Weighted Sensitivity Design of PID Controller For An Arbitrary-Order Transfer Function With Time Delay

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1. Introduction
A graphical technique for finding all proportional integral derivative (PID) controllers that stabilize a given single-input-single-output (SISO) linear time-invariant (LTI) system of any order system with time-delay has been solved. In this paper, a method is introduced that finds all PID controllers that also satisfy an $H_{\infty}$ weighted sensitivity constraint. This problem can be solved by finding all PID controllers that simultaneously stabilize the closed-loop characteristic polynomial and satisfy constraints defined by a set of related complex polynomials. A key advantage of this procedure is the fact that it does not require the plant transfer function, only its frequency response.

This paper is organized as follows. The design method is presented in Section 2. In Section 3, this method is applied to a numerical example. The results are summarized in Section 4. Finally the acknowledgments are in Section 5.

1. Design Method
Consider the SISO and LTI system shown in Figure 1, where $G_p(s)$ is the plant, $G_c(s)$ is the PID controller, and $W_p(s)$ is the sensitivity function weight. The reference input and the error signal are $R(s)$ and $Z(s)$, respectively. The plant transfer function can be written as $G_p(s) = G(s)e^{-\tau s}$, where $G(s)$ is an arbitrary order transfer function, and $\tau$ is the time delay.

![Block diagram of the system with weighted sensitivity](image)

The transfer functions in Figure 1 can all be expressed in the frequency domain. The PID controller is defined as

$$G_c(j\omega) = K_p + \frac{K_i}{j\omega} + K_d j\omega,$$  \hspace{1cm} (1)

where $K_p$, $K_i$, and $K_d$ are the proportional, integral, and derivative gains, respectively. The plant transfer function and the sensitivity weight $W_p(s)$ can be written in terms of their real and imaginary parts as

$$G_p(j\omega) = R_p(\omega) + j I_p(\omega) \quad \text{and} \quad W_p(j\omega) = A(\omega) + jB(\omega).$$  \hspace{1cm} (2)

The deterministic values of $K_p$, $K_i$, and $K_d$ for which the closed-loop characteristic polynomial is Hurwitz stable have been found in [1]. In this paper, the problem is to find all PID controllers that satisfy the weighted sensitivity constraint

$$|W_p(j\omega)S(j\omega)| \leq \gamma \quad \forall \omega,$$  \hspace{1cm} (3)

where $S(j\omega) = \frac{1}{1+G_p(j\omega)G_c(j\omega)}$ is the sensitivity function, and $\gamma$ is a positive scalar. If (3) holds, then for each value of $\omega$

$$\frac{W_p(j\omega)e^{j\theta}}{1+G_p(j\omega)G_c(j\omega)} \leq \gamma,$$  \hspace{1cm} (4)

must be true for some $\theta \in [0,2\pi]$. Consequently, all PID controllers that satisfy (3), must lie at the intersection of all controllers that satisfy (4) for all $\theta \in [0,2\pi]$.

To accomplish this, for each value of $\theta \in [0,2\pi]$ we will find all PID controllers on the boundary of (4). It is easy to show from (4), that all the PID controllers on the boundary must satisfy

$$1 + G_p(j\omega)G_c(j\omega) - \frac{1}{\gamma} W_p(j\omega)e^{j\theta} = 0.$$  \hspace{1cm} (5)

Note that (5) reduces to the frequency response of the standard closed-loop characteristic polynomial as $\gamma \to \infty$. By substituting (1), (2), and $e^{j\theta} = \cos \theta + j\sin \theta$ into (5), and solving for the real and imaginary parts to zero yields...
ωK_pR_p(ω) + K_iI_p(ω) − ω^2K_dI_p(ω) =
\frac{ω}{γ} \left( A(ω)\cos θ − B(ω)\sin θ \right) − ω
\tag{6}

ωK_pI_p(ω) − K_R_p(ω) + ω^2K_dR_p(ω) =
\frac{ω}{γ} \left( A(ω)\sin θ + B(ω)\cos θ \right).
\tag{7}

This is a three-dimensional system in terms of the controller parameters \(K_p\), \(K_i\), and \(K_d\). The boundary of (4) can be found in the \((K_p, K_i)\) plane for a fixed value of \(K_d\). After setting \(K_d\) to the fixed value \(\tilde{K}_d\), (6) and (7) can be rewritten as

\begin{bmatrix}
ωR_p(ω) & I_p(ω) \\
ωI_p(ω) & −R_p(ω)
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix}
= \begin{bmatrix}
ω^2\tilde{K}_dI_p(ω) + \frac{ω}{γ} \left( A(ω)\cos θ − B(ω)\sin θ \right) − ω \\
−ω^2\tilde{K}_dR_p(ω) + \frac{ω}{γ} \left( A(ω)\sin θ + B(ω)\cos θ \right)
\end{bmatrix}
\tag{8}

Solving (8) for all \(ω \neq 0\) and \(θ \in [0, 2π)\), gives the following equations:

\[
K_p(ω, θ, γ) = \frac{1}{γ} \left( \frac{(A(ω)\cos θ − B(ω)\sin θ)R_p(ω) + (A(ω)\sin θ + B(ω)\cos θ)I_p(ω) − R_p(ω)}{G_p(jω)} \right)
\tag{9}
\]

\[
K_i(ω, θ, γ) = ω^2\tilde{K}_d + \frac{−ω}{γ} \left( \frac{(A(ω)\sin θ + B(ω)\cos θ)R_p(ω) − (A(ω)\cos θ − B(ω)\sin θ)I_p(ω)}{G_p(jω)} \right)
\tag{10}
\]

where \(\left|G_p(jω)\right|^2 = R_p^2(ω) + I_p^2(ω)\). Setting \(ω = 0\) in (8), we obtain

\[
\begin{bmatrix}
0 & I_p(0) \\
0 & −R_p(0)
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

and conclude that \(K_p(0, θ, γ)\) is arbitrary and \(K_i(0, θ, γ) = 0\), unless \(I_p(0) = R_p(0) = 0\), which holds only when \(G_p(s)\) has a zero at the origin.

2. Example

In this section, an example demonstrates the application of this method. Consider the second order plant transfer function

\[
G_p(s) = \frac{-0.5s + 1}{(s + 1)(2s + 1)}e^{-0.6s}
\]

from [1]. The goal is to design a PID controller such that (3) is satisfied for a given \(W_p(s)\). Ideally, \(γ\) will be less than one. We use (9) and (10) with \(\tilde{K}_d = 0.02\), \(γ = 1\), \(θ \in [0, 2π)\), and \(W_p(s) = (0.9s + 0.09)/(s + 1)\).

The results of this example are shown in Figure 2. The top graph is the \(H_∞\), weighted sensitivity region. Any arbitrary PID controller from this region gives us a peak value less than one. Choosing \(K_i = 0.28\) and \(K_p = 0.19\), where \(\tilde{K}_d = 0.02\), and substituting in (1) and (3), gives

\[
\max\left|W_p(jω)S(jω)\right| = γ = 0.9045 < 1
\]

in the bottom graph, which displays the maximum of the weighted sensitivity frequency response. As can be seen, the results meet our goal.

![Weighted Sensitivity Region with PID Controller for Kd=0.02](image)

Fig. 2. The \(H_∞\) weighted sensitivity region and magnitude of weighted sensitivity frequency response.

3. Conclusion

A graphical technique is introduced for finding an \(H_∞\) weighted sensitivity region of any arbitrary order transfer function with time delay. This method is only depends on the frequency response of the system.

5. Acknowledgements

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Reference