

Unless otherwise stated, all manifolds are closed and simply connected and all G-actions are assumed to be locally smooth, effective T^2-actions.

CHAPTER 3

Orbit Structure

3.1 Orbit Space

For the entire paper, we will be concerned with smooth, effective actions of the 2-dimensional torus group $G = T^2 = SO(2) \times SO(2)$ on closed, simply connected 4-dimensional manifolds, M. We can parametrize G by

$$G = \{(e^{i\theta}, e^{i\phi}) \mid 0 \leq \theta \leq 2\pi, \ \ 0 \leq \phi \leq 2\pi\}.$$

Or more simply, $G = (\theta, \phi)$. Let m, n be relatively prime integers. Define the circle subgroup of slope (m_i, n_i) in T^2 by $G(m_i, n_i) = \{(\theta, \phi) \mid m\theta + n\phi = 0\}$. That is, $G(m_i, n_i) = \{(e^{2\pi i(-\frac{n_i}{m_i \phi})}, e^{2\pi i \phi})\}$. Define the determinant of two circle subgroups $G(m_i, n_i)$ and $G(m_{i+1}, n_{i+1})$ by the 2×2 matrix

$$\det \begin{vmatrix} m_i & m_{i+1} \\ n_i & n_{i+1} \end{vmatrix} = \varepsilon_{i+1}.$$

Two circle subgroups have trivial intersection and generate the homology of T^2 if and only if they have determinant equal to ± 1.

An orbit whose isotropy group is isomorphic to the circle group will be called a C-orbit, denoted by C. The fixed point set of the T^2-action is denoted by $\mathrm{Fix}(M, T^2)$ or F. Together, these make up the singular orbits. It is shown in Neumann [30] that the possible isotropy groups of a T^2-action consists of the nontrivial finite cyclic group $\mathbb{Z}_p$, the identity $\{e\}$, the circle group $G(m, n)$, and the entire group T^2. We will state the following lemma without proof from [2] that provides the existence of a maximum orbit type.

Theorem 3.1.1. [2] *There exists a maximum orbit type G/H for G on M, where H is conjugate to a subgroup of each isotropy subgroup. The union $M_{(H)}$ of the orbits of type G/H is open and dense in M and its image $M^*_{(H)}$ in $M^* = M/G$ is connected.*

We say that the maximum orbit type for obits in M is the *principal orbit type* and orbits of this type are called *principal orbits*. Principal orbits have trivial isotropy group as we will show in the following lemma:

Lemma 3.1.2. [23] *Principal Orbits are of orbit type T^2.*

Proof. If principal orbits were of type T^2/H, where $H \neq \{e\}$, then all points $p \in M$ have isotropy subgroup G_p such that $H \subseteq G_p$. Therefore H acts trivially on M and the action is not effective, a contradiction. □

The next question to ask about the orbit space is what does the manifold part of M^* look like? The following lemma from [2] will illustrate that the orbit space, M^*, is a manifold with or without boundary.

Lemma 3.1.3. [2] *Let G be a compact Lie Group acting locally smoothly on an n-manifold M with M^* connected and d denotes the maximum orbit dimension so that dim$(M^*) = n-d$. If $n - d \leq 2$, then M^* is a manifold (with boundary).*

Proof. Let $k = n - d$, be the codimension of the principal orbit, that is, the dimension of the orbit space. The local structure on M^* can by analyzed by induction on k. Consider a linear tube around an orbit $G(x) = G/H$ in M. Then the linear tube has the form

$$G \times_H V \text{ and } (G \times_H V)^* \cong V^*.$$

V^* is the open cone over S^*, where S is the sphere in V and $\dim(M^*) = \dim(V^*) = \dim(S^*)+1$, since the process of coning increases the dimension by 1. If $k = 0$, then M^* is a discrete set of points and if M is a sphere, then M^* is one or two points. If $k = 1$, M^* has the local structure of the open cone over one point and is therefore a 1-manifold with boundary. M^* also has the local structure of the open cone over two points and therefore is a 1-manifold without boundary. When $k = 1$, and M is a sphere, then M^* is a compact, connected 1-manifold and hence is an arc or circle. Finally, when $k = 2$, then M^* has the local structure of an open cone over an arc and hence is a 2-manifold with boundary. M^* also has the local structure of an open cone over a circle and hence is a 2-manifold with possible boundary. □

By Lemma 3.1.3, for T^2 acting on a 4-dimensional manifold, the orbit space M^* is a 2-dimensional manifold with possible boundary. We will show in the following lemma that if M is simply connected, then the orbit space is also simply connected.

Lemma 3.1.4. [2] *If M is an arc-wise connected G-space, G compact Lie, and if there is an orbit which is connected or there are fixed points of the G-action, then the fundamental group of M maps onto that of M/G. Thus if M is simply connected, then so is M^*.*

Proof. Let $\pi : M \to M^*$ be the orbit map. Suppose that $G(x)$ is connected and $x^* = \pi(x)$. Define a loop $f : I \to M^*$ at x^* such that $f(0) = f(1) = x^*$. By the homotopy lifting property, there exists a unique lift $\tilde{f} : I \to M$ such that $f = \pi \circ \tilde{f}$. Composing $\tilde{f}$ by an element of G, let $\tilde{f}(0) = x$. Define a path $k : I \to G(x)$ from $\tilde{f}(1)$ to x. Taking the path composite, $\tilde{f} \circ k : I \to M$, we have a loop, γ, in M, based at x projecting to $f \circ \mathbf{1}_{x^*} \simeq f$, which is homotopic to f. Hence we have projected the fundamental group of M onto M^*. □

By our assumption that M is simply connected, it has trivial fundamental group and $H_1(M;\mathbb{Z}) \cong 0$. Hence by Lemma 3.1.4, M^* is also simply connected. We will show in the next sections that in the presence of fixed points which help make up singular orbits, then M^* is a 2-dimensional manifold with boundary such that the only possibilities are the 2-disk, D^2, or the sphere S^2.

3.2 Fixed Points

Let T^2 act effectively on a simply connected, compact 4-manifold, M. In this section, we will see there must exist a nontrivial T^2 isotropy group. We will begin with the following special case of a well known theorem of Kobayashi [21] which tells us that the Euler characteristic of the 4-manifold equals the Euler characteristic of the fixed point set of the T^2-action:

Theorem 3.2.1. [21] *Let T^2 act effectively on a 4-dimensional manifold, M. The Euler characteristic of the fixed point set is equal to the Euler characteristic of the manifold. That is, $\chi(\mathrm{Fix}(M, T^2)) = \chi(M)$.*

Proof. Let A_i be the closure of an ε-neighborhood of N_i, where N_i is a fixed point. Let ε be so small such that every point of A_i can be joined to the nearest point of N_i by a unique geodesic of length $\leq \varepsilon$ and that $A_i \cap A_j = \emptyset$ for $i \neq j$. Then A_i is a fibre bundle over N_i whose fibers are closed solid balls of radius ε. That is, we can deformation retract A_i onto N_i. Set $A = \cup A_i$ and let B be the closure of the open set $M \setminus A$. Then $A \cap B$ is the boundary of A. Consider the exact sequence of vector spaces:

$$\cdots \to U_k \to V_k \to W_k \to U_{k-1} \to \cdots$$

Taking the alternating sum of the vector spaces, we have the formula,

$$\sum (-1)^k \mathrm{dim} U_k - \sum (-1)^k \mathrm{dim} V_k + \sum (-1)^k \mathrm{dim} W_k = 0$$

which computes the Euler number. Applying this formula to the exact sequences of homology groups induced by

$$B \to M \to (M, B) \quad \text{and} \quad A \cap B \to A \to (A, A \cap B)$$

we obtain

$$\chi(B) - \chi(M) + \chi(M, B) = 0 \quad \text{and} \quad \chi(A \cap B) - \chi(A) + \chi(A, A \cap B) = 0$$

where χ denoted the Euler number. Using excision, we find that (M, B) and $(A, A \cap B)$ have the same relative homology and hence $\chi(M, B) = \chi(A, A \cap B)$. Equating the two equations, it follows that

$$\chi(M) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Since B and $A \cap B$ do not have fixed points, Lefschetz Theorem implies that $\chi(B) = \chi(A \cap B) = 0$. Therefore, $\chi(M) = \chi(A)$. And since A_i deformation retracts onto N_i, we have

$$\chi(A_i) = \chi(N_i).$$

Hence,

$$\chi(M) = \chi(A_i) = \chi(N_i).$$

□

In fact, Theorem 3.2.1 tells us more. If M admits a T^k-action, then $\chi(M) = \chi(\text{Fix}(M, T^l))$ for any $1 \leq l \leq k$.

Lemma 3.2.2. [33] *The T^2-action has fixed points, that is $F \neq \emptyset$.*

Proof. Since M is simply connected, $H_1(M;\mathbb{Z}) = 0$. By Poincaré Duality, $H_1(M;\mathbb{Z}) \cong H^3(M;\mathbb{Z}) = 0$ and $H_3(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) = 0$. Therefore, by the Universal Coefficient Theorem, $H_2(M;\mathbb{Z})$ is torsion free. The Euler characteristic of M is thus

$$\chi(M) = 2 + \text{rank} H_2(M) \geq 2.$$

Hence $\chi(M) \geq 2$. From the results of Theorem 3.2.1, we have the Euler characteristic of the fixed point set $\chi(\text{Fix}(M, T^2)) = \chi(M)$. Hence $\chi(\text{Fix}(M, T^2)) \geq 2$ and the T^2-action has fixed points as desired. □

3.3 Conical Orbit Structure

Definition 3.3.1. (Conical Orbit Structure) *A conical orbit structure denoted by $\overline{C(Y)} = (Y \times I)/(Y \times \{0\})$, is the closed cone over a space Y and by $C(Y) = Y \times [0,1)/(Y \times \{0\})$ the open cone. Then the orbit structure of X is called* conical*, if the orbit space X^* is homeomorphic to an open cone $C(Y)$ with constant orbit type along rays, less the vertex p^*.*

The following lemma is a tool for constructing a conical obit space which provides the existence of a canonical homeomorphism.

Lemma 3.3.2. [2] *Let X be a right H-space, Y a left H-space and a right K-space and Z a left K-space. Then there is a canonical homeomorphism*

$$(X \times_H Y) \times_K Z \to X \times_H (Y \times_K Z)$$

given by $[[x, y], z] \mapsto [x, [y, z]]$.

Proof. The map is well defined since

$$[[xh^{-1}, hy]k^{-1}, kz] = [[xh^{-1}, hyk^{-1}], kz]$$

goes to

$$\begin{aligned}[xh^{-1}, [hyk^{-1}, kz]] &= [xh^{-1}, h[yk^{-1}, kz]] \\ &= [x, [y, z]].\end{aligned}$$

Continuity follows from the fact that the composition

$$(X \times Y) \times Z \to (X \times_H Y) \times Z \to (X \times_H Y) \times_K Z$$

is open since, by definition, all orbit maps are open. The inverse has the same properties. □

Since we are working with a T^2-action on a closed, simply connected 4-dimensional manifold M, we have the following classification of a neighborhood of a point in M with isotropy group T^k for $k \leq 2$. For the case of a T^3-action on M^6, see McGavran in [23] and the general case of a T^n-action on M^{2n}, see Escher and Searle in [4].

Theorem 3.3.3. *Suppose T^2 acts locally smoothly on a closed 4-dimensional manifold, M. Suppose $p \in M$ has isotropy groups T^k, where $0 \leq k \leq 2$ and $T^0 = \{e\}$, the identity element. Let X be a closed invariant neighborhood of $p \in M$ such that $X^* = \overline{C(Y)}$ with vertex p^*. Suppose $C(Y)$ is an open subset of M^* with conical orbit structure. Then X is equivariantly homeomorphic to $T^{2-k} \times D^{2+k}$.*

Proof. Let $\pi : M \to M^*$ be the orbit map. By definition of an orbit space, since $C(Y)$ is an open subset of M^* with conical orbit structure and vertex p^*, then M has an open subspace, $\pi^{-1}(C(Y)) = X \setminus \pi^{-1}(Y \times 1)$, which is a 4-dimensional manifold. Since the action is locally smooth, we have:

$$\begin{aligned}X \setminus \pi^{-1}(Y \times 1) &\cong T^2 \times_{T^k} \mathring{D}^{2+k} \\ &= (T^{2-k} \times_e T^k) \times_{T^k} \mathring{D}^{2+k}\end{aligned}$$

By Lemma 3.3.2, we have the following homeomorphism:

$$\phi : (T^{2-k} \times_e T^k) \times_{T^k} \mathring{D}^{2+k} \to T^{2-k} \times_e (T^k \times_{T^k} \mathring{D}^{2+k})$$

which is equivariant. Hence

$$\begin{aligned}(T^{2-k} \times_e T^k) \times_{T^k} \mathring{D}^{2+k} &\cong T^{2-k} \times_e (T^k \times_{T^k} \mathring{D}^{2+k}) \\ &\cong T^{2-k} \times \mathring{D}^{2+k}.\end{aligned}$$

Therefore, we have the following equivariant homeomorphism:

$$\gamma : X \setminus \pi^{-1}(Y \times 1) \to T^{2-k} \times \mathring{D}^{2+k}.$$

Letting $W = \gamma^{-1}(T^{2-k} \times D^{2+k})$, where $\partial W \cong T^{2-k} \times S^{1+k}$ and $W^* \cong (Y \times [0, 1/2])/(Y \times 0)$. Let $U = X \setminus W$, so that X is the union of U and W attached via the identity along $T^{2-k} \times S^{1+k}$. Then X^* is just W^* attached to U^* along $(\partial W)^*$ where $U^* = (\partial W)^* \times I$. Hence, $U \cong \partial W \times I$ and $X \cong T^{2-k} \times D^{2+k}$. □

Using the results of Theorem 3.3.3, we can classify orbits and describe their image in the conical orbit space.

Lemma 3.3.4. [23] *Points on the boundary of M^* are singular orbits of isotropy type T^1 or T^2. Interior points are principal orbits of isotropy type $\{e\}$.*

Proof. Let $p \in M$ and $G_p = T^n$ for $n = 1, 2$. Then a linear tube about $G(p)$ is an open neighborhood of $G(p)$ equivariantly homeomorphic to $T^2 \times_{T^n} D^{2+n} \cong T^{2-n} \times D^{2+n}$ by Theorem 3.3.3. Then $(T^{2-n} \times D^{2+n})^*$ will be an open neighborhood of $p^* \in M^*$. In each case $(T^{2-n} \times D^{2+n})^*$ is conical with vertex p^*. Hence $p^* \in \partial M^*$. If $G_p = \{e\}$, then a linear tube about $G(p)$ is of the form $T^2 \times D^2$. Then $(T^2 \times D^2)^* \cong D^2$ is an open neighborhood of p^*, so p^* is in the interior of M^*. □

By Lemma 3.2.2 and Lemma 3.3.4, the points on the boundary of the orbit space are singular orbits (circle orbits and at least two fixed points) and interior points are principal orbits. Hence, the orbit space is D^2, a simply connected 2-manifold with boundary. We can show on the boundary of the orbit space, we have isolated fixed points. Consider $p \in M$ and $p \in T^2(p)$. By the Slice Theorem 2.4.6 and Lemma 3.3.4, if we look at the T^2-action on the normal space $\nu_p = \partial(T^2 \times_{T^2} D^4) = \partial(D^4) = S^3$, we have a cohomogeneity one action. This action is still effective, therefore the fixed points are isolated. If the action was determined to be ineffective, the fixed points would no longer be isolated.

We have the following lemma showing the orbit space D^2 has no E-orbits.

Lemma 3.3.5. [2] *Suppose that G is connected with a 2-dimensional orbit space, D^2. Suppose that M is compact and connected with $H_1(M, \mathbb{Z}) = 0$ and that a singular orbit exists. Then $E = \emptyset$.*

Proof. Let A^* be an arc in M^* from some fixed point f_i^* to $y^* \in E^*$ such that the interior points of A^* correspond to principal orbits. Consider $\pi : M \to M^*$. Let A be the lifting of A^* to M with $A = \pi^{-1}(A^*)$. Then A is the union of mapping cylinders

$$A = M_\varphi \cup M_\gamma$$

where $\varphi : P \to E$ and $\gamma : P \to S$ are equivariant maps from a principal orbit to an exceptional orbit and a singular orbit, respectively. If φ is a k-fold covering map, then we have the exact sequence of φ

$$\cdots \to H^2(E;\mathbb{Z}) \to H^2(P;\mathbb{Z}) \to H^3(M_\varphi, P;\mathbb{Z}) \to H^3(E;\mathbb{Z}) = 0$$

The first two groups are infinite cyclic since P and E are connected orientable 2-manifolds and the map between them is multiplication by k since $\deg\varphi = k$. Thus $H^3(A, M_\gamma;\mathbb{Z}) \cong H^3(M_\varphi, P;\mathbb{Z}) \cong \mathbb{Z}_k$. But M_γ has S as a deformation retract and $\dim(S) \leq 1$. Thus $H^3(A;\mathbb{Z}) \cong \mathbb{Z}_k$. By Poincaré Duality, $H^3(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) = 0$, and the exact sequence

$$0 = H^3(M;\mathbb{Z}) \to H^3(A;\mathbb{Z}) \to H^4(M, A;\mathbb{Z}) \to \cdots$$

shows that $H_0(M \setminus A;\mathbb{Z}) \cong H^4(M, A;\mathbb{Z})$ has torsion, which is impossible. Hence $E = \emptyset$ □

We can prove the following lemma from Audin in [1] using the existence of tubes and the results of Theorem 3.3.3:

Lemma 3.3.6. [1] *The union of all the orbits of a given type is a sub-manifold of M.*

Proof. Let (H) be the conjugacy class of subgroups of G and let $M_{(H)}$ be the union of all the orbits of type (H) such that:

$$M_{(H)} = \{x \in M \mid G_x \in (H)\}.$$

Let $x \in M_{(H)}$, then we will show that $M_{(H)}$ is a sub-manifold in the neighborhood of the orbit of x. Look at orbits of type (H) in $G \times_H A$. If $g' \in G_{[g,a]}$, we have $g'[g,v] = [g'g,a]$ if and only if for some $h \in H$ and $a \in A$, $g'g = gh^{-1}$ and $a = ha$. Thus $G_{[g,a]} = gH_a g^{-1}$ is the conjugation class of H_a and all conjugates appear when g varies and a is fixed. The orbit of $[g,a]$ is of type (H) if and only if $H_a = H$, that is, if and only if a is a fixed point of the H-action in A. Let

$$F = \{a \in A \mid h(a) = a, \text{ for all } h \in H\}$$

be the set of these fixed points. This is a subspace of A and

$$(G \times_H A) \cap M_{(H)} = \{[g,a] \in G \times_H A \mid G_{[g,a]} \in (H)\} = G \times_H F$$

is a sub-bundle of $G \times_H A$ and hence a sub-manifold. □

Consider the case when the isotropy group is equal to the whole group, T^2. By definition, when this occurs, we have a fixed point. Therefore by Lemma 3.3.6 we have the following theorem:

Theorem 3.3.7. [1] *The set of fixed points of G is a sub-manifold of M.*

Proof. The result follows straight from the proof of Lemma 3.3.6. □

Therefore, the orbit types give a decomposition of M into sub-manifolds. Since M is compact, we know by Definition 2.4.3 and Lemma 3.3.6 that each tube about an orbit of type G/H contains only a finite number of orbit types.

We have now shown that M^* is a compact 2-manifold with nonempty boundary, that is $M^* \cong D^2$. By examining the linear tubes, the orbit space can be described just by the boundary of M^*. Interior points correspond to principal orbits, that is, orbits with trivial isotropy. The boundary of M^* has only two types of orbits where a point on the boundary may be the image of a circle orbit or it may be a fixed point. That is, orbits with T^1 isotropy or T^2 isotropy, respectively. We have also found that there are no exceptional orbits. We present the following table that lists the possible isotropy groups together with the slice, the action of the isotropy group on the slice and the image of the orbit on M^*.

Isotropy Group	Orbit	Slice	Action of Isotropy Group on Slice	Image of Orbit on M^*
$\{e\}$	T^2	D^2	rotation	interior point
$G(m,n)$	T^1	D^3	rotation	boundary point
G	fixed point	D^4	rotation in two 3-planes by $G(m_1,n_1)$ and $G(m_2,n_2)$	isolated boundary point $G(m_1,n_1)$, $G(m_2,n_2)$ with $\det \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} = \pm 1$

CHAPTER 4

Cross-Sectioning Theorem

In this chapter, we prove the Cross-Sectioning Theorem needed for the Equivariant Classification Theorem. The following results follow from [4] and [23]. The main tool needed for this classification is the existence of smooth cross-sections.

Theorem 4.0.1. [4][23] *Let T^2 act effectively on a smooth, closed, orientable 4-dimensional manifold, M such that $M^* = M/G$ is a 2-dimensional disk, D^2. Assume that M^* has only two types of orbits, all interior points are principal orbits and the points on the boundary correspond to singular orbits with connected isotropy subgroups. Then the orbit map $\pi : M \to M^*$ has a cross-section.*

4.1 Constructing Cross-Sections

The key topological tool in the proof comes from obstruction theory. If X is a connected abelian CW complex and (W, A) is a CW pair such that $H^{i+1}(W, A; \pi_i X) = 0$ for all i, then every map $A \to X$ can be extended to a map $W \to X$. We have the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \\ W & & \end{array}$$

For more information on obstruction theory refer to Chapter 7 of Davis and Kirk [3] or Chapter 4 of Hatcher [16].

In the following lemma, consider a closed $T^2 = T_1 \times T_2$ invariant subset C of the closed manifold, M, where M is as in Theorem 4.0.1. Choose C so that its orbit space under the T^2-action, $C^* \subset M^* = D^2$ is a closed conical section of M^*, with conical orbit space as in Definition 3.3.1. Hence, $C^* \cong D^2 = \overline{C(D^1)}$, is a closed cone over D^1, where D^1 is an arc.

We can choose C so that its intersection with the boundary of M^* is homeomorphic to D^1. Figure 4.1 below illustrates the orbit space C^* and its homeomorphic image D^2:

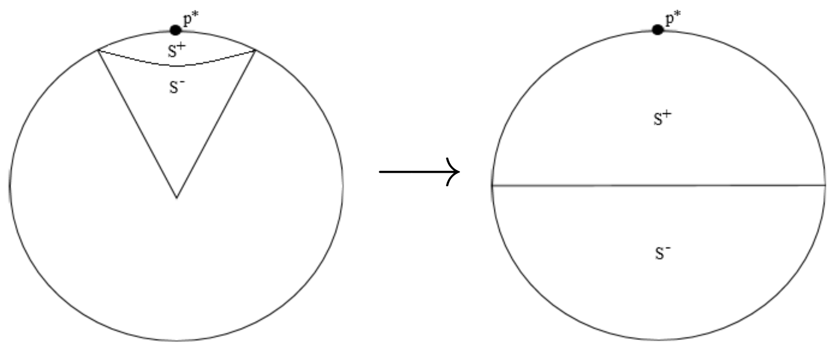


Figure 4.1: The conical section $C^* \subset M^*$ and its homeomorphic image D^2.

Denote the arc we cone over by S^- and the cone point will be denoted by $p^* \in S^+$, where $S^+ = C^* \cap \partial M^*$. Since isotropies are constant along rays from p^*, we can partition (S^+, p^*) into 1-dimensional cells denoted by U_1, U_2 provided they all intersect in p^*. Hence $p^* \in \cap_{j=1}^{2} U_j$. For each U_l, we associate the corresponding circle isotropy subgroup T_l, where $1 \leq l \leq 2$. By assumption, T_l generates T^2 and each pair of distinct circles has trivial intersection, that is, $T_1 \times T_2 = T^2$. This gives us a weighted decomposition of (S^+, p^*) which we will denote by

$$\{(U_1, T_1), (U_2, T_2)\}.$$

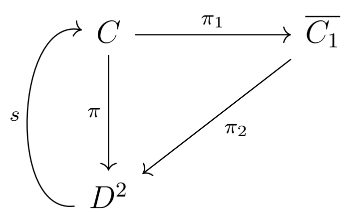
In this decomposition, each intersection of j-cells in a $(j-1)$ cell corresponds to connected isotropy subgroup of the T^2-action generated by the isotropy subgroup associated to each of the j-cells.

The simplest decomposition will be given in the following lemma where the decomposition of (S^+, p^*) is given by $\{(U_2, T_2)\}$ and is illustrated in figure 4.1 on the right. Note that by construction, all nontrivial isotropies are connected and correspond to points on S^+ and all other orbits are principal. We began the simplest case and show that we can construct a cross section for C^*.

Lemma 4.1.1. [4][23] *Let $T^2 = T_1 \times T_2$ act smoothly on a smooth, closed 4-dimensional subspace $C \subset M$, where M is a smooth, closed 4-dimensional manifold, and C has quotient space C^* as described above. Suppose $(S^+, p^*) = \{(U_2, T_2)\}$. Then there exists a cross section. Moreover, suppose a cross-section is given on a 1-cell, $A \subset S^-$, then it can be extended to a cross section over all of D^2.*

Proof. Note that the orbit space $C/T^2 = C^*$ is a closed cone with vertex p^*. Since the orbit structure is conical and $G_p = T_2$, then by Theorem 3.3.3, C is equivariantly homeomorphic to $T^1 \times D^3$, with T_2 acting orthogonally on D^3. Construct a cross section from D^2 to D^3 using the inverse of the projection map $\pi : T^1 \times D^3 \to D^2$ given via the orthogonal action of T_2 on D^3 and then construct a section from D^3 to $T^1 \times D^3$ by sending an arbitrary point $x \in D^3$ to $(t, x) \in T^1 \times D^3$ for some $t \in T^1$.

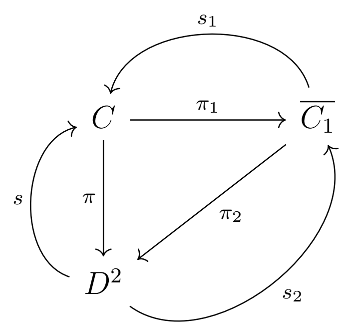
Suppose a cross section s is given on a 1-cell $A \subseteq S^-$. Let $A' = A \cap (D^2 \setminus S^+)$ and let $\pi : C \to C^* \cong D^2$ be the orbit map. Then $\pi^{-1}(D^2 \setminus S^+)$ is a principal T^2-bundle over $(D^2 \setminus S^+)$. Using the long exact sequence for relative cohomology and excision, it follows that $H^2((D^2 \setminus S^+), A') = 0$. By obstruction theory, we may assume that s is defined on $(D^2 \setminus S^+) \cup A$ and we have the following diagram where $\pi_1 : C \to \overline{C}_1 \cong C/T^1 \cong D^3$ and $\pi_2 : D^3 \to C^* \cong D^2$:



Letting $s_2 = \pi_1 \circ s$, we obtain a cross-section s_2 to π_2 defined on $(D^2 \setminus S^+) \cup A$. Since S^+ corresponds to the set of fixed points of the T_2-action on $\overline{C_1}$, we can define s_2 on all of D^2. We need to check the continuity of s_2 by first describing D^3 as $I \times S^+$, where I is an interval. Note that $\pi_2^{-1}(S^+) \cong D^1$ and $\pi_2^{-1}(I) \cong \overline{C}(T_2) \cong D^2$ and the T_2-action on $\pi_2^{-1}(I)$ is a rotation. Hence the T_2 action on $C_1 \cong D^3 \cong D^2 \times D^1$ is given by rotation on the first factor and trivial on the second.

An orbit is now described as $\{re^{i\theta}, 1 | 0 \le \theta < 2\pi\}$ and the fixed point set of T_2 on $D^2 \times D^1$ is $\{0\} \times D^1$. Let $q = (0, Se^{i\sigma}) \in \text{Fix}(T_2, D^2 \times D^1)$ and $\{q_n^*\} = \{(r_n e^{i\theta_n}, R_n e^{i\Theta_n})^*\}$ be a sequence in M^* converging to q^*. Then $r_n \to 0$, $R_n \to S$, and $\Theta_n \to \sigma$. Then the sequence $\{s_2(q_n^*)\}$ will be of the form $\{(r_n e^{i\theta_n}, R_n e^{i\Theta_n})^*\}$ which converges to $q = (0, Se^{i\sigma})$. Thus s_2 is continuous.

Now we need a cross-section to s_1 defined on $s_2(C^*) \cong D^2$. Let $s_1 = s \circ \pi_2 : (s_2(C^*) \setminus s_2(S^+ \setminus A)) \to C$. Then $\pi_1^{-1}(s_1(C^*))$ is a principal T^1-bundle over $s_2(C^*) \cong D^2$ and we have s_1 which is a cross section on π_1 defined on $s_2(C^*) \setminus s_2(S^+ \setminus A)$. Since $s_2(S^+ \setminus A)$ is a homology 1-cell on the boundary of $s_2(C^*)$, it follows that $H^2(s_2(C^*), s_2(C^*) \setminus s_2(S^+ \setminus A)) = 0$. Again by obstruction theory, s_1 can be extended to all of $s_2(C^*)$. Thus $s_1 \circ s_2$ extends s to all of C^* and we have the following diagram:



□

Using the results of Lemma 4.1.1, we can now prove Theorem 4.0.1.

Proof. First decompose the orbit space $M^* \cong D^2$ into a collection of conical sections $\{C_i^*\}_{i=1}^m$, with $C_i^* \cong D^2$ for each $i \in \{1, ..., m\}$, and such that the orbit structure for each $C_i \subset M^4$ is conical as in the Definition 3.3.1. From the proof of Lemma 4.1.1, each $C_i \cong T^{2-l} \times D^{2+l}$ for some $l \in \{1, 2\}$ and therefore a cross-section exists on each C_i^*. By obstruction theory, a cross-section given on a 1-cell $A \subset S^- \subset C_i^*$ can be extended to all of C_i^*. A partial decomposition is shown in Figure 4.2:

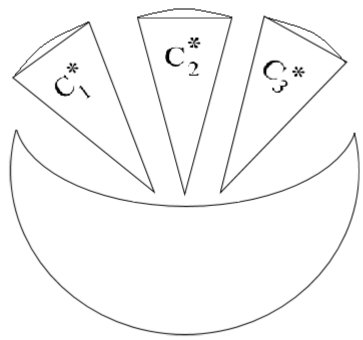


Figure 4.2: Conical Orbit Structure

To create a cross-section on all of M^*, start by defining a cross-section s on C_1^*. Attach C_2^* to C_1^* along a 1-cell A. We have a cross-section s defined on C_1^* and on $A \subset S^- \subset C_2^*$. By Lemma 4.1.1, we can extend the cross-section s to all of C_2^*. Continuing this process we can extend s to all of M^*. □

4.2 Simply Connected 4-Manifolds

Using the Cross-Sectioning Theorem, we can show that a compact, closed, connected 4-manifold M is simply connected.

Theorem 4.2.1. [23] *Let M be a compact, closed, and connected 4-manifold. Let M admit a locally smooth T^2-action with orbit map $\pi : M \to M^* \cong D^2$ and any fixed point $p \in M$ has an open neighborhood equivariantly homeomorphic to D^4 with an orthogonal T^2-action. Therefore M is simply connected.*

Proof. Decompose M^* into conical sections as illustrated in Figure 4.2. Each C_i^* represents an invariant tubular neighborhood, C_i, of a fixed point and hence is homeomorphic to D^4. Then M is obtained by attaching the C_i's together along subspaces of their boundaries homeomorphic to $D^2 \times T^1$. Now, if we remove D^4, this corresponds to removing a small closed 2-cell about the image of the fixed point in the orbit space by cross-sectioning. The cross-section now retracts to the remaining boundary. This deformation lifts equivariantly to a deformation of the manifold to a linear chain of 2-spheres. Their number equals the number of fixed points minus 2. By Van Kampen's Theorem, the fundamental group of the total space is trivial. Hence M is simply connected. $\square$

CHAPTER 5

Equivariant Classification

It follows from Chapter 4, that if T^2 acts smoothly and effectively on a closed, simply connected 4-manifold, then the orbit map $\pi : M \to M^*$ has a cross section if and only if there are no finite stability groups and if and only if all orbits are principal then the principal bundle is trivial. In this chapter, an equivariant classification for actions without exceptional orbits and with the presence of fixed points and C-orbits is given. Consider the weighted orbit spaces M_1^* and M_2^* of a T^2-action on M_1 and M_2, respectively. A weight preserving homeomorphism of M_1^* onto M_2^* is a homeomorphism of M_1^* onto M_2^* which carries the weights of M_1^* onto the weights of M_2^*. We therefore have the following Equivariant Classification Theorem:

Theorem 5.0.1. [19][33] *Let T^2 act effectively on simply connected closed 4-manifolds M_1 and M_2 such that the orbit map has a cross section. Then there exists an equivariant homeomorphism h of M_1 onto M_2 if and only if there exists a weight preserving homeomorphism h^* of M_1^* onto M_2^*. Furthermore, if M_1 and M_2 are oriented and the orientation of M_1^* and M_2^* are those induced by M_1 and M_2, then h is orientation preserving if and only if h^* is orientation preserving.*

5.1 Weighted Orbit Spaces and Orientation

Definition 5.1.1. (Weighted Orbit Space) *Let G act smoothly on a 4-manifold M with orbit space $M^* = M/G$. To each orbit in M^*, there is associated to it a certain orbit type which is characterized by the isotropy group of the points of the orbit together with the slice representation at the given orbit. This orbit space together with its orbit types and slice representation is called a* weighted orbit space.

Let T^2 act smoothly on a 4-manifold M with orbit space M^*. Each orbit in M^* has a certain orbit type associated to it that is characterized by the isotropy group of the points on the orbit together with the slice representation at the given orbit. We can think of the slice representation at a fixed point as just the action of the isotropy group on the normal disk. With the presence of a fixed point f_i^*, the action is determined by the pairs $\pm G(m_i, n_i)$ and $\pm G(m_{i+1}, n_{i+1})$, where

$$\det \begin{vmatrix} m_i & m_{i+1} \\ n_i & n_{i+1} \end{vmatrix} = \varepsilon_{i+1} = \pm 1$$

depending on orientation.

It is useful to work with a certain orientation. Let G be oriented such that an orientation of M determines an orientation of M^* and vice versa if there are no isotropy groups which reverse the orientation of the slice. The orientation of M can be described as the product orientation of the set of principal orbits. When the orbit map has a cross section, then the orientation of the image of M^*, via the cross section, is that induced by the cross-section and that the orientation of M is compatible with it.

Recall that the orbit space together with a chosen orientation, the associated orbit structure and the slice representation is called a weighted orbit space. It will completely determine the action up to orientation preserving equivariant diffeomorphism and is determined by the following orbit invariants:

(i) $g \geq 0$; the genus of the oriented 2-manifold M^*.

(ii) $s \geq 0$; the number of boundary components of M^* whose orbits are C-orbits.

(iii) $t \geq 0$; the number of boundary components of M^* each having fixed points.

(iv) ε; a specific orientation assigned to the orientable 2-manifold M^*. An orientation for T^2 is chosen, so the orientation of ε determines the orientation for M.

(v) $\langle p_i, q_i \rangle$; the ith boundary component of M^*, consisting of C-orbits with isotropy group $G(p_i, q_i)$ or equivalently $G(-p_i, -q_i)$. $\langle p_i, q_i \rangle$ denoted either parametrization together with an orientation of the boundary circles compatible with ε.

(vi) $\{m, n\}_i = \{\langle m_{i,1}, n_{i,1} \rangle, ..., \langle m_{i,t_i}, n_{i,t_i} \rangle\}$; the ith boundary component of M^* which contains t_i fixed points for $t_i \geq 2$. Here $\langle m_{i,j}, n_{i,j} \rangle = \pm(m_{i,j}, n_{i,j})$ is a parametrization of the isotropy group of the jth oriented arc of the ith boundary component. The order of the entries in $\{m, n\}$ is determined up to cyclic permutation.

We may represent the 4-manifold M in terms of its weighted orbit space using the invariants above:

$$M = \{\varepsilon; g; s; t; \langle p_1, q_1\rangle, ..., \langle p_s, q_s\rangle; \{m, n\}_1, ..., \{m, n\}_t\}.$$

Or more simply, we can write our simply connected oriented 4-manifold as:

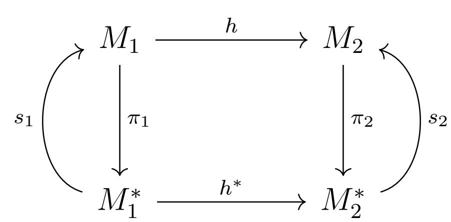
$$M = \{(m_1, n_1), (m_2, n_2), ..., (m_t, n_t)\}$$

where $t \geq 2$ is the number of fixed points and M^* is oriented from (m_i, n_i) to (m_{i+1}, n_{i+1}) for $i = 1, ..., t$.

5.2 Main Result

Let M_1^* and M_2^* represent the orbit spaces of a smooth action of T^2 on closed, simply connected 4-manifolds M_1 and M_2. A homeomorphism of M_1^* and M_2^* which caries the weights of M_1^* isomorphically onto the weights of M_2^* is called a weight preserving homeomorphism. We can now prove Theorem 5.0.1, the Equivariant Classification Theorem:

Proof. Let $\pi_i : M_i \to M_i^*$ be the orbit map and let $s_i : M_i^* \to M_i$ be the cross-section. Consider the following diagram:



Define $\bar{h} : s_1(M_1^*) \to s_2(M_2^*)$ by $\bar{h} = s_2 \circ h^* \circ \pi_1$. If m and tm are both in $s_1(M_1^*)$ for $m \in M_1$ and $t \in T^2$, then $t = \{e\}$. Hence, $\bar{h}(tm) = t\bar{h}(m)$. By Lemma 4.1.1, we can extend $\bar{h}$ uniquely to $h : T^2(s_1(M^*)) = M_1 \to M_2$ where $h(tm) = t\bar{h}(m)$ for $m \in s_1(M_1^*)$ and $t \in T^2$. We can define the inverse of h in the same way and hence, h is a homeomorphism. $\square$

By work of Fintushel [6][7], homeomorphism has been improved to diffeomorphism. Therefore, there exists an equivariant diffeomorphism h of M_1 onto M_2 if and only if there exists a weight preserving diffeomorphism h^* of M_1^* onto M_2^* as stated in Theorem 5.0.1.

CHAPTER 6

4-Manifolds

In this chapter, we define intersection form and look at a few examples. This is an important tool used for topologically classifying simply connected 4-dimensional manifolds that will be used in Chapter 7.

6.1 Intersection Form

In this section, we define a notation that interprets intersections of surfaces inside manifolds. The information follows from Perutz [39], Scorpan [42], and Weiler [43].

Definition 6.1.1. Intersection Form *Given any closed, oriented 4-manifold M, its* intersection form *is the symmetric bilinear form defined as follows:*

$$Q_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to Z$$

$$Q_M(\alpha, \beta) = (\alpha \cup \beta)[M]$$

with cohomology classes α and β.

By Poincaré duality, $H_2(M,\mathbb{Z}) \cong H^2(M,\mathbb{Z})$ and Q_M can be represented on $H_2(M,\mathbb{Z}) \times H_2(M;\mathbb{Z})$ as well. The intersection form is also *unimodular*, meaning that the matrix representing Q_M is invertible over $\mathbb{Z}$. That is, we can represent the intersection form by a matrix of determinant ± 1. Note that Q_M vanishes on any torsion element and can be defined on the free part of $H^2(M;\mathbb{Z})$. Since our manifolds are assumed to be simply connected and by Poincaré duality, torsion does not arise. Therefore, $H_2(M;\mathbb{Z})$ is a free $\mathbb{Z}$-module and there are isomorphisms $H_2(M;\mathbb{Z}) \cong \oplus m\mathbb{Z}$ where $m = b_2(M)$.

To understand the intersection form geometrically, consider the case of smooth manifolds. Then $Q_M(\alpha, \beta)$ can be interpreted as the intersection number of certain sub-manifolds in M. Further, we will represent classes $\alpha, \beta \in H_2(M;\mathbb{Z})$ by embedded surfaces S_α and S_β and redefine $Q_M(\alpha, \beta)$ as the intersection number of S_α and S_β. Therefore when M

is simply-connected, by Hurewitz's Theorem, $\pi_2(M) \cong H_2(M;\mathbb{Z})$. This implies that every second homology class can be represented as an immersed S^2 and the immersion $S^2 \to M$ intersects itself in transverse double points.

We have the following important theorem by the work of Freedman [8] in 1982 classifying 4-manifolds up to homeomorphism by their intersection form.

Theorem 6.1.2. [8] *For each symmetric bilinear unimodular form Q over $\mathbb{Z}$, there exists a closed orientable, compact, simply connected topological 4-manifold M with Q_M as its intersection form.*

6.2 Intersection Form and Connected Sums

A simple way to combine two 4-manifolds is the simplest way to combine two intersection forms. Consider the *connected sum* of two manifolds M and N denoted by

$$M\#N.$$

This is the simplest method for combining two manifolds M and N into one connected manifold. This is done by joining the two manifolds together by a tube. In the 4-dimensional case, remove a small open 4-ball in each manifold M and N. We are left with two manifolds M° and N° with boundary S^3. Identifying these 3-spheres, we obtained a closed manifold $M\#N$. The identification of the two 3-spheres is made through an orientation reversing diffeomorphism $\partial M^\circ \cong \overline{\partial N^\circ}$. Essentially, we are identifying the 3-spheres by an orientation flip. We therefore have the following lemma:

Lemma 6.2.1. [42] *If M and N have intersection forms Q_M and Q_N, then their connected sum $M\#N$ will have intersection form*

$$Q_{M\#N} = Q_M \oplus Q_N.$$

6.3 Examples

Useful examples are given that will be needed in Chapter 7.

Example 6.3.1. [13] **The Four Sphere**, S^4

For the four sphere, S^4, we have $H^2(S^4;\mathbb{Z}) \cong 0$. Therefore, the intersection form Q_{S^4} is trivial.

Example 6.3.2. [39] **The Projective Plane** $\mathbb{C}P^2$

For the complex plane, $\mathbb{C}P^2$ we have $H^2(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}$ generated by a class S_α with $S_\alpha \cdot S_\alpha = 1$. The Poincaré dual of S_α is represented by fundamental class of a projective line $L = \mathbb{C}P^1 \subset \mathbb{C}P^2$. For any other such line, L', distinct from L, the intersection $L \cap L'$ is a single point. Hence the intersection number is $+1$ and $\mathbb{C}P^2$ has intersection form,

$$Q_{\mathbb{C}P^2} = [+1].$$

The opposite orientation $\overline{\mathbb{C}P^2}$ has intersection form,

$$Q_{\overline{\mathbb{C}P^2}} = [-1].$$

Consider line bundles over spheres. The only nontrivial Chern class of line bundles is the first Chern class. Therefore, they are classified by their first Chern class, which has values in $H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$ (note that the top Chern class is equal to the Euler class of the bundle). If we compactify each fiber by adding a point at infinity, we get a S^2 bundles over S^2 with generators from the homology of the zero section (base space) and a fiber. Let L be a sphere bundle obtained in this way. The intersection form of its total space is

$$\begin{pmatrix} c_1(L) & 1 \\ 1 & 0 \end{pmatrix}$$

where $c_1(L)$ is its first Chern class. However, a S^2 bundle over S^2 is defined by a choice of gluing over the equator of the base spheres using the identification of the S^2 fibers that rotates them by 2π. That is, a S^2 bundle over S^2 is described by an equatorial gluing map $S^1 \to SO(3)$ and $\pi_1(SO(3)) = \mathbb{Z}_2$. Hence there are only two topologically distinct sphere bundles over a sphere, those with even $c_1(L)$ (trivial case) and those with odd $c_1(L)$ (nontrivial case). The two such sphere bundles are $S^2 \times S^2$ and the twisted product $S^2 \tilde{\times} S^2$.

Additionally, base fiber equators describe a circle bundle with Euler number 1, which has to be the Hopf circle bundle $S^3 \to S^2$. Hence the sphere bundle is cut into two halves by a 3-sphere. Each of these halves is a disk-bundle of Euler number 1 and can be identified with a neighborhood $\mathbb{C}P^1$ inside $\mathbb{C}P^2$, where the complement of such a neighborhood is a 4-ball.

Example 6.3.3. [39] **Trivial Sphere Bundle** $S^2 \times S^2$

The trivial sphere bundle has $H_2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}^2$*. For the product space* $S^2 \times S^2$*, we have the following intersection form,*

$$Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $c_1(L) = 0$ *since the self intersection is empty.*

Example 6.3.4. [43] **Nontrivial Sphere Bundle**

The nontrivial sphere bundle has $H_2(S^2 \tilde{\times} S^2; \mathbb{Z}) \cong \mathbb{Z}^2$*. The twisted product* $S^2 \tilde{\times} S^2$ *has intersection form,*

$$Q_{S^2 \tilde{\times} S^2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

with $c_1(L) = 1$*. A simple change of basis yields the intersection form*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = [+1] \oplus [-1].$$

We can see then that $S^2 \tilde{\times} S^2 \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$*.*

CHAPTER 7

Topological Classification of Simply Connected 4-Manifolds

We now have the necessary information to equivariantly topologically classify simply connected 4-manifolds which admit effective T^2-actions. The following results follow from the work of [10] and [33]. In this chapter, we will prove the following theorem:

Theorem 7.0.1. [33] *Let T^2 act effectively and smoothly on a closed, orientable, simply connected 4-dimensional manifold M. Then M is equivariantly diffeomorphic to one of these 4-manifolds: S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, $S^2 \times S^2$. Or an equivariant connected sum of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, $S^2 \times S^2$.*

7.1 Orbit Space

Recall by Lemma 3.2.2, for a closed, orientable, simply connected manifold M, the T^2-action has nonempty fixed point set and there are circle orbits. By Lemma 3.3.5, there are no exceptional orbits. Therefore, given $E = \emptyset$, $F \neq \emptyset$, $C \neq \emptyset$ and $M^* \cong D^2$, let $f_1, f_2, ..., f_t$ denote the fixed points of the T^2-action for $t \geq 2$ and f_i^* their image in M^*. The arc, S_i^* between f_i^* and f_{i+1}^* on ∂M^* represents an invariant 2-sphere S_i. Points on ∂M^* have either T^1- or T^2-isotropy and all interior points have trivial isotropy.

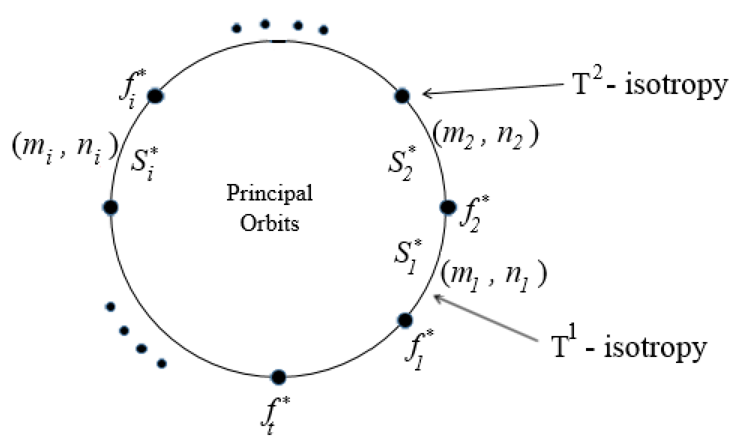


Figure 7.1: Orbit Structure

Note for $i = 2, 3, ..., t$ we have the following determinants:

$$\det \begin{vmatrix} m_{i-1} & m_i \\ n_{i-1} & n_i \end{vmatrix} = \varepsilon_i = \pm 1 \qquad \det \begin{vmatrix} m_t & m_1 \\ n_t & n_1 \end{vmatrix} = \varepsilon_1 = \pm 1$$

where each ε_i corresponds to an orientation.

In a neighborhood of a fixed point in M^*, we look at the inverse image of a small arc $L^*_{i+1,i}$ from an orbit of type $G(m_{i+1}, n_{i+1})$ to one of type $G(m_i, n_i)$ with principal orbits as the interior of M^*. By the Slice Theorem 2.4.6, we have an action of the isotropy group $G_{f_i} = T^2$ on the unit normal 3-sphere where the action must be effective. By the work of Neumann [30], the action corresponds to a cohomogeneity one action with possible orbit diagram $(\{e\}, G(1,0), G(m,n))$ for m, n relatively prime. In particular, this implies that $G(1,0)$ and $G(m,n)$ are orthogonal to each other in T^2. Therefore the possible 3-manifolds are denoted by $L(n,m)$ corresponding to: S^3 when $L(n,1)$ for $n \geq 1$, S^3 again when $L(1,0)$, and $S^1 \times S^2$ when we have $L(0,1)$. Note that $L(n,m)$ satisfies the identities:

$$L(n,m) = L(-n,-m) = \overline{L}(-n,m) = \overline{L}(n,-m) = \overline{L}(n, n-m)$$

where $\overline{L}$ means reversed orientation.

By Theorem 3.3.3 the following orbit space of M corresponding to a neighborhood of a fixed point is diffeomorphic to $D^2 \times D^2$ and has the following orbit space denoted by $W^*_{i+1,i}$:

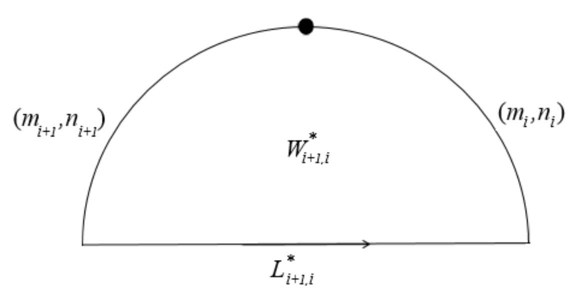


Figure 7.2: One Fixed Point Decomposition

The orbit space is decomposed as follows: the interior points of $W^*_{i+1,i}$ together with the interior points of the straight arc $L^*_{i+1,i}$ correspond to principal orbits. The remainder of the boundary has one fixed point and two intervals with isotropy groups $G(m_i, n_i)$ and $G(m_{i+1}, n_{i+1})$. In this decomposition, the straight arc, $L^*_{i+1,i}$ will be the image of the invariant 3-sphere, S^3. Therefore the original space $W_{i+1,i}$ can be seen as the cone over a 3-sphere with the action extended to the cone from the boundary 3-sphere. The space $W_{i+1,i}$ is a manifold with boundary, that is, a 4-cell.

Now consider the local orbit structure around two fixed points:

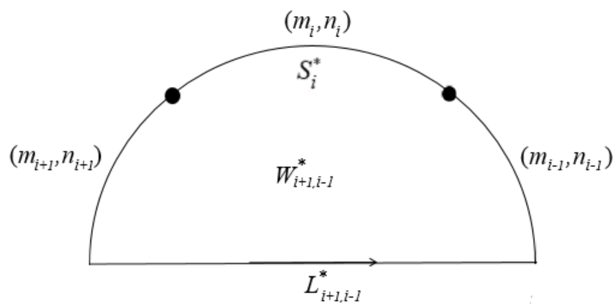


Figure 7.3: Two Fixed Points Decomposition

Interior points of $W^*_{i+1,i-1}$ together with the interior points of the straight arc $L^*_{i+1,i-1}$ represent principal orbits. The arc S^*_i is the orbit space of an invariant 2-sphere S_i in $W_{i+1,i-1}$. The remainder of the boundary has two fixed points and three intervals with isotropy group $G(m_{i-1}, n_{i-1})$, $G(m_i, n_i)$, and $G(m_{i+1}, n_{i+1})$ satisfying the following conditions:

$$\det \begin{vmatrix} m_{i-1} & m_i \\ n_{i-1} & n_i \end{vmatrix} = \varepsilon_i = \pm 1 \qquad \text{and} \qquad \det \begin{vmatrix} m_i & m_{i+1} \\ n_i & n_{i+1} \end{vmatrix} = \varepsilon_{i+1} = \pm 1.$$

To determine the inverse image of the straight arc, $L^*_{i+1,i-1}$, consider the following lemma:

Lemma 7.1.1. *The straight arc, $L^*_{i+1,i-1}$ has inverse image the oriented lens space, $L_{i+1,i-1} = L(r, s)$ for $r \neq 0, 1$.*

Proof. For the oriented lens space $L(r, s)$ let

$$\varepsilon_{i+1} = \det \begin{vmatrix} m_i & m_{i+1} \\ n_i & n_{i+1} \end{vmatrix} = 1.$$

Then

$$r = \det \begin{vmatrix} m_{i-1} & m_{i+1} \\ n_{i-1} & n_{i+1} \end{vmatrix} \qquad \text{and} \qquad s = \det \begin{vmatrix} m_i & m_{i-1} \\ n_i & n_{i-1} \end{vmatrix}.$$

From $s = m_i n_{i-1} - m_{i-1} n_i$ and $\varepsilon_{i+1} = m_i n_{i+1} - m_{i+1} n_i = 1$ we obtain for $r \neq 0$ and $r \neq 1$,

$$m_i = \frac{\varepsilon_{i+1} m_{i-1} - s m_{i-1}}{m_{i+1} n_{i-1} - n_{i+1} m_{i-1}} = \frac{m_{i+1} - s m_{i+1}}{r}.$$

Thus, $s m_{i+1} \equiv m_{i-1} \mod r$, determining s uniquely in the interval $-|r| < s < |r|$.

Now for $\varepsilon_i = m_{i-1} n_i - n_{i-1} m_i$ and $\varepsilon_{i+1} = -m_{i+1} n_i - n_{i+1} m_i$, we have for $r \neq 0$ and $r \neq 1$,

$$m_i = \frac{\varepsilon_{i+1} m_{i-1} + \varepsilon_i m_{i-1}}{m_{i+1} n_{i-1} - n_{i+1} m_{i-1}} = \frac{\varepsilon_{i+1} m_{i+1} - \varepsilon_i m_{i+1}}{r}.$$

Thus $\varepsilon_{i+1} m_{i+1} - \varepsilon_i m_{i+1} \equiv 0 \mod r$. Consequently, $s = -\varepsilon_i \varepsilon_{i+1}$ for $r \neq 0$ and $r \neq 1$. This implies

$$L_{i+1,i-1} = L(r,s) = L(r, -\varepsilon_i \varepsilon_{i+1}) = -\varepsilon_i \varepsilon_{i+1} L(r,1).$$

Therefore, up to orientation, $L(r,s) = S^3$ when $r \neq 0$ and $r \neq 1$. $\square$

In fact, the manifold $W_{i+1,i-1}$ is an oriented 2-disk bundle over S_i. That is, it is a D^2-bundle over S^2. The associated S^1-principal bundle has total space $L(r,1)$ and Euler class $-r$. In particular, when $r = 1$, the associated S^1-principal bundle is the Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$.

The characteristic class of $W_{i+1,i-1}$ is given by $\omega_i = \varepsilon_i \varepsilon_{i+1} r$ and the 2-sphere S_i, over which $W_{i+1,i-1}$ is an oriented 2-disk bundle, has self intersection number $S_i \cdot S_i$. Therefore we have the following,

$$\omega_i = S_i \cdot S_1 = \varepsilon_i \varepsilon_{i+1} r = \begin{vmatrix} m_{i-1} & m_i \\ n_{i-1} & n_i \end{vmatrix} \begin{vmatrix} m_i & m_{i+1} \\ n_i & n_{i+1} \end{vmatrix} \begin{vmatrix} m_{i-1} & m_{i+1} \\ n_{i-1} & n_{i+1} \end{vmatrix}.$$

It remains to show what happens when $r = 0$ and $r = 1$. Consider the alternative orbit structure:

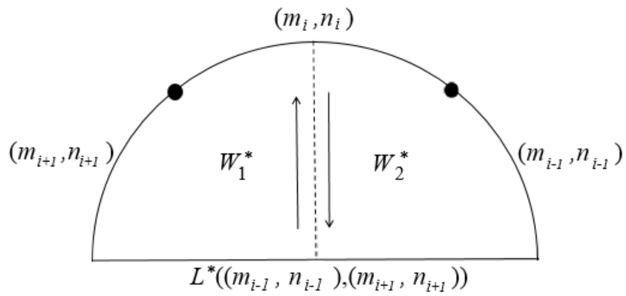


Figure 7.4: Alternate Two Fixed Points Decomposition

The 2-disk bundle over the 2-sphere S_i can be obtained by pasting together along the equator of S_i two 2-disk bundles over the upper and lower hemispheres. We need to show that $L((m_{i-1}, n_{i-1}), (m_{i+1}, n_{i+1})) = L(r, s) = -\varepsilon_i \varepsilon_{i-1} L(r, 1)$.

Lemma 7.1.2. *The inverse image of* $L^*((m_{i-1}, n_{i-1}), (m_{i+1}, n_{i+1}))$ *is* $L(r, s)$ *for* $r = 0$ *and* $r = 1$.

Proof. We can regard $L((m_{i-1}, n_{i-1}), (m_{i+1}, n_{i+1}))$ as the union of two solid tori such that we obtain the lens space $L(r, s)$ by sewing the two solid tori along their boundaries. Letting

$$\varepsilon_{i+1} = \det \begin{vmatrix} m_i & m_{i+1} \\ n_i & n_{i+1} \end{vmatrix} = 1,$$

we have

$$r = \det \begin{vmatrix} m_{i-1} & m_{i+1} \\ n_{i-1} & n_{i+1} \end{vmatrix} \qquad \text{and} \qquad s = \det \begin{vmatrix} m_i & m_{i-1} \\ n_i & n_{i-1} \end{vmatrix} = -\varepsilon_i.$$

Then,

$$\begin{aligned} L(r, s) &= L\left(\begin{vmatrix} m_{i-1} & m_{i+1} \\ n_{i-1} & n_{i+1} \end{vmatrix}, \begin{vmatrix} m_i & m_{i-1} \\ n_i & n_{i-1} \end{vmatrix} \right) \\ &= L(r, -\varepsilon_i) \\ &= -\varepsilon_i L(r, 1) \\ &= -\varepsilon_i (1) L(r, s) \\ &= -\varepsilon_i \varepsilon_{i+1} L(r, 1) \end{aligned}$$

On the other hand if we have $\varepsilon_{i+1} = -1$, then

$$t = \det \begin{vmatrix} m_i & m_{i-1} \\ n_i & n_{i-1} \end{vmatrix} = \det \begin{vmatrix} m_{i-1} & m_i \\ n_{i-1} & n_i \end{vmatrix} = -\varepsilon_i$$

and

$$w = \det \begin{vmatrix} m_{i-1} & m_{i+1} \\ n_{i-1} & n_{i+1} \end{vmatrix}.$$

Then

$$\begin{aligned} L(w,t) &= L\left(\begin{vmatrix} m_{i-1} & m_{i+1} \\ n_{i-1} & n_{i+1} \end{vmatrix}, \begin{vmatrix} m_{i-1} & m_i \\ n_{i-1} & n_i \end{vmatrix} \right) \\ &= L(r, -\varepsilon_i) \\ &= -\varepsilon_i L(r,1) \\ &= -L(r,s). \end{aligned}$$

Hence $L(r,s) = \varepsilon_i L(r,1)$ and since $\varepsilon_{i+1} = -1$, we have $-L(r,s) = -\varepsilon_i \varepsilon_{i+1} L(r,1)$ again. Therefore, for all values of r, we have $-\varepsilon_i \varepsilon_{i+1} L(r,1)$. When $r = 0$, we have $(m_{i+1}, n_{i+1}) = \pm(m_{i-1}, n_{i-1})$ and $L(0,1) = S^1 \times S^2$. When $r = 1$, we have $L(1,1) = S^3$. $\square$

By Lemma 7.1.1 and Lemma 7.1.2, the 2-sphere S_i over which W is an oriented 2-disk bundle, has self intersection number $\varepsilon_{i+1}\varepsilon_i r$ for every value of r.

We have now shown the possible orbit structures in the presence of one or two fixed points. We will use these results to classify simply connected 4-manifolds in the presence of a certain number of fixed points. Note that these orbit structure decompositions are not unique.

7.2 Two Fixed Points

In the presence of two fixed points, our orbit space has the following orbit structure:

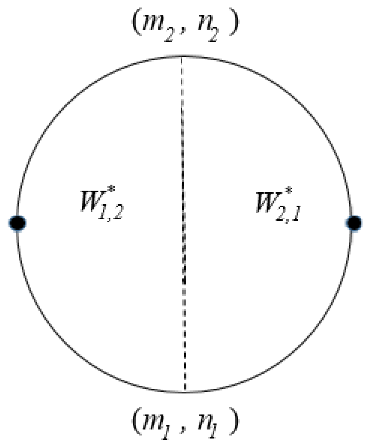


Figure 7.5: Orbit Space of Two Fixed Points

Both $W^*_{1,2}$ and $W^*_{2,1}$ have orbit structures of Figure 7.2. Since adjacent pairs are orthogonal, we can deduce that the dashed line has orbit diagram $(\{e\}, G(1,0), G(m,n))$ for m, n relatively prime. We therefore have the possible weighted orbit space with $(m_1, n_1) = (0,1)$ and $(m_2, n_2) = (1,0)$:

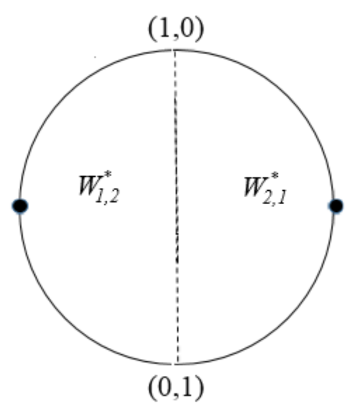


Figure 7.6: Weighted Orbit Space t=2

Filling in the orbit diagram, we have $(\{e\}, G(1,0), G(0,1))$ where the dashed line represents $L(1,0) = S^3$. Decomposing down the common boundary S^3, we have the following orbit decomposition:

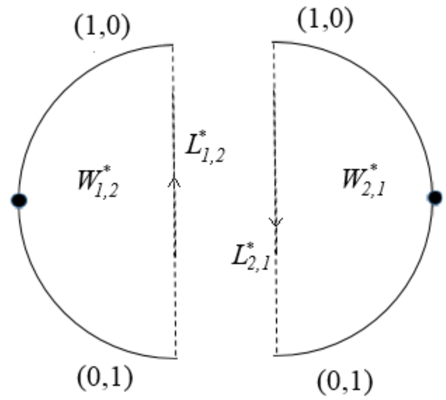


Figure 7.7: Orbit Decomposition t=2

Consider, $W^*_{1,2}$ and $W^*_{2,1}$. They satisfies the following conditions:

$$\varepsilon_1 = \det \begin{vmatrix} m_2 & m_1 \\ n_2 & n_1 \end{vmatrix} = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\varepsilon_2 = \det \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} = \det \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Hence, each $W^*_{1,2}$ and $W^*_{2,1}$ represent the inverse image of 4-cells $W_{1,2}$ and $W_{2,1}$. Gluing $W_{1,2}$ and $W_{2,1}$ along their shared boundary S^3 yields

$$W_{1,2} \cup_{S^3} W_{2,1} \cong S^4.$$

Therefore in the presence of two fixed points in M^*, we have the closed, simply connected manifold $M = S^4$ with $H_2(M;\mathbb{Z}) = H^2(M;\mathbb{Z}) \cong 0$ and $\chi(M) = 2$.

7.3 Three Fixed Points

In the presence of three fixed points, our orbit space has the following orbit structure:

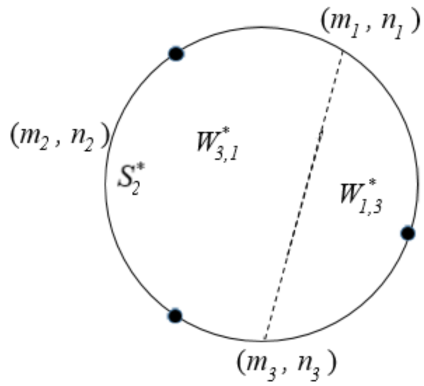


Figure 7.8: Orbit Space of Three Fixed Points

Notice $W^*_{3,1}$ has orbit structure as Figure 7.3 and $W^*_{1,3}$ has Figure 7.2 orbit structure. Since adjacent pairs are orthogonal and the dashed line has orbit diagram $(\{e\}, G(1,0), G(m,n))$ for m, n relatively prime, consider the possible weighted orbit space where $(m_1, n_1) = (1, 0)$, $(m_2, n_2) = (0, 1)$, and $(m_3, n_3) = (1, \pm 1)$:

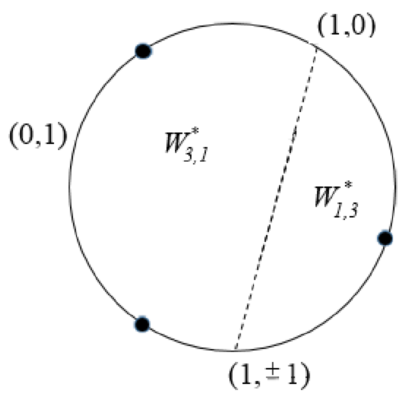


Figure 7.9: Weighted Orbit Structure t=3

Decomposing the orbit structure along the shared boundary, we have the following weighted orbit decomposition:

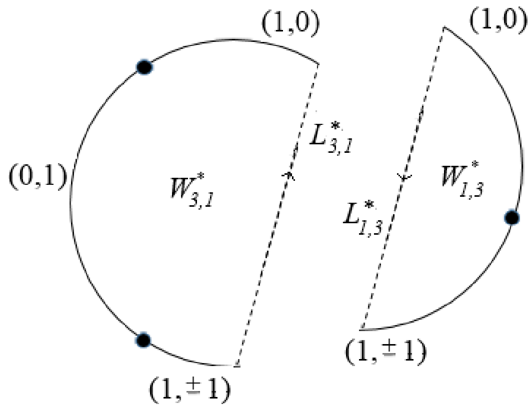


Figure 7.10: Orbit Decomposition t=3

$L_{1,3}$ has orbit diagram, $(\{e\}, G(1,0), G(1,\pm1))$, which represents $L(\pm1, 1) = S^3$. Hence $W_{1,3}$ is a 4-ball with boundary S^3.

Now, $W_{3,1}$ is a D^2-bundle over S_2. We can find its boundary $L_{3,1}$ by considering,

$$\varepsilon_2 = \det \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\varepsilon_3 = \det \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} = \det \begin{vmatrix} 0 & 1 \\ 1 & \pm1 \end{vmatrix} = -1.$$

Then,

$$-r_2 = \det \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix} = \det \begin{vmatrix} 1 & 1 \\ \pm1 & 0 \end{vmatrix} = \pm1$$

and $s = -\varepsilon_2\varepsilon_3 = 1$. Thus $L(r, s) = L(r, 1) = L(\pm1, 1) = S^3$ and $W_{3,1}$ has S^3 as boundary. We can compute its characteristic class, $\varepsilon_2\varepsilon_3 r_2$ to obtain the self intersection number of S_2. When $r_2 = 1$ we have,

$$\omega_2 = S_2 \cdot S_2 = \varepsilon_2\varepsilon_3 r_2 = -1.$$

When $r_2 = -1$, we have the following,

$$\omega_2 = S_2 \cdot S_2 = \varepsilon_2\varepsilon_3 r_2 = 1.$$

Connecting $W_{1,3}$ and $W_{3,1}$ along their shared S^3 boundary yields the intersection form,

$$\begin{aligned} Q_{W_{1,3}\#W_{3,1}} &= [0] \oplus [\pm 1] \\ &= [\pm 1]. \end{aligned}$$

When $Q_{W_{1,3}\#W_{3,1}} = [1]$ we have $\mathbb{C}P^2$ and when $Q_{W_{1,3}\#W_{3,1}} = [-1]$ we have $\overline{\mathbb{C}P^2}$. Hence, the simply connected 4-dimensional manifold in the presence of three fixed points is $M = \mathbb{C}P^2$ or $M = \overline{\mathbb{C}P^2}$ with $H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\chi(M) = 3$.

7.4 Four Fixed Points

The following is the structure of on orbit space in the presence of 4 fixed points:

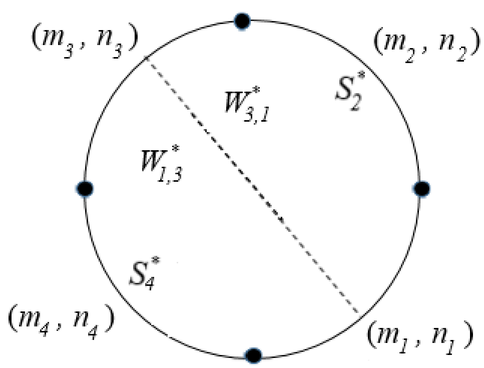


Figure 7.11: Orbit Space of 4 Fixed Points

Observe that M is the union of two D^2-bundles, $W_{1,3}$ with 0-section S_4 and $W_{3,1}$ with 0-section S_2 where each W has orbit structure as Figure 7.3. We can also regard M as the union of two D^2-bundles, $W_{2,4}$ with 0-section S_1 and $W_{4,2}$ with 0-section S_3. Consider the possible weighted orbit space in the presence of four fixed points where $(m_1, n_1) = (1, 1)$, $(m_2, n_2) = (2, 1)$, $(m_3, n_3) = (1, 0)$, $(m_4, n_4) = (0, 1)$:

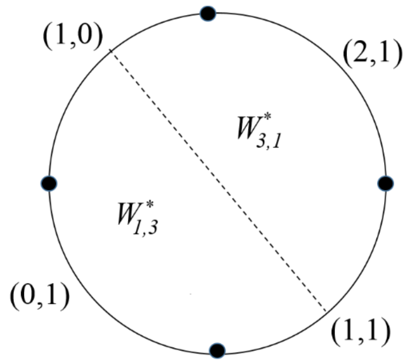


Figure 7.12: Weighted Orbit Space t=4

Decomposing the orbit structure along the shared boundary, we have the following orbit decomposition:

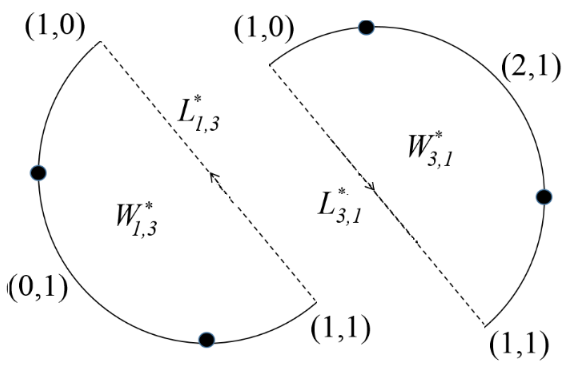


Figure 7.13: Orbit Decomposition t=4

Since $W_{1,3}$ is a D^2-bundle over S_4, we can find its boundary $L_{1,3}$ by considering,

$$\varepsilon_4 = \det \begin{vmatrix} m_3 & m_4 \\ n_3 & n_4 \end{vmatrix} = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\varepsilon_1 = \det \begin{vmatrix} m_4 & m_1 \\ n_4 & n_1 \end{vmatrix} = \det \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1.$$

Then,

$$r_2 = \det \begin{vmatrix} m_1 & m_3 \\ n_1 & n_3 \end{vmatrix} = \det \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

and $s = -\varepsilon_4\varepsilon_1 = 1$. Thus $L_{1,3} = L(r,s) = L(-1,1) = \overline{L}(1,1) = S^3$ and $W_{1,3}$ has S^3 as boundary. The characteristic class of $W_{1,3}$ is therefore $\varepsilon_4\varepsilon_1 r_2$ and hence the self intersection number of S_4 is

$$\omega_4 = S_4 \cdot S_4 = \varepsilon_4\varepsilon_1 r_2 = 1.$$

Now, $W_{3,1}$ is a D^2-bundle over S_2 and we can find its boundary $L_{3,1}$ by considering,

$$\varepsilon_2 = \det \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} = \det \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

$$\varepsilon_3 = \det \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} = \det \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Then,

$$-r_2 = \det \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix} = \det \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

and $s = -\varepsilon_2\varepsilon_3 = -1$. Thus $L_{3,1} = L(r,s) = -L(r,1) = -L(-1,1) = L(1,1) = S^3$ and $W_{3,1}$ has S^3 as boundary. The characteristic class of $W_{3,1}$ is therefore $\varepsilon_2\varepsilon_3 r_2$ and hence the self intersection number of S_2 is

$$\omega_2 = S_2 \cdot S_2 = \varepsilon_2\varepsilon_3 r_2 = -1.$$

Connecting $W_{3,1}$ and $W_{1,3}$ along their shared S^3 boundary yields the intersection matrix for generators S_2 and S_4 of $H_2(M;\mathbb{Z})$:

$$\begin{aligned} Q_{W_{3,1}\#W_{1,3}} &= \begin{pmatrix} \varepsilon_2\varepsilon_3 r_2 & 0 \\ 0 & -\varepsilon_4\varepsilon_1 r_2 \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon_2\varepsilon_3 r_2 & 0 \\ 0 & \varepsilon_2\varepsilon_3 r_2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

This is the intersection form of the simply connected 4-manifold $M = \overline{\mathbb{C}P^2}\#\overline{\mathbb{C}P^2}$ with $H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong \mathbb{Z}^2$ and $\chi(M) = 4$. Note, depending on orientation, $r_2 = \pm 1$ and the intersection matrix for generators S_2 and S_4 of $H_2(M;\mathbb{Z})$ can also be:

$$Q_{W_{3,1}\#W_{1,3}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the intersection forms of the simply connected 4-manifold $M = \mathbb{C}P^2\#\mathbb{C}P^2$. Hence $H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong \mathbb{Z}^2$ and $\chi(M) = 4$.

Let us now consider a different weighted orbit structure in the presence of 4 fixed points such that $(m_1, n_1) = (1, 0)$, $(m_2, n_2) = (p, 1)$, $(m_3, n_3) = (1, 0)$, $(m_4, n_4) = (0, 1)$:

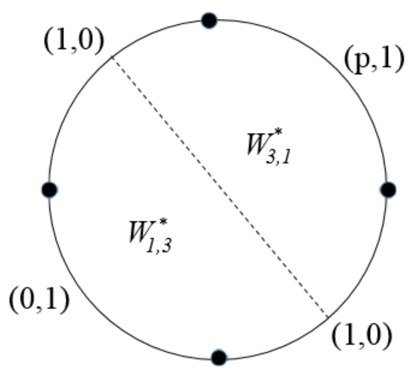


Figure 7.14: Weighted Orbit Space t=4

Again, M is the union of two D^2 bundles, $W_{3,1}$ with 0-section S_2 and $W_{1,3}$ with zero section S_4. By Lemma 7.1.2, when $(m_3, n_3) = (m_1, n_1)$, we have $r_2 = 0$. Hence the boundary of $W_{3,1}$ and $W_{1,3}$ is $S^1 \times S^2$. This is a line bundle over a sphere and we will classify the resulting 4-manifold by its first Chern class (which, in this case, will be the Euler class). Compactifying each fiber by adding a point at infinity, we get an S^2-bundle over S^2 with a generator from the homology of the zero section and a fiber. The next step is to look at $W_{2,4}$ and $W_{4,2}$, where the two halves will be pasted together to form an S^2-bundle over S_3 (or S_1) and the isotropy group $G(m_1, n_1)$ can be reduced to $SO(3)$. The S^2-bundle over S^2 is identified by equatorial gluing $S^1 \to SO(3)$. We have the following decomposition:

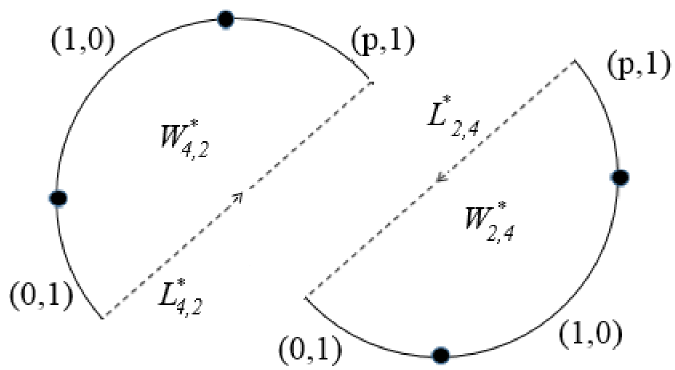


Figure 7.15: Orbit Decomposition $t = 4$

We begin by calculating the characteristic class of $W_{4,2}$ using the following matrices:

$$-r_3 = \det \begin{vmatrix} m_4 & m_2 \\ n_4 & n_2 \end{vmatrix} = \det \begin{vmatrix} 0 & p \\ 1 & 1 \end{vmatrix} = -p$$

$$\varepsilon_3 = \det \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} = \det \begin{vmatrix} p & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\varepsilon_4 = \det \begin{vmatrix} m_3 & m_4 \\ n_3 & n_4 \end{vmatrix} = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

The characteristic class of $W_{4,2}$ is therefore $\varepsilon_3\varepsilon_4 r_3$ and hence the self intersection number of S_3 is

$$\omega_3 = S_3 \cdot S_3 = \varepsilon_3\varepsilon_4 r_3 = -p.$$

Now let us calculate the characteristic class of $W_{2,4}$ using the following matrices:

$$r_3 = \det \begin{vmatrix} m_2 & m_4 \\ n_2 & n_4 \end{vmatrix} = \det \begin{vmatrix} p & 0 \\ 1 & 1 \end{vmatrix} = p$$

$$\varepsilon_1 = \det \begin{vmatrix} m_4 & m_1 \\ n_4 & n_1 \end{vmatrix} = \det \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\varepsilon_2 = \det \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} = \det \begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} = 1.$$

The characteristic class of $W_{2,4}$ is therefore $\varepsilon_1\varepsilon_2 r_2$ and hence the self intersection number of S_1 is

$$\omega_1 = S_1 \cdot S_1 = \varepsilon_1\varepsilon_2 r_2 = -p.$$

We also need to look at the self intersection number of the fiber, S_2. We have:

$$\omega_2 = \varepsilon_2\varepsilon_3 r_2 = 0.$$

Therefore the self intersection matrix for S_3 (or S_1) and S_2 of $H_2(M;\mathbb{Z})$ is:

$$\begin{aligned} Q &= \begin{pmatrix} \varepsilon_3\varepsilon_4 r_3 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_2\varepsilon_3 r_2 \end{pmatrix} \\ &\sim \begin{pmatrix} -p & -1 \\ -1 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} p & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

If p is even, we have the intersection form of the trivial bundle $M = S^2 \times S^2$. This is a simply connected 4-manifold with $H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong Z^2$ and $\chi(M) = 4$. If p is odd, we have the intersection form of $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which is a simply connected 4-manifold with $H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong Z^2$ and $\chi(M) = 4$.

We have now shown the first case of Theorem 7.0.1. When T^2 acts effectively and smoothly on a closed, orientable, simply connected 4-dimensional manifold M, then M is equivariantly diffeomorphic to S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, and $S^2 \times S^2$. We can summarize the results in the following table. These manifolds are the building blocks for topologically classifying simply connected 4-dimensional manifolds.

t	M	Conditions
2	S^4	
3	$\mathbb{C}P^2$	$\varepsilon_1\varepsilon_2\varepsilon_3 = -1$
	$-\mathbb{C}P^2$	$\varepsilon_1\varepsilon_2\varepsilon_3 = 1$
4	$\mathbb{C}P^2 \# \mathbb{C}P^2$	$\left.\varepsilon_1\varepsilon_4 = -\varepsilon_2\varepsilon_3 \right\} \Rightarrow$ $r_2 = \pm 2, \quad \pm\varepsilon_2\varepsilon_3 = 1$ or $r_2 = \pm 1, \quad \pm\varepsilon_2\varepsilon_3 = 1$
	$-\mathbb{C}P^2 \# -\mathbb{C}P^2$	$\left.\varepsilon_1\varepsilon_4 = -\varepsilon_2\varepsilon_3 \right\} \Rightarrow$ $r_2 = \pm 2, \quad \pm\varepsilon_2\varepsilon_3 = -1$ or $r_2 = \pm 1, \quad \pm\varepsilon_2\varepsilon_3 = -1$
	$S^2 \times S^2$	$\varepsilon_1\varepsilon_4 = \varepsilon_2\varepsilon_3$, both r_2 and r_3 are even (at least one is 0)
	$\mathbb{C}P^2 \# -\mathbb{C}P^2$	$\varepsilon_1\varepsilon_4 = \varepsilon_2\varepsilon_3$ either r_2 or r_3 are odd (the other one is 0)

7.5 Greater than Four Fixed Points

We will prove the last sentence of Theorem 7.0.1 with the following lemma. The proof follows directly from [33].

Lemma 7.5.1. *If there are more than 4 fixed points, then M is an equivariantly connected sum of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, and $S^2 \times S^2$.*

Proof. An equivariant sum of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, and $S^2 \times S^2$ is formed by removing open tubular neighborhoods of circle orbits of the same isotropy type from each and identifying the resulting boundaries by an orientation preserving equivariant diffeomorphism. We proceed by showing that if $t > 4$, then there exists an arc $L^*_{i,j}$ such that $L_{i,j} = \pm S^3$ and hence $M = W_{i,j} \# W_{j,i}$. By induction, we will show that an automorphism of T^2 alters the action without changing the manifold. Consider the possible orbit structure in the presence of more than 4 fixed points:

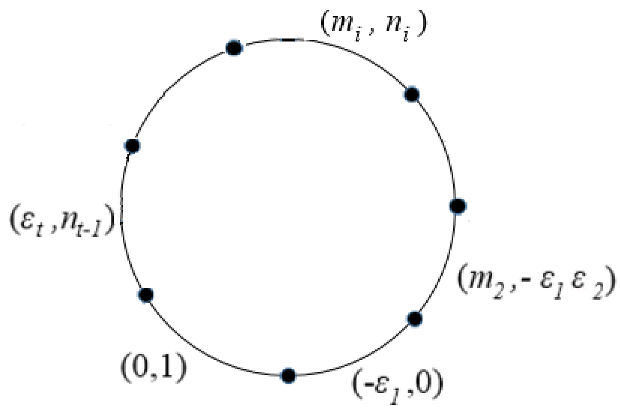


Figure 7.16: $t > 4$ Orbit Structure

Assume that $(m_t, n_t) = (0, 1)$ and $(m_1, n_1) = (-\varepsilon_1, 0)$. Then

$$\det \begin{vmatrix} -\varepsilon_1 & m_2 \\ 0 & n_2 \end{vmatrix} = \varepsilon_2 \qquad \text{and} \qquad \det \begin{vmatrix} m_{t-1} & 0 \\ n_{t-1} & 1 \end{vmatrix} = \varepsilon_t$$

where $n_2 = -\varepsilon_1\varepsilon_2$ and $m_{t-1} = \varepsilon_t$. Consider the sequence

$$\left|\frac{-\varepsilon_1}{0}\right|, \left|\frac{m_2}{\varepsilon_1\varepsilon_2}\right|, \cdots, \left|\frac{m_i}{n_i}\right|, \cdots, \left|\frac{\varepsilon_t}{n_{t-1}}\right|, \left|\frac{0}{1}\right|$$

where $|m_1/0|$ means ∞.

(*i*) If $m_i = 0$, then $|n_i| = 1$. Suppose $m_i = 0$ for $i < t-1$. Then choose $L^*_{1,i}$ if $i \neq 2$ and $L^*_{t-1,2}$ if $i = 2$. In either case L is a 3-sphere S^3.

(*ii*) If $n_i = 0$, then $|m_i| = 1$. Suppose $n_i = 0$, for $i > 2$. Then choose $L^*_{t,i}$ if $i < t-1$ and $L_{t-1,2}$ if $i = i-1$. Again, L is a 3-sphere S^3.

(*iii*) If for some i, $|m_i/n_i| = 1$, then $m_i = \pm 1$, and $n_i = \pm 1$. Choose $L^*_{t,i}$ unless $i = t-1$, in which case choose $L^*_{1,i}$. Again L is a 3-sphere S^3.

(*iv*) Our goal is to show at least one of conditions (*i*) − (*iii*) holds. Suppose that no conditions hold, then there is a first integer $j-1$ such that $|m_{j-1}/n_{j-1}| > 1$ and $|m_j/n_j| < 1$ for $2 \leq j-1 \leq t-3$. Since

$$\varepsilon_j = \det \begin{vmatrix} m_{j-1} & m_j \\ n_{j-1} & n_j \end{vmatrix} = m_{j-1}n_j - m_jn_{j-1},$$

we have $m_{j-1}n_j = \varepsilon_j + m_j n_{j-1}$ and in particular $|m_{j-1}||n_j| \leq 1 + |m_j||n_{j-1}|$. But

$$(|n_j| + 1)(|n_{j-1}| + 1) \leq 1 + |m_j||n_{j-1}|.$$

This yields a contradiction as $|n_{j-1}| > 0$ and $|m_j| > 0$. □

We have just shown that for adjacent pairs, (m_i, n_i) and (m_{i-1}, n_{i-1}) one can always find a different pair (m_j, n_j) equal to either $\pm(m_i, n_i)$, $\pm(m_{i-1}, n_{i-1})$, or $(\varepsilon m_i \pm m_{i-1}, \varepsilon n_i \pm n_{i-1})$, for $t > 4$.

Since the decompositions are not unique, here are two examples of what the weighted orbit decomposition could look like in the presence of 5 fixed points.

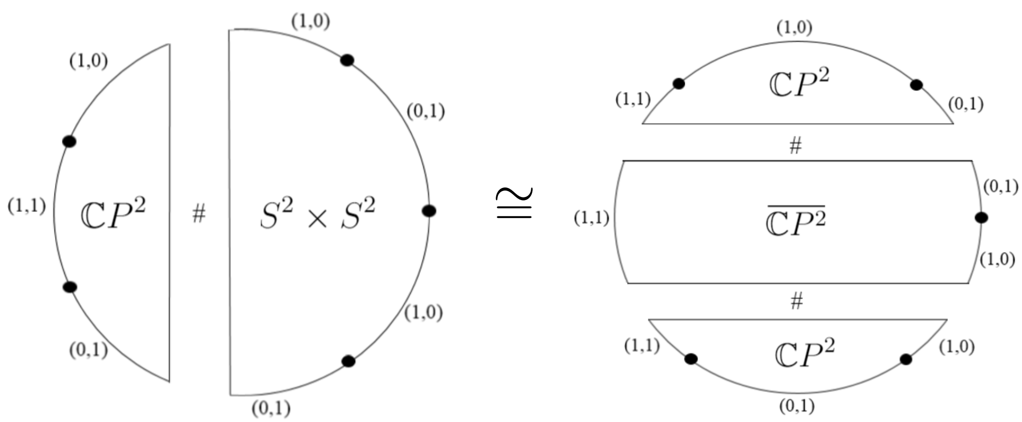


Figure 7.17: $\mathbb{C}P^2 \# S^2 \times S^2 \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \mathbb{C}P^2$

CHAPTER 8

Conclusion

The main goal of this thesis was to topologically classify effective T^2-actions on simply connected 4-dimensional manifolds. A key result was the existence of a cross-section, which allowed us to obtain an equivariant classification theorem. We have shown that in the presence of fixed points, we obtain the following simply connected 4-manifolds

$$S^4, \mathbb{C}P^2, \overline{\mathbb{C}P^2}, \text{ and } S^2 \times S^2,$$

or an equivariant connected sums of

$$\mathbb{C}P^2, \overline{\mathbb{C}P^2}, \text{ or } S^2 \times S^2.$$

Further work to obtain an equivariant classification of non-simply connected 4-dimensional manifolds admitting T^2-actions was done by Melvin [27], Orlik and Raymond [34], and Pao [35]. Later, Oh [31] studied cohomogeneity two torus actions with 5- and 6-dimensional manifolds classifying them to equivariantly diffeomorphism. In a joint work McGavern and Oh [26] gave a partial classification of cohomogeneity three torus actions on simply connected 5- and 6-manifolds.

These results have also had an impact on classification of compact Riemannian manifolds with positive and non-negative sectional curvature. One observes that manifolds with positive and non-negative sectional curvature admits a large number of symmetries. An approach to classifying these manifolds has been to assume the existence of "large" symmetry groups with the advantage that "large" can be defined many ways. For example, one definition of "large" is that the rank of the isometry group is large, that is, there exists a large torus action. This approach was inspired by the work of Hsiang and Kleiner [17] where they showed that simply connected 4-dimensional manifolds with positive curvature admitting an effective, isometric circle action are homeomorphic to either S^4 or $\mathbb{C}P^2$. If the

4-dimensional manifold is assumed to have only non-negative curvature, then by Kleiner [20] and Searle and Yang [41], M must be homeomorphic to one of $S^4, \mathbb{C}P^2, \mathbb{C}P^2 \# \pm \mathbb{C}P^2$ or $S^2 \times S^2$. Homeomorphism can be improved to diffeomorphism by using the work of Fintushel [6][7] and the fact that the Poincaré Conjecture has been solved by Perelman [36] [37] [38]. In fact, a classification up to equivariant diffeomorphism was later obtained by Grove and Wilking [15].

The next step in understanding the abelian symmetries of a manifold was achieved in the work of Grove and Searle [14]. They showed that n-manifolds of positive curvature can admit T^k isometries actions where $k \leq \lfloor \frac{n+1}{2} \rfloor$ and in the case of maximal symmetry rank, a classification is obtained.

It is then natural to consider the case where the manifold is non-negatively curved. Under additional assumptions, the manifold is simply connected and it has been conjectured that the maximal symmetry rank in $\lfloor \frac{2n}{3} \rfloor$ and in the case of equality, such manifolds are products of spheres modulo linear torus actions. This is work due to Escher and Searle [4]. The conjecture has been verified in dimensions 2 and 3 up to equivariant homeomorphism by Galaz-García and Searle [11] and up to equivariant diffeomorphism in dimensions 4 and 6 by Galaz-García and Kerin [10] and up to equivariant diffeomorphism in dimensions 7 through 9 in [4]. Under additional hypotheses, the conjecture has been verified in higher dimensions up to equivariant diffeomorphism by [4].

REFERENCES

LIST OF REFERENCES

[1] M. Audin, *Torus Actions on Sympletic Manifolds*, Progress in Mathematics, **93**, Birkhäuser Verla, (2004).

[2] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press **48** (1972).

[3] J. F. Davis and P. Kirk, *Lecture Notes in Algebraic Topology*, Graduate Studies in Mathematics, **35**, American Mathematics Society, Providence, RI, (2001).

[4] C. Escher and C. Searle, *Non-negative Curvature and Torus Actions*, arXiv preprint, arXiv:1506.08685v2[math.DG], 13 Apr 2016.

[5] C. Escher and C. Searle, *Non-negatively Curved* 6*-Dimensional Manifolds of Almost Maximal Symmetry Rank*, in preparation.

[6] R. Fintushel, *Classification of Circle Actions on 4-Manifolds*, Amer. Math. Soc., **242** (1978), 377-390.

[7] R. Fintushel, *Circle Actions on Simply Connected 4-Manifolds*, Trans. Amer. Math. Soc., **230** (1977), 147-171.

[8] M. Freedman, *The Topology of Four-Dimensional Manifolds*, J. Diff. Geom. **17** (3) (1982), 357–453.

[9] F. Galaz-García, PhD Thesis, University of Maryland, College Park (2009).

[10] F. Galaz-García and M. Kerin, *Cohomogeneity Two Torus Actions on Non-Negatively Curved Manifolds of Low Dimension*, Math. Zeitschrift **276** (1-2) (2014), 133–152.

[11] F. Galaz-García and C. Searle, *Low-Dimensional Manifolds With Non-Negative Curvature and Maximal Symmetry Rank*, Proc. Amer. Math. Soc. **139** (2011), 2559–2564.

[12] F. Galaz-García and C. Searle, *Nonnegatively Curved* 5*-Manifolds With Almost Maximal Symmetry Rank*, Geom. Topol. **18** no. 3 (2014), 1397–1435.

[13] R.E. Gompf and A.I Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Mathematics, **20**, American Mathematics Society, Providence, RI, (1999).

[14] K. Grove and C. Searle, *Positively Curved Manifolds With Maximal Symmetry Rank*, Jour. of Pure and Appl. Alg. **91** (1994), 137–142.

[15] K. Grove and B. Wilking, *A Knot Characterization and* 1*-Connected Nonnegatively Curved 4- Manifolds With Circle Symmetry*, Geom. Topol. **18** no. 5 (2014), 3091–3110.

[16] A. Hatcher *Algebraic Topology*, Cambridge University Press, (2015).

[17] W. Y. Hsiang and B. Kleiner, *On the Topology of Positively Curved* 4*-Manifolds with Symmetry*, J. Diff. Geom. **29** (1989), 615–621.

[18] L. Kennard, *Lecture Notes for Math 260P: Group actions*, https://www.math.upenn.edu/ wziller/math661/LectureNotesLee.pdf, (2013).

[19] S. Kim, D. McGavran and J. Pak, *Torus Group Actions on Simply Connected Manifolds*, Pacific J. Math., **53** no.2 (1974), 435–444.

[20] B. Kleiner, PhD thesis, U.C. Berkeley (1989).

[21] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer Berlin Heidelberg (1972).

[22] J. M. Lee *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics, **218**, Springer, New York, (2012).

[23] D. McGavran, PhD Thesis, Wayne State University (1974).

[24] D. McGavran, *T^3-Actions on Simply Connected 6-Manifolds. I*, Trans. Amer. Math. Soc., **220** (1976), 59–85.

[25] D. McGavran, *T^n-Actions on Simply Connected* $(n+2)$*-Manifolds*, Pacific J. Math., **71** no.2 (1977), 487–497.

[26] D. McGavran and H. S. Oh, *Torus Actions on 5– and 6–Dimensional Manifolds*, Indiana University J. Math., **31** no. 3 (1982), 363–376.

[27] P. Melvin, *On 4-Manifolds with Singular Torus Actions*, Mathematische Annalen **256** (1981), 255–276.

[28] D. Montgomery and C. T. Yang, *The Existence of a Slice*, Ann. of Math., **65** no. 1 (1957), 108–116.

[29] P. S. Mostert, *On a Compact Lie Group Acting on a Manifold*, Ann. of Math., **65** no. 2 (1957), 447–455.

[30] W. D. Neumann, *3-Dimensional G-Manifolds With 2-Dimensional Orbits*, 1968 Proc. Conf. on Transformation Groups (New Orleans, La., 1967), Springer, New York, 220–222.

[31] H. S. Oh, *Toral Actions on* 5-*Manifolds*, Trans. Am. Math. Soc. **278** (1983), 233–252.

[32] P. Orlik and F. Raymond, *Actions of* $SO(2)$ *on* 3-*Manifolds*, Proc. Conf. Transformation Groups, Springer, New York, (1968), 297–318.

[33] P. Orlik and F. Raymond, *Actions of the Torus on* 4-*Manifolds, I*, Trans. Amer. Math. Soc., **152** (1970), 531–559.

[34] P. Orlik and F. Raymond, *Actions of the Torus on* 4-*Manifolds, II*, Topology, **13** (1974), 89–112.

[35] P. S. Pao, *The Topological Structure of 4-Manifolds with Effective Torus Actions. I*, Trans. Am. Math. Soc. **227** (1977), 279–317.

[36] G. Perelman, *The Entropy Formula for the Ricci Flow and its Geometric Applications*, arXiv:math.DG/0211159, 11 Nov 2002.

[37] G. Perelman, *Ricci Flow With Surgery on Three-Manifolds*, arXiv:math.DG/0303109, 10 Mar 2003.

[38] G. Perelman, *Finite Extinction Time for the Solutions to the Ricci Flow on Certain Three-Manifolds*, arXiv:math.DG/0307245, 17 Jul 2003.

[39] T. Perutz, *Smooth 4-Manifolds*, https://www.ma.utexas.edu/users/perutz/PDF_files/Smooth4manifolds.pdf, (2006).

[40] F. Raymond, *Classification of the Actions of the Circle on* 3-*Manifolds*, Trans. Amer. Math. Soc., **131** (1968), 51–78.

[41] C. Searle and D. Yang, *On the Topology of Non-Negatively Curved Simply-Connected 4-Manifolds With Continuous Symmetry*, Duke Math. J. **74** no. 2 (1994), 547–556.

[42] A. Scorpan, *The Wild World of 4-Manifolds*, American Mathematics Society, (2005).

[43] M. Weiler, *4-Manifolds: Classification and Examples*, https://math.berkeley.edu/∼morganw/notes/sgts_talk_2.pdf, (2014).