DECENTRALIZED CONTROL OF INTERCONNECTED SINGULARLY PERTURBED SYSTEMS

A Dissertation by

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DECENTRALIZED CONTROL OF INTERCONNECTED SINGULARLY PERTURBED SYSTEMS

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To my parents, my wife, my kids, my brothers and sisters, and my dear friends
ABSTRACT

Designing a controller for large scale systems usually involve a high degree of complexity and computation ability. When talking about decentralized interconnected systems, the complexity and computation are not the only snags encountered when designing the controllers; channel bandwidth is another constraint to design procedures. The stability and optimization process can shift dramatically when an external noise is affecting the measurements at the output sensors or disturbing the controlled input.

The investigated model in this research of a large scale stochastic decentralized interconnected system was a reduced order quasi-steady state model of the original system using singular perturbation techniques. Two methods of analyzing and minimizing the cost function of the full order system were proposed. The first method was to design a controller by standard stochastic control theory techniques using LQG approach and Kalman filter design. In the second method, game theory strategies were applied to the decentralized interconnected singularly perturbed systems. A Stackelberg game was designed and implemented to the reduced order model with one of the controllers designated leader in the game. The minimization of conditions and constraints reaches a solution which applies Lyapunov equations coupled with constraints equations to optimize the performance index of the reduced-order and the full order system.
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NOTATION

\( x \) Slow subsystem states variable in interconnected systems
\( z_i \) Fast subsystem \( i \) states variable in interconnected systems
\( A, B, C, D \) Reduced order fast model variables
\( u_i \) System input \( i \)
\( y_i \) System output \( i \)
\( F \) Feedback gain
\( \varepsilon \) Singular Perturbation small, real valued, positive parameter
\( \bar{x}, \bar{z}, \bar{u} \) Quasi-steady state system variables
\( x^s, u^s, y^s \) Reduced order slow model variables
\( x^f, u^f, y^f \) Reduced order fast model variables
\( G \) Process noise multiplier matrix
\( w \) Process noise
\( v \) Measurement noise
\( W \) Intensity matrix of process noise vector \( w \)
\( V \) Intensity matrix of measurement noise vector \( v \)
\( H \) Measurement noise multiplier matrix
\( L \) Observer parameter
\( P, K \) Lyapunov and Riccati equations variables
\( E\{.\} \) Expected value
\( tr\{A\} \) The trace of the matrix \( A \)
\( \dot{x} \) First derivative of \( x \) with respect to time
\( \frac{\partial y}{\partial x} \) Partial derivative of \( y \) with respect to \( x \)
\( A > 0 \) Matrix \( A \) is p.d.
\( A \geq 0 \) Matrix \( A \) is p.s.d.
CHAPTER 1
INTRODUCTION

Systems in engineering involve a number of interacting elements that perform certain tasks or functions in a desired way. Often systems can run out of control or drift from their desired behavior, and a controlling unit is required to maintain the functionality and guarantee the stability of the system. This is where control systems come into the picture to make existing systems work better according to certain conditions and specific goals. Usually the objective of a control system is to force the system to operate in a desired way to enhance the performance. A system is said to be stable when it can continuously function according to its intended purpose. If system stability is not guaranteed, the system will be unreliable and generally useless. Enhancing a system is not only a measure of stability; robustness and optimality are very important in system performance, and a combination of these measures can be enhanced using control theory techniques [1] – [5].

1.1 Background

Regular control methodology requires synchronous and complete information exchange between sensors in the output of the plant and actuators in the plant input via a control feedback loop. Some systems’ setups involve large-scale systems interconnected together in such a way as to require huge channel bandwidth for controller communication, complex controller design, and expensive implementation. To overcome such limitation of interconnected large-scale systems, decentralized control setup can simplify controller design and reduce the order of the system, communication channel bandwidth, and implementation cost.
1.1.1 Decentralized control

In a centralized control system, complete system information and the states data is assumed to be known during implementation. Usually, systems involve a single centralized controller when there is only one system involved. If more than one system to be controlled, all systems must communicate their data with the central controller which makes the decisions and send commands to each system’s actuators. Many real world control problems are classified as large scale systems and may not to be solved using a single problem approach. The reason is that the complexity and dimensionality of analyses and design procedures of the system grow in a rate much more than the size of the system itself. Other difficulties may arise in large scale control systems like data communication between all systems’ sensors and controlling unit can be difficult or impossible especially for geographically separated systems. For long distance data communication, synchronization and delay could significantly affect the performance of the centralized control system. The main reasons of complexity that leads to difficulties in centralized control for large scale systems can be summarized as:

**Dimensionality**: the behavior and properties of the overall system cannot be completely described by a mathematical model. Also, high computation requirement and decision making speed may exceed the capability of the centralized controller.

**Data availability**: the information about the system dynamics is not available for all subsystems. That is when it is known that no single control unit can read all subsystems’ measurements and send control decisions to all actuators in the system.

**Lack of controllability**: one controller may not have enough authorities to control some parts of the overall system. Subsystems may be completely independent or physically separated and do not provide permission or ability be controlled by the centralized control unit.
**Communication constraints:** for centralized control, the communication network requires a huge bandwidth to transfer all measurements and decisions between subsystems and the centralized controller. Sometimes communication between the subsystems and the centralized controller cannot be implemented. Also, the cost and reliability of communication links and the delay in information exchange can affect the performance of the centralized controller.

One definition of large scale systems says, a problem is considered a large-scale problem if it is necessary to divide into a manageable sub-problems. Because the system to be controlled is large and the problem to be solved is complex, decentralized control was introduced to simplify the problem. So, decentralized control consists of multiple sub-systems in which each sub-system has its own control unit that operates locally and makes its own decisions based on its own measurements.

Decentralized control is widely used in the control of interconnected power systems, distribution networks, traffic systems, computer and communication networks, and aerospace systems. The idea of decentralization is by decomposing a problem into smaller sub-problems, solve these sub-problems separately and combine their solutions as the solution of original full problem. This is done in control systems by dividing the analysis and synthesis of the overall system into uncoupled or weakly-coupled sub-problems, with complete observability and connectivity to states sensors. As a result, the overall plant is controlled by several independent controllers which all together represent a decentralized controller.

In decentralized control, interconnected subsystems are controlled separately and independently using decentralized control strategies. Each subsystem can be controlled locally using its own states for state feedback control. This makes controllers communicate locally within each subsystem individually.
1.1.2 Singular perturbation

To formulate a control problem, a mathematical model of the system is necessary to analyze and design the controller. This mathematical representation can involve some small parameters with negligible effect. The first step in simplifying the mathematical model and reducing the order of the controlled system is to neglect these parameters. When these parameters are only significant for a very short period due to the fast convergence of the subsystems, singular perturbation can be considered to separate the problem and reduce the order of the overall system. Considerable research effort has been concentrated toward singular perturbation for long time.

Singular perturbation utilizes the separation of the slow and fast system dynamics of the whole system to reduce the order of the composite system and simplify the problem. This is done by reducing the order of the system to the order of the slow subsystem only for the slow dynamics of the system and it splits the fast dynamics behavior into several independent fast sub problems.

1.1.3 Output feedback

State feedback technique assumes the availability of all states of each subsystem to the local controller, which is not always applicable in real world systems. Usually, state feedback control techniques are used to control and control a system. In the case when system states are not available, control can be implemented using output feedback by observing and estimating the states or by static output feedback. [1] – [5].

The unavailability of the states is not the only problem involved in using output feedback techniques; the noise in output measurement and disturbance to controller input can have a considerable effect on system performance and even lead to instability. Stochastic control techniques may be used to describe the system with uncertainties [6], [7], [8].
Many procedures for analyzing and designing control systems have been investigated over the past five decades. These procedures include modeling, describing and controlling dynamic systems. In large-scale systems, this setup may not be efficient or even applicable due to the decentralized nature of the physical system or model representation. It is a common simplification technique in working on mathematical models to use simpler reduced models with decreased accuracy compared to the original full model. The reduced model will reduce the computation requirements and simplify the structure of the system. Model reduction can be classified into two methods; one is reduction by aggregation, and the other is reduction using perturbation techniques. The first is not part of this research and will not be discussed here, but the latter is the key concept of model reduction used in this research [9].

Decentralized large-scale systems may consist of two or more subsystems with different time responses. That is when some subsystems converge to steady-state much faster than others. Singular perturbation is a mathematical technique used for solving differential equations with different time scales. It plays a prominent role in solving such control problems in real world systems. In a decentralized system, each subsystem has its own local input and output information. This setup can utilize the separation between slow and fast overall system dynamics using singular perturbation techniques which significantly simplify controller design and reduce the order of the whole system. This reduction can minimize the complexity of control algorithms and implementation cost [10].

1.2 Literature Survey

In the late 1960s, singular perturbation became a common mathematical model reduction tool for the control systems analysis and design. Singular perturbation interested many researchers as a mathematical technique to simplify and reduce the order of large-scale control systems [9]–
In [14], Chemouil and Wahdan investigated large-scale system decomposition according to time scale properties to get an output feedback control for the full order system. They designed a controller that is simple in calculation with reasonable suboptimality. Linear time-invariant systems have been studied using output feedback control with singular perturbation by Khalil in [15]. In this paper, the author has stabilized the system while preserving its two-time-scale structure via sequential procedure by designing fast and slow controllers separately and combining the two controllers as a two-time-scale stabilizing controller for the singularly perturbed system. In [16] Moerder and Calise studied the implementation singular perturbation techniques to design a reduced order output feedback regulator for linear system with ill-conditioned dynamics. The controllers gain designed can be calculated separately for the reduced-order model to stabilize the full-order system. Rosenbrock [17] has suggested an optimal control solution for the regulator problem with a quadratic performance index in steady state. Later, Shaked [18] investigated the stability of linear quadratic deterministic optimal regulator (LQR) with a sufficient small singular perturbation parameters. Ishihara and Takeda [19] have extended Shaked’s research with a useful prediction estimator to simplify the discrete-time linear quadratic Gaussian (LQG) design. For a non-deterministic system, singular perturbation was applied to find the stochastic control for the LQG problem for two time scale (TTS) systems by Haddad and Kokotovic [20]. The optimal control was approximated by a near optimal control of the combined slow and fast controls computed in separate time scales. Kokotovic [21] has provided a full detailed paper of singular perturbation techniques applied to control problems. Standard singular perturbation problems, controllability, stability and optimal linear state regulators for linear, nonlinear and stochastic systems were investigated by Kokotovic. Oloomi et al. [22] have designed a static output feedback
regulator for a linear shift-invariant singularly perturbed discrete-time system by deriving approximate quadratic performance index.

A near optimal solution was derived for a singularly perturbed large-scale interconnected discrete-time system with uncertain parameters by Zhou and Challoo [23]. Corless et al. [24] has studied and designed controller using singular perturbation techniques for linear time varying systems with uncertainty. Stepanyan and Nguyen [25] designed a controller for a plant with slow actuators using singular perturbation techniques with tracking error guaranteed to be bound to a small parameter in the actuator’s dynamics.

1.3 Contribution and Motivation

Decentralization usually implemented when the system’s dimension is very large, subsystems are weakly coupled, or subsystems have contradicting objectives. It is necessary to decentralize the control system if one subsystem lacks the observability of other subsystems. Or the computation capability of the centralized controller is not enough to carry out all control decisions.

Cost and complexity reduction is not the only advantage of decentralized systems. As they consist of various subsystems with a local controller for each that have full access and full control within the subsystem level and perform a specific task as a part of the global system objective. Each controller may have access to a portion of the global information to monitor the function of a group of controllers that share a single global objective. This highlights on another main advantage of decentralized control which is the parallelism in computations where controllers perform tasks related to their own subsystems at the same time. Usually, fully decentralization systems do not permit information exchange among the decision makers and the local control units.
are completely independent which lead in more reliability and robustness because of the distributed nature of the decentralized system and the better signal transmission within the subsystem level.

This research, motivated by several previous researches, is concerned with stability and optimization of output feedback in decentralized singularly perturbed systems by using either an observer based output feedback controller or a static output feedback system. The unavailability of the main system or even subsystems states is common and more practical in physical systems, so it is a dominant assumption in this research. However, designing a static output feedback controller will be even more efficient and less expensive than implementing an observer beside the controller, a fact that motivated the author to consider static output feedback in some areas within the scope of this research. Because noise is unavoidable in real life, a further research to investigate stochastic singularly perturbed interconnected systems is considered. Finally, when considering a two or more interconnected controlled systems, usually the controller behavior would make decisions selfishly aiming to minimize the local cost function. Often an expensive centralized controller is needed to maintain the overall system performance index in addition to the individual controller’s performance indices. In most applications interconnected controllers need to cooperate with each other so that there will be a compromised optimization between the individual cost and overall system cost. A game theory application or specifically a Stackelberg game strategy implementation on such interconnected singularly perturbed system would maintain the compromise between both local and global cost optimization.
CHAPTER 2
PROBLEM STATEMENT

2.1 Overview

A decentralized control system consists of a main system that is controlled by one or more subsystems. The controller synthesis for decentralized systems incorporates a local controller for each subsystem, which controls the main system accordingly. As shown in Figure 2.1, subsystems are not interconnected; they work independently, so each subsystem is controlled by its own output and not affected directly by the output of any of the other subsystems or the main system. This setup reduces the order of the designed controller rather than considering the order of the composite system which leads to simple calculations in the design process.

\[
\begin{align*}
\dot{z}_i &= A_{i0} x + A_{ii} z_i + B_i u_i \\
y_i &= C_i z_i
\end{align*}
\]

\[
\begin{align*}
\dot{z}_j &= A_{j0} x + A_{jj} z_j + B_j u_j \\
y_j &= C_j z_j
\end{align*}
\]

\[
\dot{x} = A_{00} x + A_{0i} z_i + A_{0j} z_j
\]

Figure 2.1 Decentralized system setup with two subsystems.

In the setup shown in Figure 2.1 the controllers \( F_i \) and \( F_j \) control subsystems \( z_i \) and \( z_j \) respectively with output feedback gain, which in turn control the main system \( x \). Then the controller of the composite system can be determined once the gains \( F_i \) and \( F_j \) are calculated.
In many real life applications the subsystems converge faster than the main system when the subsystems are electrical controllers for mechanical or chemical main systems. In this case singular perturbation is considered to reduce the order of the full scale system.

In the system setup shown in Figure 2.1, the main system $x$ is controlled by the states subsystems $z_i$ and $z_j$. The output of subsystem $z_i$ is $y_i$ and the control input signal is $u_i$ with feedback gain $F_i$. In singular perturbation we define the fast dynamics to be the response of the fast converging subsystems with the slow system response to be constant during the transient time. The slow or steady state dynamic is defined when the response of the fast systems converge to zero and the only response that affects the system is that of the slow system. The system in consideration is linear time-invariant decentralized singularly-perturbed system.

2.2 Conditions and Limitations

Because of the way the system is set up, the states and output of the main system are not accessible. The states of the subsystems are not available either. So, the only variable available for control decision is the output of each subsystem. When the subsystem is controlled with output feedback gain control, the main system is controlled accordingly. Some limitations exist for this setup to be controlled successfully.

2.3 Mathematical Model and Preliminary Analysis

Consider the linear time-invariant decentralized singularly-perturbed model in (2.1) representing the system shown in Figure 2.1
where \( x \in \mathbb{R}^n \), \( z_i \in \mathbb{R}^{m_i} \), and \( z_j \in \mathbb{R}^{m_j} \) are the state variables of the slow main system and fast subsystems \( z_i \) and \( z_j \) respectively. Inputs \( u_i \in \mathbb{R}^{r_i} \) and \( u_j \in \mathbb{R}^{r_j} \) are the control input vectors of subsystems \( z_i \) and \( z_j \), and outputs \( y_i \in \mathbb{R}^{p_i} \) and \( y_j \in \mathbb{R}^{p_j} \) are the control output of subsystems \( z_i \) and \( z_j \). \( A_{00}, A_{0i}, A_{ii}, A_{0j}, B_i, B_j, C_i \) and \( C_j \) are constant matrices with appropriate matching dimensions and it is assumed that the singular perturbation parameter \( \varepsilon \ll 1 \) is a small positive number.

Now let the following quadratic performance index for the model in (2.1)

\[
J(t) = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_{t}^{t_f} (x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau)) d\tau
\]

where \( S, Q \geq 0 \) and \( R > 0 \).

### 2.4 Model Analysis

Singular perturbation technique will be used to analyze the behavior of the system in two-time scales.

#### 2.4.1 Slow System Dynamics

In singular perturbation, when \( \varepsilon \) approaches zero, we assume that the fast subsystems state variables, \( z_i \) and \( z_j \) have reached quasi-steady state. Hence, the system order is reduced to the order of the main system, which is equal to the dimension of the slow state variable \( x \). When \( \varepsilon \to 0 \), then \( \varepsilon \dot{z} = 0 \), which means that only the slow part dynamics affect the system. Now, let \( \bar{x} \),
\( \bar{z} \) and \( \bar{u} \) to represent the quasi-steady state variables of the system. The slow variables of the system can be set as

\[
x^s = \bar{x}, \quad u^s = \bar{u} \quad \text{and} \quad y^s = \bar{y}
\]

then (2.1) reduces to

\[
\begin{align*}
\dot{x} & = A_{00} \bar{x} + A_{0i} \bar{z}_i + A_{0j} \bar{z}_j \\
0 & = A_{i0} \bar{x} + A_{il} \bar{z}_l + B_l \bar{u}_i \\
0 & = A_{j0} \bar{x} + A_{lj} \bar{z}_j + B_j \bar{u}_j
\end{align*}
\]

(2.3)

By solving the second and third equations of the system (2.3) for \( \bar{z}_i \) and \( \bar{z}_j \) respectively we get the following

\[
\bar{z}_i = -A_{il}^{-1} (A_{i0} \bar{x} + B_i \bar{u}_i) \quad (2.4)
\]

where \( A_{il} \) are non-singular. By substituting \( \bar{z}_i \) and \( \bar{z}_j \) for \( \bar{x} \) in (2.3) we get

\[
\begin{align*}
\dot{x} & = A_{00} \bar{x} - A_{0i} A_{il}^{-1} (A_{i0} \bar{x} + B_i \bar{u}_i) - A_{0j} A_{lj}^{-1} (A_{j0} \bar{x} + B_j \bar{u}_j) \\
& = A_{00} \bar{x} - A_{0i} A_{il}^{-1} A_{i0} \bar{x} - A_{0i} A_{lj}^{-1} B_i \bar{u}_i - A_{0j} A_{lj}^{-1} A_{j0} \bar{x} - A_{0j} A_{lj}^{-1} B_j \bar{u}_j \\
& = (A_{00} - A_{0i} A_{il}^{-1} A_{i0} - A_{0j} A_{lj}^{-1} A_{j0}) \bar{x} - A_{0i} A_{lj}^{-1} B_i \bar{u}_i - A_{0j} A_{lj}^{-1} B_j \bar{u}_j
\end{align*}
\]

(2.5)

and

\[
\bar{y}_i = C_i \bar{z}_i
\]

(2.6)

Now the slow dynamic of the system can be presented as

\[
\begin{align*}
\dot{x}^s & = \hat{A} x^s + \hat{B}_i u^s_i + \hat{B}_j u^s_j \\
y_i^s & = \hat{C}_i x^s + \hat{D}_i u^s_i
\end{align*}
\]

(2.7)

(2.8)

(2.9)

(2.10)

where
\[
\dot{A} = A_{00} - A_{0l}A_{ll}^{-1}A_{l0} - A_{0j}A_{jj}^{-1}A_{j0} \quad (2.11)
\]
\[
\dot{B}_i = -A_{0l}A_{ll}^{-1}B_i \quad (2.12)
\]
\[
\dot{C}_i = -C_iA_{ll}^{-1}A_{l0} \quad (2.13)
\]
\[
\dot{D}_i = -C_iA_{ll}^{-1}B_i \quad (2.14)
\]

### 2.4.2 Fast System Dynamics

By subtracting the slow system variables from the actual system variables we can get the fast system variables as

\[
z^f = z - \bar{z}, \quad u^f = u - \bar{u}, \quad y^f = y - \bar{y}, \text{ where } z^f, u^f \text{ and } y^f \text{ are the fast subsystem variable, fast input vector and fast output vector respectively. In the transient phase where the dynamics of the fast subsystems did not reach the quasi-steady state, the slow part of the system appear as constant to the overall system dynamics. We can rewrite the fast system’s dynamics in (2.1) and (2.3) as}
\]

\[
\varepsilon \dot{z}_i = A_{ii}x + A_{ii}z_i + B_i u_i \quad (2.15)
\]
\[
\varepsilon \dot{\bar{z}}_i = A_{ii}\bar{x} + A_{ii}\bar{z}_i + B_i \bar{u}_i \quad (2.16)
\]

By subtracting (2.16) from (2.15) we get

\[
\varepsilon \dot{z}_i - \varepsilon \dot{\bar{z}}_i = A_{ii}(x - \bar{x}) + A_{ii}(z_i - \bar{z}_i) + B_i(u_i - \bar{u}_i) \quad (2.17)
\]
\[
\varepsilon \dot{z}_i^f = A_{ii}z_i^f + B_i u_i^f \quad (2.18)
\]

Similarly we can subtract the output in (2.8) from that in (2.1) to get

\[
y_i^f = y_i - \bar{y}_i = C_i(z_i - \bar{z}_i) = C_i z_i^f \quad (2.19)
\]

### 2.5 Objective and Design Specifications

The objective of this research is to develop a stabilizing, robust and optimal control of reduced-order output feedback controller. This controller should be able to stabilize a linear time-
invariant decentralized singularly-perturbed system. Besides states unavailability, the lack of direct control to the main system is a major issue in the problem formulated in section 2.3.

The proposed technique will use singular perturbation approach to reduce the order of the original model to be applicable to large-scale systems. It will be more practical to use the proposed techniques when a fast subsystem controls a slow main system such as a chemical or electromechanical system. The new technique in this research will help to simplify system analysis and controller design and can be expanded for multi-subsystems. In addition to optimizing the performance index, the proposed technique also provides a region of robustness and guarantees stability.

The key idea in singular perturbation techniques is the negligence of effect of the fast subsystems. In optimal control design, the criteria to be analyzed is a standard LQR performance index for a deterministic model and LQG problem for the stochastic model. The first part of the design process assumes the subsystems are independent and not affecting each other. An extension to the problem will be studied using game theory techniques if the two fast-subsystems are dependent and cooperative or competitive.
CHAPTER 3

APPROACH

3.1 Optimal Control

Optimization in optimal control theory is meant to enhance a performance index by maximizing performance measures or minimizing a cost function. A system’s performance index can be optimized using a linear quadratic (LQ) problem for linear systems with quadratic cost function.

3.1.1 LQ Problem

For a general dynamic system

\[
\dot{x} = \frac{dx(t)}{dt} = a(x,u)
\]

(3.1)

with a general cost-to-go function

\[
J(t) = h(x_f) + \int_{t}^{t_f} g(x,u) d\tau
\]

(3.2)

This cost represents the cost of the system operating from \(t\) to \(t_f\), where \(h(x_f)\) is the terminal cost of \(x_f\) at \(t_f\). The goal is to find the optimal cost from any time \(t\) to the end when \(t = t_f\) in addition to the terminal cost. Minimization is done with respect to \(u\) for the time from \(t\) to \(t_f\).

\[
J^*(t) = \min_{u(\tau)} \left\{ h(x_f) + \int_{t}^{t_f} g(x,u) d\tau \right\}
\]

(3.3)

The integral term in (3.3) can be split into two
\[
\int_t^{t+\Delta t} g(x,u) d\tau = \int_t^{t_f} g(x,u) d\tau + \int_t^{t+\Delta t} g(x,u) d\tau
\]  

(3.4)

where \(\Delta t\) is very small. Assuming that the second term in the right hand side of (3.4) is optimal and equal to \(J^*(t + \Delta t)\), then (3.3) can be

\[
J^*(t) = \min_{u(\tau)} \left\{ J^*(t) + \int_t^{t+\Delta t} g(x,u) d\tau \right\}
\]

(3.5)

Using Taylor series expansion, we can write

\[
J^*(t + \Delta t) = J^*(t) + J^*_t \Delta t + J^*_x \Delta \dot{x} + \text{Higher order terms}
\]

(3.6)

where

\[
J^*_t = \frac{\partial J^*(t)}{\partial t}
\]

(3.7)

\[
J^*_x = \frac{\partial J^*(t)}{\partial x}
\]

(3.8)

\[
\Delta \dot{x} = \frac{dx(t)}{dt} \Delta t = a(x,u) \Delta t
\]

(3.9)

So, (3.5) becomes

\[
J^*(t) = \min_{u(\tau)} \left\{ J^*(t) + \int_t^{t+\Delta t} \left[ J^*_t + J^*_x a(x,u) + \text{Higher order terms} + g \right] d\tau \right\}
\]

(3.10)

When \(\Delta t\) is very small, higher order terms in can be neglected and \(u(\tau)\) at \(t < \tau < t + \Delta t\) will be like \(u(t)\). Moreover, \(J^*(t)\) will cancel in both sides and the only terms that will remain part of minimization are those that are functions of \(u\). and (3.10) reduces to the Hamilton-Jacobi-Bellman (HJB) equation

\[
0 = J^*_t + \min_{u(t)} \left\{ J^*_x a(x,u) + g(x,u) \right\} = J^*_t + \mathcal{H}(u^*)
\]

(3.11)
The partial differential equation can be solved along with a boundary condition which is the terminal cost $J^*(t_f) = h(x_f)$.

A special case of this general optimization process is the linear system with quadratic cost function or the LQ problem. The system equation is

$$\dot{x} = a(x,u) = Ax + Bu \quad (3.12)$$

and the cost function is quadratic

$$J(t) = \frac{1}{2} x_f^T S x_f + \frac{1}{2} \int_0^{t_f} (x^T Q x + u^T R u) dt \quad (3.13)$$

Then the Hamiltonian

$$\mathcal{H}(u^*) = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + J^*_x \{Ax + Bu\} \quad (3.14)$$

For minimization, partial derivative with respect to $u$ can be taken

$$0 \equiv \frac{\partial \mathcal{H}}{\partial u} = Ru + B^T J^*_x \quad (3.15)$$

yields to the optimal control input

$$u^* = -R^{-1}B^T J^*_x \quad (3.16)$$

and the optimal Hamiltonian $\mathcal{H}^*$

$$\mathcal{H}^* = \mathcal{H}(u^*) = \frac{1}{2} x^T Q x - \frac{1}{2} J^*_x B R^{-1} B^T J^*_x + J^*_x A x \quad (3.17)$$

For a linear system and a quadratic cost function, it is expected that

$$J^*_x = K(t)x(t) \quad (3.18)$$

$$J(t) = \frac{1}{2} x^T(t)K(t)x(t) \quad (3.19)$$

where $K(t)$ is a unknown symmetric non-singular matrix. Then,

$$0 = J^*_t + \mathcal{H}(u^*) \quad (3.20)$$
equation (3.20) is the HJB equation which can be written as

$$0 = \frac{1}{2} x^T \dot{K} x + \frac{1}{2} x^T Q x - \frac{1}{2} x^T K B R^{-1} B^T K x + x^T K A x$$  \hspace{1cm} (3.21)$$

or

$$-\dot{K} = Q + K B R^{-1} B^T K + K A + A^T K$$  \hspace{1cm} (3.22)$$

which is the Riccati equation. The solution $K$ is used to find the optimal value for feedback gain and consequently, the control input $u$ and optimal cost. For the total cost of the system, the steady state representation of (3.22) is called the Algebraic Riccati Equation (ARE) that has the form

$$0 = Q + K B R^{-1} B^T K + K A + A^T K$$  \hspace{1cm} (3.23)$$

the solution $K$ of this Riccati equation can be used in (3.25) to calculate the total cost of the system

$$J = \frac{1}{2} x^T (0) K x (0)$$  \hspace{1cm} (3.24)$$

3.1.2 State Feedback Optimal Cost

For the state feedback gain controller; regarding the system (2.1), the feedback gain is as follows

$$u = -F x$$  \hspace{1cm} (3.25)$$

using this gain in the performance index shown in (2.2)

$$J(t) = \frac{1}{2} x^T (t_f) S x (t_f) + \frac{1}{2} \int _t ^{t_f} (x^T (\tau) Q x (\tau) + (F x (\tau))^T R (F x (\tau))) d\tau$$  \hspace{1cm} (3.26)$$

where $P > 0$ and $Q, R \geq 0$. Using the fact $(F x (\tau))^T = x^T (\tau) F^T$. 

\[ J(t) = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_t^{t_f} (x^T(\tau) Q x(\tau) + x^T(\tau) F^T R F x(\tau)) d\tau \]  

(3.27)

\[ J(t) = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_t^{t_f} (x^T(\tau)(Q + F^T R F) x(\tau)) d\tau \]  

(3.28)

\[ J(t) = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_t^{t_f} (x^T(\tau) Q_c x(\tau)) d\tau \]  

(3.29)

where

\[ Q_c = Q + F^T R F \]  

(3.30)

and the total cost

\[ J(t) = \frac{1}{2} x^T P x \]  

(3.31)

where \( P \) is the solution of the Lyapunov equation

\[ 0 = P A + A^T P + Q_c = P A + A^T P + Q + F^T R F \]  

(3.32)

with optimal value when choosing

\[ F^* = -R^{-1} B^T K \]  

(3.33)

where \( K \) is the solution of the algebraic Riccati equation

\[ 0 = K A + A^T K + Q - K B R^{-1} B^T K \]  

(3.34)

### 3.1.3 Singularly Perturbed System Cost

System (2.1) can be rewritten in the following augmented form [26]

\[ \dot{x} = \tilde{A} \tilde{x} + \tilde{B} \tilde{u} \]  

(3.35)

where
\[ \ddot{x} = \begin{bmatrix} \dot{x} \\ \dot{z}_i \\ \dot{z}_j \end{bmatrix} \]  \hspace{1cm} \text{(3.36)}

\[ \tilde{A} = \begin{bmatrix} \tilde{A}_{00} & \tilde{A}_{0i} & A_{02} \\
\tilde{A}_{i0}/\varepsilon & \tilde{A}_{ii}/\varepsilon & 0 \\
\tilde{A}_{j0}/\varepsilon & 0 & \tilde{A}_{jj}/\varepsilon \end{bmatrix} \]  \hspace{1cm} \text{(3.37)}

\[ \tilde{B} = \begin{bmatrix} 0 & 0 & 0 \\
0 & B_i/\varepsilon & 0 \\
0 & 0 & B_j/\varepsilon \end{bmatrix} \]  \hspace{1cm} \text{(3.38)}

\[ \tilde{u} = \begin{bmatrix} u_i \\
u_j \end{bmatrix} \]  \hspace{1cm} \text{(3.39)}

\[ \tilde{y} = \begin{bmatrix} y_i \\
y_j \end{bmatrix} \]  \hspace{1cm} \text{(3.40)}

\[ \tilde{c} = \begin{bmatrix} 0 & c_i & 0 \\
0 & 0 & c_j \end{bmatrix} \]  \hspace{1cm} \text{(3.41)}

The cost-to-go function from \( t \) to \( t_f \).

\[ J(t) = \frac{1}{2} \ddot{x}^T(t_f) \tilde{S} \ddot{x}(t_f) + \frac{1}{2} \int_t^{t_f} \left( \ddot{x}^T(\tau) \tilde{Q} \ddot{x}(\tau) + \tilde{u}^T(\tau) \tilde{R} \tilde{u}(\tau) \right) d\tau \]  \hspace{1cm} \text{(3.42)}

where the symmetric matrices \( \tilde{Q}, \tilde{S} \geq 0 \) and \( \tilde{R} > 0 \).

Consider a general state feedback controller

\[ \tilde{u} = -\tilde{F} \ddot{x} \]  \hspace{1cm} \text{(3.43)}

The system model and cost become

\[ \dot{\ddot{x}} = (\tilde{A} - \tilde{B} \tilde{F}) \ddot{x} = \tilde{A}_c \ddot{x} \]  \hspace{1cm} \text{(3.44)}

and
\[ J(t) = \frac{1}{2} \ddot{x}(t) \dddot{x}(t) + \frac{1}{2} \int_t^{t_f} \left( \dddot{x}(\tau) \dddot{\tilde{x}}(\tau) + (\dddot{r} \dddot{x}(\tau) + (\dddot{r} \dddot{x}(\tau) \right) d\tau \] (3.45)

\[ J(t) = \frac{1}{2} \ddot{x}(t) \dddot{x}(t) + \frac{1}{2} \int_t^{t_f} \left( \dddot{x}(\tau) \dddot{\tilde{x}}(\tau) \right) d\tau \] (3.46)

where \( \dddot{q}_c = \dddot{q} + \dddot{r} \dddot{r} \dddot{r} \) and the terminal cost is

\[ J(t_f) = \frac{1}{2} \ddot{x}(t_f) \dddot{x}(t_f) \] (3.47)

Using the Lyapunov equation, for the feedback gain, \( F \), we have the cost-to-go function

\[ J(t) = \frac{1}{2} \ddot{x}^T P \ddot{x} \] (3.48)

where

\[-\dot{P} = P \dddot{A}_c + \dddot{A}_c^T P + \dddot{q}_c \] (3.49)

\[ P(t_f) = \tilde{S} \] (3.50)

the optimal feedback control gain \( \dddot{F}^* \)

\[ \dddot{F}^* = -\dddot{R}^{-1} \dddot{B}^T \dddot{R} \] (3.51)

where \( \dddot{R} \) is the solution of the Riccati Equation (3.52)

\[ 0 = \dddot{R} \dddot{A} + \dddot{A}^T \dddot{R} + \dddot{q} - \dddot{R} \dddot{B} \dddot{R}^{-1} \dddot{B}^T \dddot{R} \] (3.52)

and the optimal performance index

\[ \dddot{J}^* = \frac{1}{2} \dddot{r}(0)^T \dddot{R}(0) \dddot{r}(0) \] (3.53)

### 3.2 Control Strategy

With the unavailability of states to the controller, two standard strategies are used for controller synthesis.
3.2.1 Observer-Based Controller

Consider the augmented system (3.35) to be connected as shown in Figure 3.1 where

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (3.54)  
\[ y = Cx \]  \hspace{1cm} (3.55) 

The state observer is modified as

\[ \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) = (A - LC)\hat{x} + Bu + Ly \]  \hspace{1cm} (3.56) 

Observer dynamics error can be defined as

\[ e = x - \hat{x} \]  \hspace{1cm} (3.57) 

\[ \dot{e} = \dot{x} - \dot{\hat{x}} = (Ax + Bu) - (A - LC)\dot{\hat{x}} + Bu + Ly \]  \hspace{1cm} (3.58) 

\[ \dot{e} = Ax + Bu - (A\hat{x} - LC\hat{x} + Bu + L(Cx)) \]  \hspace{1cm} (3.59) 

\[ \dot{e} = Ax + Bu - A\hat{x} + LC\hat{x} - Bu - LCx \]  \hspace{1cm} (3.60) 

\[ \dot{e} = (A - LC)x - (A - LC)\hat{x} \]  \hspace{1cm} (3.61) 

\[ \dot{e} = (A - LC)(x - \hat{x}) = (A - LC)e \]  \hspace{1cm} (3.62) 

choose \( L \) such that \((A - LC)\) is stable. Because \( x \) is not available, a state feedback can be taken from \( \hat{x} \) where

\[ u = -F\hat{x} \]  \hspace{1cm} (3.63) 

By substituting the value of \( u \) in (3.63) for that in (3.54)
\[
\dot{x} = A x + B(-F\hat{x}) = A x - BF\hat{x} \quad (3.64)
\]

Similarly by substituting (3.63) and (3.55) in (3.56)

\[
\dot{x} = (A - LC)\hat{x} + B(-F\hat{x}) + L(Cx) = (A - LC - BF)\hat{x} + (LC)x \quad (3.65)
\]

Hence, the closed loop composite system will be

\[
\begin{bmatrix}
\dot{\hat{x}} \\
\dot{\hat{\xi}}
\end{bmatrix} =
\begin{bmatrix}
A & -BF \\
LC & A - LC - BF
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{\xi}
\end{bmatrix} 
(3.66)
\]

\[
y = [C \quad 0]
\begin{bmatrix}
\hat{x}
\end{bmatrix} 
(3.67)
\]

by using the defined error, we have

\[
\dot{x} = Ax - BF\hat{x} = Ax - BF(x - e) = (A - BF)x + BFe \quad (3.68)
\]

So,

\[
\begin{bmatrix}
\dot{\hat{x}} \\
\dot{e}
\end{bmatrix} =
\begin{bmatrix}
A - BF & BF \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix} 
(3.69)
\]

\[
y = [C \quad 0]
\begin{bmatrix}
e
\end{bmatrix} 
(3.70)
\]

we can say

\[
\begin{bmatrix}
x \\
e
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
I & -I
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{\xi}
\end{bmatrix} = T
\begin{bmatrix}
\hat{x}
\end{bmatrix} 
(3.71)
\]

Using the estimated state \(\hat{x}\), a controller can be designed separately as the separation principle guarantees the success of the designed controller to be applied to the estimated states.

The feedback gain and control input have the form

\[
u = -Fx \quad (3.72)
\]

where

\[
F = R^{-1}B^{T}P \quad (3.73)
\]

and

\[
-\dot{P} = PA + A^{T}P + Q \quad \text{with} \quad P(t_{f}) = S \quad (3.74)
\]

and the optimal feedback gain
\[ F^* = R^{-1}B^TK \]  \hspace{1cm} (3.75)

where

\[ -\dot{K} = KA + A^T K + Q - KBR^{-1}B^TK \]  \hspace{1cm} (3.76)

and the optimal performance index

\[ J^* = \frac{1}{2}x^T(0)K(0)x(0) = \frac{1}{2}tr\{K(0)\Sigma\} \]  \hspace{1cm} (3.77)

where

\[ \Sigma = x(0)x^T(0) \]  \hspace{1cm} (3.78)

### 3.2.2 Static Output Feedback

A more static and more reliable output feedback control method without the need of an observer (Kalman Filter) for estimating the states of the system is the static output feedback gain. This method utilizes the information provided by the output signal only and without approximating the states of the system to calculate the feedback gain. In this case the feedback gain will be on the dimensions of the output signal and will no longer be associated with the states vector \( x \). For the static output feedback control, consider the system (3.79) shown in Figure 3.2.

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (3.79)

\[ y = Cx \]  \hspace{1cm} (3.80)

Because neither the of states is available and no observer is present, the controller will make decisions according to the output signal only and will have the form

\[ u = -Fy \]  \hspace{1cm} (3.81)

By using the quadratic performance index
\[ J(t) = \frac{1}{2} \int_{t}^{\infty} \left( x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau) \right) d\tau \] (3.82)

\[ \dot{x} = Ax + B(-FCx) = (A - BFC)x = A_c x \] (3.83)

Similarly, the performance index will be expressed as

\[ J(t) = \frac{1}{2} \int_{t}^{\infty} \left( x^T(\tau)Q_c x(\tau) \right) d\tau \] (3.84)

where

\[ Q_c = Q + C^T F^T RFC \] (3.85)

Instead of dynamically selecting \( K \) to minimize \( J \) to the constraint (3.83), a static minimization can be found by selecting a positive semi-definite \( P \) so that

\[ J = \frac{1}{2} x^T(0)Px(0) = \frac{1}{2} tr\{P\Sigma\} \] (3.86)

where

\[ 0 = A_c^T P + PA_c + Q_c \] (3.87)

and \[ \Sigma = x(0)x^T(0) \] (3.88)

Equation (3.87) is a condition for (3.86) and will be referred to as \( \kappa \).
\[ \kappa = A_c^T P + P A_c + Q_c \]  \hspace{1cm} (3.89)

The performance index in (3.86) then can be minimized by introducing the Lagrangian multiplier \( Z \) to the Hamiltonian \( \mathcal{H} \) in (3.90) satisfying (3.89)

\[ \mathcal{H} = tr\{ P \Sigma \} + tr\{ \kappa Z \} \]  \hspace{1cm} (3.90)

For optimization, let’s set partial derivatives of \( \mathcal{H} \) with respect to all independent variables \( P, Z, \) and \( F \) to zero

\[ 0 \equiv \frac{\partial \mathcal{H}}{\partial Z} = Q_c + A_c^T P + PA_c \]  \hspace{1cm} (3.91)

\[ 0 \equiv \frac{\partial \mathcal{H}}{\partial P} = \Sigma + A_c Z + Z A_c^T \]  \hspace{1cm} (3.92)

\[ 0 \equiv \frac{\partial \mathcal{H}}{\partial F} = 2RFCZC^T - 2B^T PZC^T \rightarrow F = R^{-1} B^T PZC^T (CZC^T)^{-1} \]  \hspace{1cm} (3.93)

assuming \( R \) is positive definite and \( CZC^T \) is invertible matrix these three equations can be solved numerically. First set an initial value of \( F \) then solve the Lyapunov equations (3.91) and (3.92) for \( P \) and \( Z \) respectively. Then solve equation (3.93) for \( F \) using the values of \( P \) and \( Z \). Using the calculated \( F \), the calculation can be repeated for next iteration by updating the values of \( A_c \) and \( Q_c \) and continue until a convergence occurs or a threshold error is satisfied.

3.3 **Stochastic Control**

3.3.1 **Kalman Filter**

The Kalman filter is an optimal stochastic state estimator used when the states of a system are not available for measurement. Designing a Kalman filter is similar to standard observer design with the difference in the output stochastic uncertainty of the system.

Consider the system
\[ \dot{x} = Ax + Bu + w \]  
\[ y = Cx + v \]  
where \( w \): is the input disturbance with mean = 0 and a positive semi-definite intensity = \( W \).

\( v \): is the measurement noise with mean = 0 and a positive definite intensity = \( V \).

A stochastic observer can be shown as
\[ \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \]  
\[ \dot{e} = (A - LC)e + w - Lv \]  
where \( e = x - \hat{x} \). The problem here is to choose \( L \) such that the mean square error is minimized

i.e. \( \min_L \{ e^T e \} \). With the variance of the error \( P \) has the form
\[ P = E\{ [e - \bar{e}][e - \bar{e}]^T \} \]

The expected value of the error square can be written as
\[ E\{ee^T\} = \bar{e}\bar{e}^T + P \]

So,
\[ tr \ E\{ee^T\} = tr \ [\bar{e}\bar{e}^T] + tr[P] \]  
\[ tr \ E\{e^Te\} = tr \ [\bar{e}^T \bar{e}] + tr[P] \]
\[ E\{e^Te\} = \bar{e}^T \bar{e} + tr[\Sigma] \]

For the initial condition uncertainty, we can choose \( E\{\bar{x}(0)\} = \bar{x}_0 \) such that \( \bar{e}(0) = 0 \), and \( \bar{e}(t) = 0 \) for all \( t \). Next minimize \( tr[P] \) \[27\]. Where \( \bar{x} \) and \( \bar{e} \) are the average values of \( x \) and \( e \) respectively.

In (3.97) we have \( w - Lv \) has intensity \( W + LVLT \), and \( P \) is given by
\[ \dot{P} = (A - LC)P + P(A - LC)^T + W + LVLT \quad \text{with} \quad P(0) = \Sigma \]

minimization is achieved by choosing
\[ L = KC^TV^{-1} \]  
\[ (3.104) \]

where \( K \) is the solution of the Riccati Equation

\[ \dot{K} = AK + KA^T + W - KC^TV^{-1}CK, \quad \text{with} \quad K(0) = \Sigma \]  
\[ (3.105) \]

### 3.3.2 Linear Quadratic Gaussian

Consider the system

\[ \dot{x} = Ax(t) + Bu(t) + Gw(t) \]  
\[ (3.106) \]

with the following performance index

\[ J = E \left\{ \frac{1}{2} x^T(t_f)Sx(t_f) + \frac{1}{2} \int_0^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \right\}. \]  
\[ (3.107) \]

where \( w \) is the input disturbance with mean = 0 and a positive semi-definite intensity = \( W \), \( S \) and \( Q \) are symmetric positive semi-definite and \( R \) is a symmetric positive definite matrix.

Let

\[ u(t) = -Fx(t) \]  
\[ (3.108) \]

where \( F \) is a general feedback gain and can be time-varying. The closed loop system can be written as

\[ \dot{x} = A_c x(t) + Gw(t) \]  
\[ (3.109) \]

\[ J = E \left\{ \frac{1}{2} x^T(t_f)Sx(t_f) + \frac{1}{2} \int_0^{t_f} [x^T(t)Q_c x(t)]dt \right\} \]  
\[ (3.110) \]

where

\[ A_c = A - BF \]
\[ Q_c = Q + F^TRF \]  
\[ (3.111) \]

for

\[ x(t) = \Phi(t)x(0) + \int_0^t \Phi(t - \tau)Gw(\tau)d\tau \]  
\[ (3.112) \]
the performance index $J$ can be written as

$$J = E \left[ \frac{1}{2} \left[ \Phi(t_f)x(0) + \int_0^{t_f} \Phi(t_f - \tau)Gw(\tau) d\tau \right]^T S \left[ \Phi(t_f)x(0) + \int_0^{t_f} \Phi(t_f - \tau)Gw(\tau) d\tau \right] \right] +$$

$$\int_0^{t_f} \Phi(t_f - \tau)Gw(\tau) d\tau \right]^T Q \left[ \Phi(t)x(0) + \int_0^t \Phi(t - \tau)Gw(\tau) d\tau \right] dt \right]$$

(3.113)

assuming $x(0)$ and $w$ are independent, (3.117) reduces to

$$J = E \left[ \frac{1}{2} x^T(0) \Phi^T(t_f)S \Phi(t_f)x(0) + \frac{1}{2} \int_0^{t_f} \Phi(t_f - \tau)Gw(\tau) d\tau \right] +$$

$$\int_0^{t_f} \Phi(t_f - \tau)Gw(\tau) d\tau \right]^T S \left[ \int_0^{t_f} \Phi(t_f - \tau)Gw(\tau) d\tau \right] +$$

$$\frac{1}{2} \int_0^{t_f} x^T(0) \Phi^T(t)Q \Phi(t)x(0) dt + \frac{1}{2} \int_0^{t_f} \left[ \int_0^t \Phi(t - \tau)Gw(\tau) d\tau \right] dt \right]$$

(3.114)

by changing integral limits

$$J = \frac{1}{2} tr \left\{ \left[ \Phi^T(t_f)S \Phi(t_f) + \int_0^{t_f} \Phi^T(t)Q \Phi(t) dt \right] \Sigma \right\}$$

$$+ \frac{1}{2} tr \left\{ GVG^T \int_0^{t_f} \left[ \Phi^T(t_f - \tau)S \Phi(t_f - \tau) \right] \right\}$$

(3.115)

$$+ \int_\tau^{t_f} \Phi^T(t - \tau)Q \Phi(t - \tau)G \ d\tau \right\}$$

or
\[ J = \frac{1}{2} tr[P(0)\Sigma] + \frac{1}{2} tr \left[ GWG^T \int_0^{t_f} P(\tau) d\tau \right] \]  
(3.116)

where \( \Sigma = E\{x(0)x^T(0)\} \), \( W \) is the intensity of the disturbance \( w \), and \( P \) is the solution of the Lyapunov equation

\[-\dot{P} = PA + A^T P + Q\]  
(3.117)

with the boundary condition \( P(t_f) = S \).

\( P \) is optimized by selecting

\[ F^* = R^{-1}B^T K \]  
(3.118)

where \( K \) is the solution of the Riccati equation

\[-\dot{K} = KA + A^T K + Q - KB R^{-1} B^T K\]  
(3.119)

with the boundary condition \( K(t_f) = S \). So, the optimal performance index is

\[ J^* = \frac{1}{2} tr[K(0)\Sigma] + \frac{1}{2} tr \left[ GWG^T \int_0^{t_f} K(\tau) d\tau \right] \]  
(3.120)

### 3.4 Game Theory

#### 3.4.1 What is game theory?

Game theory is concerned with optimizing the cost or profit among two or more players. It involves a strategic decision making technique where the decision depends not just on how one player chooses among different decisions, but on the choices that other players made. Those players can be cooperative or non-cooperative with each other. Hence, conflicts will be studied for both competing players and cooperative players, for which the focus will be on optimizing individual player’s decisions. Although it is primarily used in economics for modeling market share and pricing techniques, the method can be used for any other decision dependent discipline.
In control systems a decision is the goal of any controller design, game theory can be applied when there are two or more controllers that have mutual impact regardless of common or conflicting goals. In the context of this research, each controller will be considered a ‘player’ and those players will be cooperating or competing to optimize the overall performance of the system [28] - [31].

### 3.4.2 Definitions

A brief description of some of the basic game theory elements with the corresponding element in the control system described in this research is presented here [32]:

**Game**: A strategic mutual actions made between two or more players where they have common or conflicting goals. This corresponds to the overall performance of the control system.

**Player**: The key element in every game who takes actions or makes decisions. Controllers take the role of players in a decentralized control system.

**Action**: The decision made or action taken by the player. The action in control systems is represented by the controlled input.

**Payoff**: Sometimes called *utility*, is the gain or loss that a player wants to optimize by making the best decisions. The objective of any optimal control design is to minimize the cost function described by equation (2.2).

**Strategy**: A set of decisions that a player can make within the game. Feedback gain or designed controller is the strategy in control systems.

### 3.4.3 Game setup

A game consists of a finite number of players. Each player has a set of strategies to choose from and an objective to optimize the total or individual outcome or performance index. The strategy of each player is the plan of what the player should do for every possible situation. Because decisions of players affect other players, a strategy should take into consideration the other player’s
action in the optimization process, which may make the strategy very complicated. There are two types of games. When each player makes decisions autocratically, or the players have conflicting interests, this is called a non-cooperative game. In cooperative games, players cooperate to optimize a total performance index together. In this research, a non-cooperative two-person game will be considered [33, 34].

3.4.4 Game theory in control systems

A non-cooperative game consists of three components: the number of players, set of strategies, and utility functions or performance index. Considerable research has been done in several game-theoretical strategies and techniques for solving non-cooperative games. Nash equilibrium is a widely known equilibrium solution concept to minimize the cost function and reach steady state equilibrium. Nash equilibrium characterizes the state when a player cannot improve performance given that other players’ actions are fixed [32].

In this research, Nash equilibrium solution is considered to optimize the defined system. A game can be modeled to minimize the objective function $J(x,u_i,u_j)$ in order to get optimal $J^*(x,u_i^*,u_j^*)$.

where

\[ J_i^*(x,u_i^*,u_j^*) \leq J_i(x,u_i,u_j^*) \quad (3.121) \]
\[ J_j^*(x,u_i^*,u_j^*) \leq J_j(x,u_i^*,u_j) \quad (3.122) \]

Suppose we have the following system

\[ \dot{x} = Ax + B_i u_i + B_j u_j, \quad \text{with } x(0) = x_0 \quad (3.123) \]
\[ y_i = C_i y_i \quad (3.124) \]

with a output feedback input
\[ u_i = -F_iC_ix \quad (3.125) \]
\[ u_j = -F_jC_jx \quad (3.126) \]

The given cost functions for each controller input
\[ J_i = \frac{1}{2} \int_0^\infty (x^TQ_xx + u_i^TR_{ii}u_i + u_j^TR_{ij}u_j)dt \quad (3.127) \]

where \( R_{ij} \geq 0, \) \( Q_x \geq 0, \) \( R_{ii} > 0, \) \( C_i \) have full row rank,
then
\[ \dot{x} = (A - B_iC_iF_i - B_jC_jF_j)x = A_cx \quad (3.128) \]
so,
\[ J_i = \frac{1}{2} \int_0^\infty x^T(Q_x + C_i^TF_i^TR_{ii}F_iC_i + C_j^TF_j^TR_{ij}F_jC_j)x \ dt = \frac{1}{2} \int_0^\infty x^TQ_xx \ dt \quad (3.129) \]

From section (3.1.1) we found the solution for this LQR problem is
\[ J_i = \frac{1}{2} tr(P_i\Sigma), \quad (3.130) \]
where \( \Sigma = x(0)x^T(0), \) and \( P_i \geq 0 \) is the solution of \( \kappa_i \) defined as
\[ \kappa_i \equiv A_c^TP_i + P_iA_c + Q_x + C_i^TF_i^TR_{ii}F_iC_i + C_j^TF_j^TR_{ij}F_jC_j = 0 \quad (3.131) \]

Now, let’s introduce the symmetric undetermined \( Z_i \) as a co-state variable for input \( i \) into
the following Hamiltonian function
\[ \mathcal{H}_i = tr(P_i\Sigma) + tr(\kappa_iZ_i) \quad (3.132) \]
To minimize \( \mathcal{H} \), partial derivative with respect to the variables \( Z_i, P_i \) and \( F_i \) are set to zero.
\[ \frac{\partial \mathcal{H}}{\partial Z_i} = \kappa_i^T = A_c^TP_i + P_iA_c + Q_x + C_i^TF_i^TR_{ii}F_iC_i + C_j^TF_j^TR_{ij}F_jC_j \equiv 0 \quad (3.133) \]
\[ \frac{\partial \mathcal{H}}{\partial P_i} = A_cZ_i + Z_iA_c^T + \Sigma \equiv 0 \quad (3.134) \]
\[
\frac{\partial \mathcal{H}}{\partial F_i} = -B_i^T P_i Z_i C_i^T + R_i F_i Z_i C_i^T \equiv 0
\] (3.135)

Where \( P_i \), \( R \)'s and \( Q_x \) are symmetric. As game players, each subsystem \( i \) will consider the feedback gain of the other player’s decision \((F_j)\) constant. By solving these equations iteratively, Nash equilibrium can be found.

Stackelberg is another famous game-theoretic optimizing strategy considered in this research where one input (player) takes the lead so the other player will dependently respond with the best decision possible, taking into consideration the leader’s strategy. So, for the leader, there is no information about the follower status. Hence, at the starting point, the cost will be exactly like Nash equilibrium results shown in (3.133), (3.134) and (3.135) for controller \( i \). After that, the follower will minimize its cost using the results of the leader plus the effect of the decision made by the other player. Mathematically, this can be formulated by taking the leader optimal solution as constraints for the follower optimization problem.

Basically, each subsystem tends to minimize its own performance index [35]. Because of the decentralized setup of the system, the two subsystems dynamics are coupled. The objective is to find an optimal control \( u_i \) for each subsystem that is able to minimize the common performance index. This is done by taking into consideration the effect of the other subsystem control in the performance index shown in (3.132).

Using Stackelberg strategy in game theory, the leader will go first (controller \( i \)) and a Hamiltonian function to be minimized is defined as

\[
\mathcal{H}_i = tr(P_i \Sigma) + tr(\kappa_i Z_i)
\] (3.136)

The objective is to minimize \( \mathcal{H}_i \), and partial derivatives with respect to \( P_i \) and \( Z_i \) will be taken. As \( \kappa_i \) is explicitly dependent on \( F_i \) and \( F_j \), partial derivatives of the Hamiltonian function are taken with respect to \( F_i \) and \( F_j \). Because of the leader-follower strategy of Stackelberg, the
leader will start without the knowledge of the follower’s decision by assuming \( F_j \) constant and \( Z_i \) is an undetermined symmetric matrix with appropriate dimensions. So, the leader chooses \( F_i \) to minimize \( J_i \) by differentiating \( \mathcal{H}_i \) with respect to \( P_i, Z_i \) and \( F_i \) respectively

\[
0 = \kappa_i^T = A_c^T P_l + P_l A_c + Q_x + C_i^T F_i^T R_{ii} F_i + C_j^T F_j^T R_{ij} F_j \\
0 = \psi = A_c Z_i + Z_i A_c^T + \Sigma \\
0 = \phi = -B_i^T P_l Z_i C_l^T + R_{ii} F_i Z_i C_l^T
\]

(3.137)

where \( P_i \) and \( Q_x \) are symmetric.

The follower now is to minimize \( J_j \) while taking in consideration the reaction of the leader. This can be arranged by taking the leader’s solution as constraints to be met while minimizing \( J_j \).

The Hamiltonian function should contain original terms \( tr(P_j \Sigma) \) and \( tr(\kappa_j Z_j) \) in addition to the constraints \( \kappa_i, \psi \) and \( \phi \) and by introducing the co-state variables \( \lambda, \gamma \) and \( \beta \) to result in the following extended Hamiltonian function [36]

\[
\mathcal{H}_j = tr(P_j \Sigma) + tr(\kappa_j Z_j) + tr(\kappa_i \lambda) + tr(\psi \gamma) + tr(\phi \beta^T) + tr(\beta^T \phi^T)
\]

(3.138)

We previously have \( P_j, \kappa_j \) similar to those in (3.131), \( \kappa_i, \psi \) and \( \phi \) are defined in (3.137).

The minimization is done by taking partial derivative of \( \mathcal{H}_j \) with respect to the leader variables \( Z_i, P_l \) and \( F_i \), the follower variables \( Z_j, P_j \) and \( F_j \), and the “fictitious follower [36]” \( \lambda, \gamma \) and \( \beta \) to end up with nine minimization conditions.
OPTIMAL CONTROL FOR STOCHASTIC SINGULARLY PERTURBED SYSTEMS

4.1 Deterministic System

For deterministic systems, the reduced order model or the quasi-steady state system (2.1) where the slow part of the fast subsystems shown in (2.4) can be written as

\[ \bar{z}_i = -A_{ii}^{-1}A_{i0}\bar{x} - A_{ii}^{-1}B_i\bar{u}_i \]  \hspace{1cm} (4.1)

These can be applied to the performance index (2.2). The performance index of the quasi-steady state model will have the form

\[ J = \frac{1}{2} \int_0^\infty \begin{bmatrix} \bar{x}^T & \bar{z}_i^T & \bar{z}_j^T \end{bmatrix} \begin{bmatrix} Q_x & 0 & 0 \\ 0 & Q_{zi} & 0 \\ 0 & 0 & Q_{zj} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{z}_i \\ \bar{z}_j \end{bmatrix} + \begin{bmatrix} 0 & \bar{u}_i^T & \bar{u}_j^T \end{bmatrix} \begin{bmatrix} 0 \\ \bar{u}_i^T \\ \bar{u}_j^T \end{bmatrix} \) \hspace{1cm} (4.2)

\[ J = \frac{1}{2} \int_0^\infty (\bar{x}^T \hat{Q} \bar{x} + 2\bar{x}^T M_i \bar{u}_i + 2\bar{x}^T M_j \bar{u}_j + \bar{u}_i^T \hat{R}_i \bar{u}_i + \bar{u}_j^T \hat{R}_j \bar{u}_j) dt \]  \hspace{1cm} (4.3)

where

\[ \hat{Q} = Q_x + A_{i0}^T A_{ii}^{-1}Q_{zi}A_{ii}^{-1}A_{i0} + A_{j0}^T A_{jj}^{-1}Q_{zj}A_{jj}^{-1}A_{j0} \] \hspace{1cm} (4.4)

\[ M_i = A_{i0}^T A_{ii}^{-1}Q_{zi}A_{ii}^{-1}B_i \] \hspace{1cm} (4.5)

\[ M_j = A_{j0}^T A_{jj}^{-1}Q_{zj}A_{jj}^{-1}B_j \] \hspace{1cm} (4.6)

\[ \hat{R}_i = R_i + B_i^T A_{ii}^{-1}Q_{zi}A_{ii}^{-1}B_i \] \hspace{1cm} (4.7)

\[ \hat{R}_j = R_j + B_j^T A_{jj}^{-1}Q_{zj}A_{jj}^{-1}B_j \] \hspace{1cm} (4.8)

To get rid of the cross product terms (2\(\bar{x}^T M_i \bar{u}_i\) and 2\(\bar{x}^T M_j \bar{u}_j\)), let

\[ \bar{u}_i = \bar{u}_0i + F_0i \bar{x} \] \hspace{1cm} (4.9)
and
\[ \bar{u}_j = \bar{u}_0 j + F_0 j \bar{x} \]  

(4.10)

by plugging into (4.3) we get
\[ J = \frac{1}{2} \int_{0}^{\infty} (\bar{x}^T Q_r \bar{x} + \bar{u}_0^T R_0 \bar{u}_0) dt \]  

(4.11)

where
\[ Q_r = \hat{Q} + 2M_i F_{0i} + 2M_j F_{0j} \]  

(4.12)

\[ \bar{u}_0 = \begin{bmatrix} \bar{u}_{0i} \\ \bar{u}_{0j} \end{bmatrix} \]  

(4.13)

and
\[ R_0 = \begin{bmatrix} \hat{R}_i & 0 \\ 0 & \hat{R}_j \end{bmatrix} \]  

(4.14)

and the optimal control input of the system will take the form
\[ u_0^* = -F_0^* \bar{x} \]  

(4.15)

where
\[ F_0^* = R_0^{-1} B_0^T K \]  

(4.16)

and \( K \) is the solution of the Riccati equation
\[ 0 = KA_c + A_c^T K + K B R_0^{-1} B^T K + Q_r \]  

(4.17)

4.2 Stochastic Systems

When the system in (2.1) is involved with some external noise from both measurements readings and input disturbance, it will take the form
\[
\begin{align*}
\dot{x} &= A_{00}x + A_{0i}z_i + A_{0j}z_j \\
\varepsilon \dot{z}_i &= A_{i0}x + A_{ii}z_i + B_i u_i + G_i w_i \\
\varepsilon \dot{z}_j &= A_{j0}x + A_{jj}z_j + B_j u_j + G_j w_j \\
y_i &= C_i z_i + H_i v_i \\
y_j &= C_j z_j + H_j v_j 
\end{align*}
\]

(4.18)

where \(w_i\) and \(v_i\) are the input disturbance and output measurement noise to subsystem \(i\) respectively. \(G_i\) and \(H_i\) are constant scaling matrices corresponds to noise for subsystem \(i\). The performance index of subsystem \(i\) is given

\[
J_i = \frac{1}{2} \mathbb{E} \left\{ \int_0^\infty (x^T Q x + z_i^T Q z_i + u_i^T R u_i) dt \right\}
\]

(4.19)

The reduced order model for system (4.18) can be obtained the same way for the deterministic system.

\[
\begin{align*}
\dot{x} &= A_{00}x + A_{0i} \bar{z}_i + A_{0j} \bar{z}_j \\
0 &= A_{i0}x + A_{ii} \bar{z}_i + B_i \bar{u}_i + G_i w_i \\
0 &= A_{j0}x + A_{jj} \bar{z}_j + B_j \bar{u}_j + G_j w_j \\
\bar{y}_i &= C_i \bar{z}_i + H_i v_i \\
\bar{y}_j &= C_j \bar{z}_j + H_j v_j 
\end{align*}
\]

(4.20)

By solving the second and third equations of the system (4.20) for \(\bar{z}_i\) and \(\bar{z}_j\) respectively we get the following

\[
\bar{z}_i = -A_{ii}^{-1}(A_{i0}x + B_i u_i + G_i w_i)
\]

(4.21)

where \(A_{ii}\) are non-singular. By plugging \(\bar{z}_i\) and \(\bar{z}_j\) from (4.21) in (4.20) we get the reduced order system

\[
\begin{align*}
\dot{\bar{x}} &= \hat{A} \bar{x} + \sum \left[B_i \bar{u}_i + \bar{G}_i w_i\right] \\
\bar{y}_i &= \hat{C}_i \bar{x} + \bar{D}_i \bar{u}_i + \bar{G}_i w_i + H_i v_i 
\end{align*}
\]

(4.22)
\[
\hat{A} = A_{00} - A_{0i} A_{ii}^{-1} A_{i0} - A_{0j} A_{jj}^{-1} A_{j0}
\]
\[
\hat{B}_i = - A_{0i} A_{ii}^{-1} B_i
\]
\[
\hat{C}_i = - C_i A_{ii}^{-1} A_{i0}
\]
\[
\hat{D}_i = - C_i A_{ii}^{-1} B_i
\]
\[
\hat{G}_i = - A_{0j} A_{jj}^{-1} G_j
\]
\[
\tilde{G}_i = - C_i A_{ii}^{-1} G_i
\]

The performance index of subsystem \(i\) is given by
\[
J_i = \frac{1}{2} E \left\{ \int_0^\infty (\hat{x}^T Q_x \hat{x} + \hat{z}_i^T Q_{zi} \hat{z}_i + \bar{u}_i^T R_{li} \bar{u}_i) dt \right\}
\]
\[
(4.29)
\]

By substituting the values of \(\hat{z}_i\) (4.21) we get the reduced performance index as
\[
J_i = \frac{1}{2} E \left\{ \int_0^\infty \{ \bar{x}^T \bar{Q} \bar{x} + 2 \bar{x}^T M_i \bar{u}_i + \bar{u}_i^T R_{cli} \bar{u}_i \} dt \right\} + J_w
\]
\[
(4.30)
\]

where \(J_w\) is a term contains noise and not affected by control [20], and
\[
\bar{Q} = Q_x + A_{i0}^T A_{ii}^{-1} T Q_{zi} A_{ii}^{-1} A_{i0}
\]
\[
M_i = B_i^T A_{ii}^{-1} T Q_{zi} A_{ii}^{-1} A_{i0}
\]
\[
R_{cli} = R_{li} + B_i^T A_{ii}^{-1} T Q_{zi} A_{ii}^{-1} B_i
\]

To get rid of the cross product term in (4.30), the feedback input for subsystem \(i\) is chosen to be
\[
\bar{u}_i = F_i \bar{x} = -R_{cli}^{-1} M_i^T \bar{x}
\]
\[
(4.34)
\]

So (4.30) will reduce to
\[
J_i = \frac{1}{2} E \left\{ \int_0^\infty \{ \bar{x}^T Q_x \bar{x} + \bar{u}_i^T R_{cli} \bar{u}_i \} dt \right\}
\]
\[
(4.35)
\]

where
\[
\ddot{u}_i = \ddot{u}_{0i} + F_{0i} \ddot{x}
\] (4.36)

For the system

\[
\dddot{x} = A_c \ddot{x} + \ddot{w}
\] (4.37)

where

\[
Q_r = \dddot{Q} - M_{0l}R_{cii}^{-1}M_{0l}^T
\] (4.38)

\[
F_{0l} = -R_{cii}^{-1}M_{l}^T
\] (4.39)

\[
A_c = A_r - F_{0l}^T M_{l}^T F_{0l}
\] (4.40)

\[
A_r = \dot{A} - \dot{B}_i R_{cii}^{-1} M_{l}^T
\] (4.41)

The optimal cost of the system is

\[
J_l = \frac{1}{2} tr\{P \Sigma\} + \frac{1}{2} tr\{P \hat{G}_i W \hat{G}_i^T\}
\] (4.42)

where \(P\) is the solution of

\[
- \dot{P} = PA_c + A_c^T P + Q_r + F_{0l}^T R_{cii} F_{0l}
\] (4.43)

and the optimal feedback gain

\[
F_{0l}^* = -R_{cii}^{-1} \hat{B}_i^T K
\] (4.44)

where \(K\) is the solution of

\[
0 = A^T K + KA + Q_r - K \hat{B}_i R_{cii}^{-1} \hat{B}_i^T K
\] (4.45)

### 4.3 Kalman Filter Design

For the system (4.22) there is no access to the main state \(x^s\), instead we can use an estimate of the state variable using Kalman filter. Because of the decentralized nature of the system setup, we will not be able to use the same estimate of the state variable \(x^s\) for both subsystems. Hence, there will be two estimates for the main system states \(x^s\) using two observers (Kalman filters) connected to each subsystem. The observed system
\[
\begin{align*}
\dot{x}_i^s &= A_0x_i^s + B_0u_i^s + B_0u_j^s + L(y_i^s - \hat{y}_i^s) \\
\dot{\hat{y}}_i^s &= E\{y_i^s\} = E(\hat{C}_i x_i^s + \hat{D}_i u_i^s + v_{oi}) = \hat{C}_i E\{x_i^s\} + \hat{D}_i u_i^s = \hat{C}_i \hat{x}_i^s + \hat{D}_i u_i^s
\end{align*}
\] (4.46)

where \( \hat{x}_i^s \) is the estimate of the quasi-steady state variable of the main system \( X \) through the Kalman filter connected to subsystem \( i \). The error between \( \hat{x}_i^s \) and the actual \( x_i^s \) is then calculated

\[
\dot{e}_i = \hat{x}_i - \hat{x}_i^s = A_c e_i + w_0 - L v_i
\] (4.47)

where \( e_i \) is the error of the estimated state \( x \) using Kalman filter connected to subsystem \( i \).

and

\[
w_0 = \hat{G}_i w_i + \hat{G}_j w_j
\] (4.48)

The error variance is defined as

\[
P = E\{e_i e_i^T\}
\] (4.49)

then

\[
\dot{P} = E\{\dot{e}_i e_i^T\} + [E\{\dot{e}_i e_i^T\}]^T
\] (4.50)

The first term in (4.50)

\[
E\{\dot{e}_i e_i^T\} = E\{(A_c e + w_0 - L v_i)e_i^T\}
\] (4.51)

\[
= A_c E\{e_i e_i^T\} + \frac{1}{2} W_0 - \frac{1}{2} L V_i L^T
\] (4.52)

as \( R_{w_0 e} = \frac{1}{2} W_0 \), \( R_{v_i e} = -\frac{1}{2} V_i L^T \) where \( W_0 \) and \( V_i \) are intensities of \( w_0 \) and \( v_i \) respectively. Now, with \( E\{e_i e_i^T\} = P \)

\[
\dot{P} = A_c P + PA_c^T + W_0 - LV_i L^T
\] (4.53)

For this well-known Lyapunov Equation, minimization is achieved by choosing

\[
L = KC^T V_i^{-1}
\] (4.54)

And \( K \) is the solution of the Riccati Equation

\[
K = A_c K + KA_c^T + W_0 - KC^T V_i^{-1} C K, \quad \text{with} \quad K(0) = \Sigma
\] (4.55)

By setting \( \dot{P} \) to zero, (4.53) will become

\[
0 = A_c P + PA_c^T + W_0 - LV_i L^T
\] (4.56)
Then we use \((K)\) as the optimal value of \(P\), when 
\[
0 = A_c K + KA_c^T + W_0 - KC^T V_i^{-1}CK
\]

\[ \tag{4.57} \]

### 4.4 Static Output Feedback

In section 2.4, the quasi-steady state model of the stochastic singularly perturbed decentralized large scale system (2.1) was shown as
\[
\begin{align*}
\dot{x}_i &= \hat{A}x_i + \hat{B}_i \bar{u}_i + \hat{G}_i w_i + \hat{B}_j \bar{u}_j + \hat{G}_j w_j \\
\bar{y}_i &= \hat{C}_i x_i + \hat{D}_i \bar{u}_i + \hat{G}_i w_i + H_i v_i
\end{align*}
\]

\[ \tag{4.58} \]

The performance index of subsystem \(i\) is given by the expected value of \(J_i\)
\[
J_i = \frac{1}{2} \mathbb{E}\left\{ \int_0^\infty \left( \bar{x}^T \hat{Q} \bar{x} + \bar{u}_i^T R_{cii} \bar{u}_i + \bar{u}_j^T R_{cij} \bar{u}_j + 2 \bar{x}^T M_i \bar{u}_i + 2 \bar{x}^T M_j \bar{u}_j \right) dt \right\} \]

\[ \tag{4.59} \]

where
\[
\begin{align*}
\hat{Q} &= Q_x + A_{i0}^T A_{ii}^{-1} Q_{zi} A_{ii}^{-1} A_{i0} + A_{j0}^T A_{jj}^{-1} Q_{zj} A_{jj}^{-1} A_{j0} \\
R_{cii} &= R_{ii} + B_i^T A_{ii}^{-1} Q_{zi} A_{ii}^{-1} B_i \\
R_{cij} &= R_{ij} + B_j^T A_{jj}^{-1} Q_{zj} A_{jj}^{-1} B_j \\
M_i &= A_{i0}^T A_{ii}^{-1} Q_{zi} A_{ii}^{-1} B_i \\
M_j &= A_{j0}^T A_{jj}^{-1} Q_{zj} A_{jj}^{-1} B_j
\end{align*}
\]

\[ \tag{4.60} \]

The static output feedback input \(u_i^s\) for each subsystem \(i\) can be written as
\[
u_i^s = -F_i \bar{y}_i
\]

\[ \tag{4.61} \]

Using the value of \(\bar{y}_i\) shown in (4.58) and the cross product term elimination, the control input will be
\[
\bar{u}_i = \Omega \bar{u}_{0i} + \Omega F_{0i} \hat{C}_i \bar{x}_i + \bar{w}_i
\]

\[ \tag{4.62} \]

where
\[
\Omega = \left( I - F_{0i} D_i \right)^{-1}
\]

\[ \tag{4.63} \]
\[ \bar{w}_i = \Omega F_{0i} \hat{G}_i w_i + \Omega F_{0i} \Omega H_i v_i \tag{4.64} \]

The system and performance index of the closed loop system are shown in (4.65) and (4.66) respectively

\[ \dot{x}_i = A_c \bar{x}_i + \bar{w}_i \tag{4.65} \]

and

\[ J_i = \frac{1}{2} E \left\{ \int_0^\infty \left( \bar{x}_i^T Q_c \bar{x}_i + \bar{u}_{0i}^T \hat{R}_{cii} \bar{u}_{0i} + \bar{u}_{0j}^T \hat{R}_{cij} \bar{u}_{0j} \right) dt \right\} \tag{4.66} \]

where

\[
\begin{align*}
A_c &= A_r - \bar{B}_i F_{0i} \hat{C}_i - \bar{B}_j F_{0j} \hat{C}_j \\
A_r &= \hat{A} - \bar{B}_i R_{cii}^{-1} M_i^T - \bar{B}_j R_{cij}^{-1} M_j^T \\
\bar{w}_i &= \bar{B}_i \bar{w}_i + \bar{B}_j \bar{w}_j \\
Q_c &= Q_r + \hat{C}_i^T F_{0i}^T R_{cii} F_{0i} \hat{C}_i + \hat{C}_j^T F_{0j}^T R_{cij} F_{0j} \hat{C}_j \\
Q_r &= \hat{Q} + M_i R_{cii}^{-1} M_i^T + M_j R_{cij}^{-1} M_j^T \\
\hat{R}_{cii} &= \Omega^T R_{cii} \Omega \\
\hat{R}_{cij} &= \Omega^T R_{cij} \Omega
\end{align*}
\]

The solution of the performance index for this system found to be

\[ J_i = \frac{1}{2} tr(P_i \Sigma) \tag{4.74} \]

where \( \Sigma = E \{ x(0)x^T(0) \} \), \( P_i \geq 0 \) is the solution of

\[ 0 = A_c^T P_i + P_i A_c + Q_c \tag{4.75} \]

Because of the decentralized setup of the system, the two subsystems dynamics are uncoupled. The objective is to find an optimal control \( u_i \) for each subsystem \( i \) that is able to minimize the common performance index. To minimize (4.74) satisfying (4.75) we may introduced the Hamiltonian function (4.76) and a symmetric positive definite slack variable \( Z_i \).
Minimization by partially differentiating with respect to $Z_i$, $P_i$, and $F_{0i}$ we get

\[ 0 = \Sigma + A_c Z_i + Z_i A_c^T \]  \hfill (4.77)

\[ 0 = A_c^T P_i + P_i A_c + Q_c \]  \hfill (4.78)

\[ 0 = R_{cli} F_{0i} \hat{C}_i Z_i \hat{C}_i^T - \hat{B}_i^T P_i Z_i \hat{C}_i^T \]  \hfill (4.79)

Equation (4.79) can be rewritten as

\[ F_{0i} = R_{cli}^{-1} \hat{B}_i^T P_i Z_i \hat{C}_i^T (\hat{C}_i Z_i \hat{C}_i^T)^{-1} \]  \hfill (4.80)

Now, a numerical solution to this problem converges to optimal feedback gain by solving the Lyapunov equations (4.77) and (4.78), and then the feedback gain value in (4.80) can be updated.

4.5 Numerical Example

a) State Feedback Controller

Consider the following decentralized singularly perturbed system

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} z_1 + \begin{bmatrix} -3 & 2 \\ -4 & -1 \end{bmatrix} z_2 \\
\varepsilon \dot{z}_1 &= \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} x + \begin{bmatrix} -5 & 4 \\ -3 & -1 \end{bmatrix} z_1 + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u_1 + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w_1 \\
\varepsilon \dot{z}_2 &= \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} x + \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix} z_2 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_2 + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w_2 \\
y_1 &= [0 \ 1] z_1 + v_1 \\
y_2 &= [1 \ 0] z_2 + v_2
\end{align*}
\]

with, $\varepsilon = 0.1$

\[
W_1 = W_2 = 0.1 \\
R_{11} = R_{12} = 1 \\
Q_x = Q_{z_1} = Q_{z_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
the optimal full order model feedback gains found to be

\[
F_1^* = [-0.63 \quad -0.03 \quad 0.28 \quad -0.56 \quad -0.01 \quad 0.00],
\]

\[
F_2^* = [0.39 \quad -0.48 \quad 0.02 \quad 0.01 \quad -0.44 \quad 0.33],
\]

with the solution of the Riccati equation \( P \)

\[
P(0) =
\begin{bmatrix}
0.33 & -0.10 & 0.01 & -0.02 & -0.03 & 0.01 \\
-0.10 & 0.32 & 0.00 & 0.00 & 0.02 & -0.02 \\
0.01 & 0.00 & 0.01 & 0.00 & 0.00 & 0.00 \\
-0.02 & 0.00 & 0.00 & 0.03 & 0.00 & 0.00 \\
-0.03 & 0.02 & 0.00 & 0.00 & 0.03 & -0.01 \\
0.01 & -0.02 & 0.00 & 0.00 & -0.01 & 0.02
\end{bmatrix}
\]

and the close loop poles are

\[
p = \begin{bmatrix}
-37.18 + j30.75 \\
-37.18 - j30.75 \\
-34.91 \\
-9.88 + j6.46 \\
-9.88 - j6.46 \\
-10.43
\end{bmatrix}
\]

with \( W_1 = W_2 = 0.1 \), Full order system optimal cost is

\[
J_1^* = 0.236285
\]

The reduced order model

\[
\dot{x} = \begin{bmatrix}
-2.71 & -4.31 \\
3.56 & -10.76
\end{bmatrix} \ddot{x} + \begin{bmatrix}
0.59 \\
2.24
\end{bmatrix} \ddot{u}_1 + \begin{bmatrix}
0.02 \\
-0.05
\end{bmatrix} w_1
\]

\[
\ddot{y}_1 = \begin{bmatrix}
-1 & 0.53
\end{bmatrix} \ddot{x} - 0.94 \ddot{u}_1 + 0.01 w_1 + v_1
\]

and the optimal feedback gain for this reduced order model is

\[
F_1^* = [-0.65 \quad 0.03]
\]

\[
F_2^* = [0.34 \quad 1.34]
\]

To use these feedback gains to calculate the cost when applied to the full system, a following zeros to be added for each of them to match dimensions of the full order model as shown in (4.81)
The optimal feedback gains for the full order system is

\[
\begin{align*}
F_{1}^* &= [-0.65 \hspace{0.1cm} 0.03 \hspace{0.1cm} 0 \hspace{0.1cm} 0 \hspace{0.1cm} 0 \hspace{0.1cm} 0] \\
F_{2}^* &= [0.34 \hspace{0.1cm} 1.34 \hspace{0.1cm} 0 \hspace{0.1cm} 0 \hspace{0.1cm} 0 \hspace{0.1cm} 0]
\end{align*}
\]

By applying this feedback gain to the full order model, the closed loop system optimal cost is found to be

\[J_1^* = 0.274238\]

The final \(P_l\) calculated for the closed loop system is

\[
P(0) =
\begin{pmatrix}
0.41 & -0.11 & 0.02 & -0.02 & -0.07 & 0.04 \\
-0.11 & 0.38 & 0.00 & -0.01 & 0.02 & -0.02 \\
0.02 & 0.00 & 0.01 & -0.01 & 0.00 & 0.00 \\
-0.02 & -0.01 & -0.01 & 0.03 & 0.00 & 0.00 \\
-0.07 & 0.02 & 0.00 & 0.00 & 0.06 & -0.03 \\
0.04 & -0.02 & 0.00 & 0.00 & -0.03 & 0.03
\end{pmatrix}
\]

and the close loop poles are

\[
p = \begin{pmatrix}
-28.67 + j28.56 \\
-28.67 - j28.56 \\
-35.12 \\
-6.23 + j4.88 \\
-6.23 - j4.88 \\
-10.08
\end{pmatrix}
\]

The response of the output and states of the system are shown in Figure 4.2 and Figure 4.1 respectively. The optimal cost of the reduced order model is very close to that of the full order system within the order of \(\epsilon^2\).
Figure 4.1 Output response for state feedback controller.

Figure 4.2 States responses for state feedback controller.
b) Static Output Feedback Controller

The optimal static output feedback gain for the reduced order model is

\[
F_1^* = -0.2626
\]
\[
F_2^* = 0.2755
\]

To use this feedback gain to the full order model, following zeros will be added for dimension matching to be

\[
F_{full,1}^* = [0.26 -0.14 0 0 0 0]
\]
\[
F_{full,2}^* = [0.00 0.90 0 0 0 0]
\]

By applying this feedback gain to the full order model, the closed loop system optimal cost is found to be

\[
J_1^* = 0.411447
\]

and the final P calculated for the closed loop system

\[
P(0) = \begin{bmatrix}
0.43 & -0.10 & 0.03 & -0.03 & -0.05 & 0.02 \\
-0.10 & 0.69 & 0.00 & -0.03 & 0.04 & -0.06 \\
0.03 & 0.00 & 0.01 & -0.01 & 0.00 & 0.00 \\
-0.03 & -0.03 & -0.01 & 0.03 & 0.00 & 0.00 \\
-0.05 & 0.04 & 0.00 & 0.00 & 0.04 & -0.02 \\
0.02 & -0.06 & 0.00 & 0.00 & -0.02 & 0.03 \\
\end{bmatrix}
\]

and the closed loop system poles are

\[
p = \begin{bmatrix}
-28.90 + j28.59 \\
-28.90 - j28.59 \\
-35.11 \\
-5.15 + j10.40 \\
-5.15 - j10.40 \\
-11.78 \\
\end{bmatrix}
\]

The output and states response of the system are shown in Figure 4.4 and Figure 4.3 respectively.
Figure 4.3 Output response for static output feedback controller.

Figure 4.4 States responses for static output feedback controller.
CHAPTER 5

OPTIMAL DESIGN USING GAME THEORETIC APPROACH

In section 2.4, the quasi-steady state model of the singularly perturbed decentralized large scale system (2.1) was shown as

\[
\begin{align*}
\dot{x} &= \hat{A}\tilde{x} + \hat{B}_i\tilde{u}_i + \hat{B}_j\tilde{u}_j \\
\tilde{y}_i &= \hat{C}_i\tilde{x} + \hat{D}_i\tilde{u}_i
\end{align*}
\] (5.1)

Input \(\tilde{u}\) for each subsystem can be written in terms of \(x\) as

\[
\tilde{u}_i = -\hat{F}_i\tilde{x}_i = -\hat{F}_i\tilde{x}
\] (5.2)

where \(\hat{F}_i = (I_{r_i} - \hat{F}_iA_{ii}^{-1}B_i)^{-1}\hat{F}_iA_{ii}^{-1}A_{i0}\), \((I_{r_i} - \hat{F}_iA_{ii}^{-1}B_i)\) is invertible, \(I_{r_i}\) is the identity matrix of dimension \(r_i\), and \(r_i\) is the number of columns in \(B_i\).

The close loop representation of the system is shown in (5.3) and the objective is to minimize performance index in (5.4)

\[
\dot{x}^s = A_c x
\] (5.3)

\[
J_i = \frac{1}{2} \int_0^\infty x^T Q_c x \, dt
\] (5.4)

where

\[
A_c = A_r - \hat{B}_i\hat{F}_i - \hat{B}_j\hat{F}_j
\] (5.5)

\[
Q_c = Q_r + \hat{F}_i^T R_{ii} \hat{F}_i + \hat{F}_j^T R_{ij} \hat{F}_j
\] (5.6)

The solution of the performance index for this system is found to be

\[
J_i = \frac{1}{2} tr(P_i \Sigma)
\] (5.7)

where \(\Sigma = x(0)x^T(0)\) and the symmetric matrix \(P_i \geq 0\) is the solution of

\[
0 = A_c^T P_i + P_i A_c + Q_c
\] (5.8)
5.1 State Feedback

Each subsystem tends to minimize its own performance index [35]. Because of the decentralized setup of the system, the two subsystems’ dynamics are coupled. The objective is to find an optimal control \( u_i \) for subsystem \( i \) that is able to minimize the common performance index. This is accomplished by taking into consideration the effect of the other subsystem control in the performance index shown in (5.7).

5.1.1 Nash Game

In game theory, the controllers need know each other in order to perform minimization considering each other. First, controller \( i \) starts with no knowledge of the feedback gain of controller \( j \). With initial guess of the feedback gain \( F_j \), controller \( i \) will minimize the optimal cost function of the singularly perturbed system given by (5.9) and calculate the feedback gain \( F_i \) using equations (5.11) and (5.12).

\[
J_i = \frac{1}{2} tr\{P\Sigma\} \quad \text{(5.9)}
\]

where \( P \) is the solution of

\[
-\dot{\hat{P}} = PA_c + A_c^T P + Q_r + F_i^T R_{c\ell} F_i + F_j^T R_{c\ell} F_j \quad \text{(5.10)}
\]

and the optimal feedback gain

\[
F_i^* = -R_{c\ell}^{-1} \hat{\bar{B}}_i^T K \quad \text{(5.11)}
\]

where \( K \) is the solution of

\[
0 = A^T K + KA + Q_r - K\hat{\bar{B}}_i R_{c\ell}^{-1} \hat{\bar{B}}_i^T K \quad \text{(5.12)}
\]

The feedback gain \( F_j \) for controller \( j \) will be calculated using the result of the previous step and performing the same minimization procedure. Now, two calculated feedback gains \( F_i \) and \( F_j \) are available can be used to recalculate \( F_i \) and \( F_j \) again with better estimates than the initial guess.
of $F_j$. This procedure can be repeated until an equilibrium occurs which represents the Nash solution of this minimization problem.

### 5.1.2 Stackelberg Game

In Stackelberg game, the follower needs to get some information about the follower other than the feedback gain only. The follower will make decision based on the leader decision that will affect its minimization. So, the follower minimization algorithm will have an additional condition from the leader’s solution. The Lyapunov equation (5.10) of the leader’s solution reflects the effect of the leader’s decision on the follower minimization. This effect will be added as an additional constraint $\kappa_i$

$$\kappa_i = A_c^T P_i + P_i A_c + Q_c$$  \hspace{1cm} (5.13)

where

$$Q_c = Q_r + F_i^T R_{iit} F_i + F_j^T R_{ij} F_j$$  \hspace{1cm} (5.14)

and

$$A_c = A_r - \hat{B}_i F_i - \hat{B}_j F_j$$  \hspace{1cm} (5.15)

To avoid cancellation in calculating the feedback gain $F_i$, the control input will be in the form

$$\bar{u}_i = -F_i \bar{x}_i = -\hat{F}_i S_i \bar{x}_i$$  \hspace{1cm} (5.16)

where $S_i$ is a non-singular non-identity weighting matrix. Equations (5.14) and (5.15) will be updated to

$$Q_c = Q_r + S_i^T \hat{F}_i^T R_{iit} \hat{F}_i S_i + S_j^T \hat{F}_j^T R_{ij} \hat{F}_j S_j$$  \hspace{1cm} (5.17)

and

$$A_c = A_r - \hat{B}_i S_i \hat{F}_i - \hat{B}_j S_j \hat{F}_j$$  \hspace{1cm} (5.18)
Using the same optimization procedure used in section 4.4, introduce a Hamiltonian function (5.19) having the original cost function added to the additional constraint $\kappa_i$ and an introduced slack variable $Z_i$

$$\mathcal{H}_i = tr(P_i \Sigma) + tr(\kappa_i Z_i) \tag{5.19}$$

Minimization obtained by setting partial derivatives of $\mathcal{H}_i$ with respect to $P_i, Z_i$ and $F_i$ found in section 4.4 to zero

$$0 = \Sigma + A_c Z_i + Z_i A_c^T \tag{5.20}$$

$$0 = Q_c + A_c^T P_i + P_i A_c \tag{5.21}$$

$$0 = -B_i^T P_i Z_i S_i^T + R_{cii} \hat{F}_i S_i Z_i S_i^T \tag{5.22}$$

To solve these equations, initial value for $\hat{F}_i$ and $\hat{F}_j$ will be set, then equations (5.20) and (5.21) can be solved using these initial values. After that, the feedback gain $\hat{F}_i$ is updated from equation (5.22) as shown in (5.23)

$$\hat{F}_i = R_{cii}^{-1} B_i^T P_i Z_i S_i^T (S_i Z_i S_i^T)^{-1} \tag{5.23}$$

Values of $A_c$ and $Q_c$ will be updated accordingly and the same procedure is done for the other controller, but using the obtained value for $\hat{F}_i$ to get an sub-optimized value for $\hat{F}_j$ to be used again along with $\hat{F}_i$ to update the values of $Z_i$ and $P_i$ for the next iteration. These steps are repeated iteratively until an equilibrium occurs approaching an optimal value for both $\hat{F}_i$ and $\hat{F}_j$.

5.1.3 Numerical Example

a) Nash Solution

Consider the singularly perturbed system discussed in section 4.5

$$\dot{x} = \begin{bmatrix} -4 & -12.46 \\ 4 & -16.38 \end{bmatrix} \bar{x} + \begin{bmatrix} 0.12 \\ -1.35 \end{bmatrix} \bar{u}_1 + \begin{bmatrix} 0.02 \\ -0.05 \end{bmatrix} \bar{v}_1$$

$$\bar{y} = \begin{bmatrix} -1 & 0.53 \end{bmatrix} \bar{x} + 0.41 \bar{u}_1 + 0.01 \bar{w}_1 + v_1$$
With, $\varepsilon = 0.1$

$W_1 = W_1 = 0.1$

$R_{11} = R_{12} = R_{21} = R_{22} = 1$

$Q_x = Q_{z_1} = Q_{z_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and the optimization algorithm in (5.20)-(5.22), the values of $F_1$ and $F_2$ were calculated.

After equilibrium occurs, the final value for $P_1$

$$P_1 = \begin{bmatrix} 0.40 & -0.10 \\ -0.10 & 0.47 \end{bmatrix}$$

and the optimal feedback gains $F_1$ and $F_2$ are

$$F_1 = [-0.65 \quad -0.10]$$

$$F_2 = [\quad 0.33 \quad 1.26]$$

To use these feedback gains to calculate the cost when applied to the full system, following zeros must be added for each of them to match dimensions of the full order model. So, the feedback gain $F_{full,1}$ and $F_{full,2}$ will be

$$F_{full,1}^* = [-0.65 \quad -0.10 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$F_{full,2}^* = [\quad 0.33 \quad 1.26 \quad 0 \quad 0 \quad 0 \quad 0]$$

Table 5.1 shows the convergence of feedback gains $F_1$ and $F_2$ and the cost of the system using a Nash strategy optimization method. The optimal cost for the system analyzed was found to be

$$J_1^* = 0.268711$$

Output and state responses of the system are shown in Figure 5.1 and Figure 5.2 respectively. The final $P$ calculated for the closed loop system
\[ P(0) = \begin{bmatrix}
0.53 & -0.15 & 0.03 & -0.02 & -0.09 & 0.05 \\
-0.15 & 0.40 & -0.01 & -0.01 & 0.03 & -0.03 \\
0.03 & -0.01 & 0.01 & -0.01 & 0.00 & 0.00 \\
-0.02 & -0.01 & -0.01 & 0.03 & 0.00 & 0.00 \\
-0.09 & 0.03 & 0.00 & 0.00 & 0.06 & -0.03 \\
0.05 & -0.03 & 0.00 & 0.00 & -0.03 & 0.03 \\
\end{bmatrix} \]

and the closed loop system poles are

\[ p = \begin{bmatrix}
-28.49 + j28.43 \\
-28.49 - j28.43 \\
-35.03 \\
-6.45 + j5.10 \\
-6.45 - j5.10 \\
-10.08 \\
\end{bmatrix} \]

### TABLE 5.1

CONVERGENCE IN STATE FEEDBACK CONTROLLER WITH NASH STRATEGY

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( F_1 )</th>
<th>( F_2 )</th>
<th>Subsystem cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[ 0.00  0.00]</td>
<td>[0.00  0.00]</td>
<td>0.33119506</td>
</tr>
<tr>
<td>1</td>
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<td>[0.33  1.34]</td>
<td>0.27294584</td>
</tr>
<tr>
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<td>0.26809288</td>
</tr>
<tr>
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<td>[0.33  1.26]</td>
<td>0.26895017</td>
</tr>
<tr>
<td>4</td>
<td>[−0.65 −0.10]</td>
<td>[0.33  1.26]</td>
<td>0.26885034</td>
</tr>
<tr>
<td>5</td>
<td>[−0.65 −0.10]</td>
<td>[0.33  1.26]</td>
<td>0.26884798</td>
</tr>
</tbody>
</table>
Figure 5.1 Output response for state feedback controller with Nash strategy.

Figure 5.2 States responses for state feedback controller with Nash strategy.
a) **Stackelberg Solution**

For the same system and the optimization algorithm in (5.20)-(5.22), the values of $F_1$ and $F_2$ were calculated. After equilibrium occurs, the final value for $P_1$

$$P_1 = \begin{bmatrix} 0.45 & -0.21 \\ -0.21 & 0.60 \end{bmatrix}$$

and the optimal feedback gains $F_1^*$ and $F_2^*$ are

$$F_1^* = \begin{bmatrix} -0.54 \\ -0.21 \end{bmatrix}$$

$$F_2^* = \begin{bmatrix} -0.21 \\ 1.45 \end{bmatrix}$$

To use these feedback gains to calculate the cost when applied to the full system, following zeros must be added for each of them to match dimensions of the full order model. So, the feedback gain $F_{full,1}^*$ and $F_{full,2}^*$ will be

$$F_{full,1}^* = \begin{bmatrix} -0.54 \\ -0.21 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_{full,2}^* = \begin{bmatrix} -0.21 \\ 1.45 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Table 5.2 shows the convergence of feedback gains $F_1$ and $F_2$ and the cost of the system using a Nash strategy optimization method with

$$S_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The final $P$ calculated for the closed loop system

$$P(0) = \begin{bmatrix} 0.35 & -0.09 & 0.02 & -0.02 & -0.03 & 0.01 \\ -0.09 & 0.36 & -0.01 & 0.00 & 0.03 & -0.03 \\ 0.02 & -0.01 & 0.01 & -0.01 & 0.00 & 0.00 \\ -0.02 & 0.00 & -0.01 & 0.03 & 0.00 & 0.00 \\ -0.03 & 0.03 & 0.00 & 0.00 & 0.04 & -0.02 \\ 0.01 & -0.03 & 0.00 & 0.00 & -0.02 & 0.02 \end{bmatrix}$$

and the closed loop system poles are
\[
p = \begin{bmatrix}
-28.35 + j28.28 \\
-28.35 - j28.28 \\
-34.94 \\
-6.23 + j8.39 \\
-6.23 - j8.39 \\
-10.89
\end{bmatrix}
\]

The optimal cost for the system analyzed was found to be \( J_1^* = 0.266586 \)

Output and state responses of the system are shown in Figure 5.3 and Figure 5.4 respectively.

**TABLE 5.2**

CONVERGENCE IN STATE FEEDBACK CONTROLLER WITH STACKELBERG STRATEGY

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( F_1 )</th>
<th>( F_2 )</th>
<th>System cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[0.00 0.00]</td>
<td>[0.00 0.00]</td>
<td>0.33119506</td>
</tr>
<tr>
<td>1</td>
<td>[-0.65 0.03]</td>
<td>[-0.21 1.47]</td>
<td>0.26585957</td>
</tr>
<tr>
<td>2</td>
<td>[-0.52 -0.20]</td>
<td>[-0.20 1.45]</td>
<td>0.26598982</td>
</tr>
<tr>
<td>3</td>
<td>[-0.54 -0.21]</td>
<td>[-0.21 1.45]</td>
<td>0.26658574</td>
</tr>
<tr>
<td>4</td>
<td>[-0.54 -0.21]</td>
<td>[-0.21 1.45]</td>
<td>0.26657484</td>
</tr>
</tbody>
</table>
Figure 5.3 Output response for state feedback controller with Stackelberg strategy.

Figure 5.4 States responses for state feedback controller with Stackelberg strategy.
5.2 Static Output Feedback

5.2.1 Nash Game

For static output feedback controller, the closed loop system shown in (4.65) is used with the performance index (4.66) controlled by input (4.62). The total cost of the system is

\[
J_i = \frac{1}{2} \text{tr}(P_i \Sigma)
\]

where \( \Sigma = E\{x(0)x^T(0)\} \), \( P_i \geq 0 \) is the solution of

\[
0 = A_c^TP_i + P_iA_c + Q_c
\]

Now introduce the Hamiltonian function (5.26) and a symmetric positive definite slack variable \( Z_i \) to the original cost function

\[
H_i = \text{tr}(P_i \Sigma) + \text{tr}(\kappa_i Z_i)
\]

Minimization by partially differentiating with respect to \( Z_i, P_i, \) and \( F_i \) we get

\[
0 = \Sigma + A_c Z_i + Z_i A_c^T
\]

\[
0 = A_c^TP_i + P_iA_c + Q_c
\]

\[
0 = R_{cil}F_{0i} \hat{C}_i Z_i \hat{C}_i^T - B_i^TP_i Z_i \hat{C}_i^T
\]

where

\[
A_c = A_r - \hat{B}_iF_{0i} \hat{C}_i - \hat{B}_jF_{0j} \hat{C}_j
\]

\[
Q_c = Q_r + \hat{C}_i^T F_{0i} R_{cil} F_{0i} \hat{C}_i + \hat{C}_j^T F_{0j} R_{cij} F_{0j} \hat{C}_j
\]

\[
Q_r = \hat{Q} - M_iR_{cil}^{-1}M_i^T - M_jR_{cij}^{-1}M_j^T
\]

Equation (5.29) can be rewritten as

\[
F_{0i} = R_{cil}^{-1} \hat{B}_i^TP_i Z_i \hat{C}_i^T \left( \hat{C}_i Z_i \hat{C}_i^T \right)^{-1}
\]

To solve these equations, set initial value for \( F_{0i} \) and \( F_{0j} \) then a numerical solution to this problem converges to optimal feedback gain for subsystem \( i \) by solving the Lyapunov equations.
(5.27) and (5.28), and then the feedback gain value in (5.33) can be updated. After that the values of $A_c$ and $Q_c$ should be updated and the new $P_l$ and $Z_l$ recalculated. Repeating these steps will converge to an optimum for the first iteration regarding subsystem $i$. The same procedure is done for the other controller but using the obtained value for $F_{0i}$ to get an optimized value for $F_{0j}$ to be used again along with $F_{0i}$ to update the values of $Z_l$ and $P_l$ for the next iteration approaching an optimal value for both $F_{0i}$ and $F_{0j}$.

### 5.2.2 Stackelberg Game

The Stackelberg strategy in game theory requires the leader to go first ($i$), just like in Nash equilibrium strategy, and a Hamiltonian function to be minimized is defined as

$$
\mathcal{H}_i = tr(P_i \Sigma) + tr(\kappa_i Z_i)
$$

(5.34)

With the objective to minimize $\mathcal{H}_i$, partial derivatives with respect to $P_i$ and $Z_i$ will be taken. As discussed in section 3.4 the leader chooses and $F_i$ to minimize $J_i$ by considering $F_j$ constant and differentiating $\mathcal{H}_i$ with respect to $P_i$, $Z_i$ and $F_i$ respectively to get the results in equations (5.27)-(5.29).

Optimizing the follower cost in static output feedback controller requires minimizing not only for the follower variables, but the leader conditions too. The effect of the leader (subsystem $i$) is taken from the solution of the leader optimization functions (5.27)-(5.29). So, in Stackelberg game, the follower will consider these minimization results as constraints for minimizing the defined Hamiltonian $\mathcal{H}_j$ in (3.138). These equations will be used as constraints knowing that (5.27) is the equation of $\kappa_i$. Equations (5.28) and (5.29) will be referenced as in (5.35)

$$
\psi = A_c Z_i + Z_i A_c^T + \Sigma
$$

$$
\phi = R_{cii} F_{0i} \hat{C}_i Z_i \hat{C}_i^T - \hat{B}_i^T P_i Z_i \hat{C}_i^T
$$

(5.35)
The follower now is to minimize $J_j$ while taking into consideration the leader’s solution as constraints with the Lyapunov equation for the follower controller (5.36)

$$\kappa_j = A_c^T P_j + P_j A_c + Q_c \quad (5.36)$$

The Hamiltonian function will include the follower variables, the leader conditions, and the introduced slack variables in the following extended form

$$\mathcal{H}_j = tr(P_j \Sigma) + tr(\kappa_j Z_j) + tr(\kappa_i \lambda) + tr(\psi \gamma) + tr(\phi \beta^T) + tr(\beta \phi^T) \quad (5.37)$$

A minimization can be found by partially differentiating $\mathcal{H}_j$ with respect to $P_j, Z_j$ and $F_j$ as the follower variables, $P_i, Z_i$ and $F_i$ as the leader variables, and the introduced slack variables $\lambda, \gamma$ and $\beta$.

The partial derivatives of the Hamiltonian function (5.37) are equated to zero as shown in (5.38)-(5.46)

$$0 = \Sigma + Z_j A_c^T + A_c Z_j \quad (5.38)$$

$$0 = Q_c + A_c^T P_j + P_j A_c \quad (5.39)$$

$$0 = -\bar{B}_i^T P_j Z_j \hat{C}_j^T - \bar{B}_i^T P_i \lambda \hat{C}_i^T - \bar{B}_i^T \gamma Z_i \hat{C}_i^T + R_{c,jj} F_{0j} \hat{C}_j Z_j \hat{C}_j^T + R_{c,ij} F_{0j} \hat{C}_i \lambda \hat{C}_i^T \quad (5.40)$$

$$0 = A_c \lambda + \lambda A_c^T + \Lambda \quad (5.41)$$

$$0 = A_c^T \gamma + \gamma A_c + \Gamma \quad (5.42)$$

$$0 = -\bar{B}_i^T P_j Z_j \hat{C}_i^T + R_{c,ij} F_{0i} \hat{C}_i Z_j \hat{C}_i^T - \bar{B}_i^T P_i \lambda \hat{C}_i^T + R_{c,ii} F_{0i} \hat{C}_i \lambda \hat{C}_i^T - \bar{B}_i^T \gamma Z_i \hat{C}_i^T + R_{c,ii} \beta \hat{C}_i Z_i \hat{C}_i^T \quad (5.43)$$

$$0 = Q_c + A_c^T P_i + P_i A_c \quad (5.44)$$

$$0 = \Sigma + A_c Z_i + Z_i A_c^T \quad (5.45)$$

$$0 = -\bar{B}_i^T P_i Z_i \hat{C}_i^T + R_{c,ii} F_{0i} \hat{C}_i Z_i \hat{C}_i^T \quad (5.46)$$

where
\[ \Lambda = -\hat{B}_i \beta \hat{C}_i Z_i - Z_i \hat{C}_i^T \beta^T \hat{B}_i^T \]  
\[
\Gamma = -P_i \hat{B}_i \beta \hat{C}_i + \hat{C}_i^T F_{0i} R_{cii} \beta \hat{C}_i - \hat{C}_i^T \beta^T \hat{B}_i^T P_i + \hat{C}_i^T \beta^T R_{cii} F_{0i} \hat{C}_i \]  
(5.47)  
(5.48)

The slack variable \( \beta \) acts like a fictitious feedback gains. To solve these nine equations, an initial value for feedback gains \( F_{0i} \), \( F_{0j} \) and \( \beta \) then based in these values solve the Lyapunov equations (5.38)-(5.39), (5.41)-(5.42) and (5.44)-(5.45) for the values of \( Z_j \), \( P_j \), \( \lambda \), \( \gamma \), \( P_i \) and \( Z_i \) respectively. Using these values, (5.46) can be solved for \( F_{0i} \) as in (5.49)

\[ F_{0i} = R_{cii}^{-1} \hat{B}_i^T P_i Z_i \hat{C}_i^T (\hat{C}_i Z_i \hat{C}_i^T)^{-1} \]  
(5.49)

Next, the value for \( F_{0i} \) can be used to calculate \( \beta \) using equation (5.43) which can be rewritten as shown in (5.50)

\[ \beta = R_{cii}^{-1} (\hat{B}_i^T P_i Z_j \hat{C}_j^T + \hat{B}_i^T P_i \lambda \hat{C}_i^T + \hat{B}_i^T \gamma Z_i \hat{C}_i^T - R_{cij} F_{0i} \hat{C}_i Z_j \hat{C}_i^T) 
- R_{cii} F_{0i} \hat{C}_i \lambda \hat{C}_i^T (\hat{C}_i Z_i \hat{C}_i^T)^{-1} \]  
(5.50)

Finally, \( F_{0j} \), to be calculated using equation (5.40), cannot be solved directly by rearranging matrices and inverses. One way to solve this equation is by rewriting it in the Sylvester equation format shown in (5.51)

\[ \mathcal{A} F_{0j} + F_{0j} \mathcal{B} = \mathcal{C} \]  
(5.51)

where

\[ \mathcal{A} = (R_{cij})^{-1} R_{cjj} \]  
(5.52)

\[ \mathcal{B} = S_j \lambda S_j^T (S_j Z_j S_j^T)^{-1} \]  
(5.53)

\[ \mathcal{C} = (R_{cij})^{-1} (\hat{B}_j^T P_j Z_j \hat{C}_j^T + \hat{B}_j^T P_j \lambda \hat{C}_j^T + \hat{B}_j^T \gamma Z_i \hat{C}_i^T) (\hat{C}_i Z_i \hat{C}_i^T)^{-1} \]  
(5.54)

This equation can be solved by numerically solving (5.55)

\[ F_{0j} = M F_{0j} N + \mathcal{L} \]  
(5.55)

where
\[ M = (I - A)^{-1}(I + A) \]  
\[ N = (I + B)(I - B)^{-1} \]  
\[ L = -2(I - A)^{-1}C(I - B)^{-1} \]  

(5.56)  
(5.57)  
(5.58)  

This solution is restricted by the assumption that both the symmetric matrices \( A \) and \( I \) in (5.41) and (5.42) are positive definite. For a less conservative solution, the calculation of \( \lambda \) and \( \gamma \) should be done with additional conditioning. One way to generalize the solution of the Lyapunov equation (5.42) is by introducing a symmetric positive definite adjusting matrix \( \Sigma \) along with the constraint function (5.59) and the variables \( \gamma \) and \( \hat{\gamma} \)

\[ 0 = A^T\gamma + \gamma A_c + \Sigma \gamma + 2\hat{C}_i^T\beta^T R_{ci} \beta \hat{C}_i \]  
(5.59)  

If we say,

\[ \tilde{\gamma} = \gamma + P_i + \hat{\gamma} \]  
(5.60)  

then

\[ 0 = A^T\tilde{\gamma} + \tilde{\gamma} A_c + \tilde{\Sigma} \]  
(5.61)  

where

\[ \tilde{\Sigma} = \Sigma + \alpha^T \delta \alpha \]  
(5.62)  

\[ \tilde{\Sigma} = Q_r + \hat{C}_i^T \beta^T R_{ci} \beta \hat{C}_i + \hat{C}_j^T F_{0j} R_{clj} F_{0j} \hat{C}_j \]  
(5.63)  

\[ \hat{\beta} = F_{0i} + \beta \]  
(5.64)  

\[ \alpha = \begin{bmatrix} I \\ \beta \hat{C}_i \end{bmatrix} \]  
(5.65)  

\[ \delta = \begin{bmatrix} \Sigma \gamma & -P_i \tilde{B}_i \\ -\tilde{B}_i^T P_i & R_{ci} \end{bmatrix} \]  
(5.66)  

The selection of \( \Sigma \gamma \) is based on making \( \tilde{\Sigma} \) a symmetric positive definite matrix.
A similar approach is used to generalize the solution of the Lyapunov equation (5.41). Let’s introduce symmetric positive definite adjusting matrices $\Sigma_\lambda$ and $R_\lambda$ along with the constraint function (5.67) and the variables $\bar{\lambda}$ and $\bar{\lambda}$

$$0 = \bar{\lambda}A_c^T + A_c\bar{\lambda} + \Sigma_\lambda + \bar{B}_i\beta R_\lambda \beta^T \bar{B}_i^T$$  \hfill (5.67)

If we say,

$$\bar{\lambda} = \lambda + Z_i + \hat{\lambda}$$  \hfill (5.68)

then

$$0 = A_c\bar{\lambda} + \bar{\lambda}A_c^T + \bar{\Sigma}$$  \hfill (5.69)

where

$$\bar{\Sigma} = \Sigma_\lambda + \hat{\alpha}^T \bar{\delta} \hat{\alpha}$$  \hfill (5.70)

$$\hat{\alpha} = \begin{bmatrix} 1 \\ \beta^T \bar{B}_i^T \end{bmatrix}$$  \hfill (5.71)

$$\bar{\delta} = \begin{bmatrix} \Sigma & -Z_i \hat{c}_i^T \\ -\hat{c}_i Z_i & \bar{R}_\lambda \end{bmatrix}$$  \hfill (5.72)

$\Sigma_\lambda$ and $R_\lambda$ are selected to make $\bar{\Sigma}$ a symmetric positive definite matrix.

### 5.2.3 Numerical Example

#### b) Nash Solution

Consider the singularly perturbed system discussed in section 4.5

$$\dot{x} = \begin{bmatrix} -4 & -12.46 \\ 4 & -16.38 \end{bmatrix} x + \begin{bmatrix} 0.12 \\ -1.35 \end{bmatrix} u_1 + \begin{bmatrix} 0.02 \\ -0.05 \end{bmatrix} w_1$$

$$\ddot{y} = \begin{bmatrix} -1 & 0.53 \end{bmatrix} \ddot{x}_1 + 0.41 \ddot{u}_1 + 0.01 w_1 + v_1$$

With, $\varepsilon = 0.1$

$$W_1 = W_1 = 0.1$$

$$R_{11} = R_{12} = R_{21} = R_{22} = 1$$
\( Q_x = Q_{z1} = Q_{z2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \),

and the optimization algorithm in (5.27)-(5.29), the values of \( F_1 \) and \( F_2 \) were calculated.

After equilibrium occurs, the final value for \( P_1 \)

\[
P_1 = \begin{bmatrix} 0.03 & 0.03 \\ 0.03 & 0.04 \end{bmatrix}
\]

and for the calculated optimal feedback gains \( F_1 \) and \( F_2 \) to be used with the full order system, following zeros must be added for each of them to match dimensions of the full order model. So, the optimal value for feedback gains \( F_{full,1}^* \) and \( F_{full,2}^* \) are

\[
F_{full,1}^* = \begin{bmatrix} 0.22 & -0.12 & 0 & 0 & 0 & 0 \\ 0.00 & 0.29 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

When applying these feedback gains to the full order system the closed loop matrix \( P_1 \) is

\[
P_1 = \begin{bmatrix}
0.42 & -0.11 & 0.02 & -0.02 & -0.05 & 0.02 \\
-0.11 & 0.53 & 0.00 & -0.02 & 0.05 & -0.05 \\
0.02 & 0.00 & 0.01 & -0.01 & 0.00 & 0.00 \\
-0.02 & -0.02 & -0.01 & 0.03 & 0.00 & 0.00 \\
0.05 & 0.05 & 0.00 & 0.00 & 0.04 & -0.02 \\
0.02 & -0.05 & 0.00 & 0.00 & -0.02 & 0.03 \\
\end{bmatrix}
\]

and the closed loop system poles are

\[
p = \begin{bmatrix}
-28.88 + j28.54 \\
-28.88 - j28.54 \\
-35.09 \\
-5.13 + j9.54 \\
-5.13 - j9.54 \\
-11.89 \\
\end{bmatrix}
\]

which results in the optimal cost

\[
J_1^* = 0.345174
\]
Table 5.3 shows the convergence of the output feedback gains $F_1$ and $F_2$ and the cost of the system using a Nash strategy optimization method. Output and state responses of the system are shown in Figure 5.5 and Figure 5.6 respectively.

TABLE 5.3

CONVERGENCE IN OUTPUT FEEDBACK CONTROLLER WITH NASH STRATEGY

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>Closed loop system cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$[0.00 \ 0.00]$</td>
<td>$[0.00 \ 0.00]$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$[-0.25 \ 0.13]$</td>
<td>$[0.00 \ -0.31]$</td>
<td>0.34581636</td>
</tr>
<tr>
<td>1</td>
<td>$[-0.30 \ 0.16]$</td>
<td>$[0.00 \ -0.31]$</td>
<td>0.34522902</td>
</tr>
<tr>
<td>2</td>
<td>$[-0.32 \ 0.17]$</td>
<td>$[0.00 \ -0.31]$</td>
<td>0.34517900</td>
</tr>
<tr>
<td>3</td>
<td>$[-0.33 \ 0.18]$</td>
<td>$[0.00 \ -0.31]$</td>
<td>0.34517473</td>
</tr>
</tbody>
</table>
Figure 5.5 Output response for output feedback controller with Nash strategy.

Figure 5.6 States responses for output feedback controller with Nash strategy.
a) Stackelberg Solution

For the same system and using the optimization algorithm in (5.38)-(5.46), the values of $F_1$ and $F_2$ were calculated. After equilibrium occurs, the final value for $P_1$

$$P_1 = \begin{bmatrix} 0.24 & -0.11 \\ -0.11 & 0.35 \end{bmatrix}$$

and the optimal feedback gains $F_1^*$ and $F_2^*$ are

$$F_1^* = \begin{bmatrix} 0.33 & -0.17 \end{bmatrix}$$

$$F_2^* = \begin{bmatrix} 0.00 & -0.34 \end{bmatrix}$$

To use these feedback gains to calculate the cost when applied to the full system, following zeros must be added for each of them to match dimensions of the full order model. So, the feedback gain $F_{full,1}$ and $F_{full,2}$ will be

$$F_{full,1}^* = \begin{bmatrix} 0.33 & -0.17 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{full,2}^* = \begin{bmatrix} 0.00 & -0.34 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The final $P$ calculated for the closed loop system

$$P(0) = \begin{bmatrix} 0.38 & -0.10 & 0.02 & -0.02 & -0.04 & 0.02 \\ -0.10 & 0.41 & 0.00 & -0.01 & 0.04 & -0.04 \\ 0.02 & 0.00 & 0.01 & -0.01 & 0.00 & 0.00 \\ -0.02 & -0.01 & -0.01 & 0.03 & 0.00 & 0.00 \\ -0.04 & 0.04 & 0.00 & 0.00 & 0.04 & -0.02 \\ 0.02 & -0.04 & 0.00 & 0.00 & -0.02 & 0.03 \end{bmatrix}$$

and the closed loop system poles are

$$p = \begin{bmatrix} -28.93 + j28.61 \\ -28.93 - j28.61 \\ -35.15 \\ -5.32 + j8.50 \\ -5.32 - j8.50 \\ -11.34 \end{bmatrix}$$
Table 5.2 shows the convergence of feedback gains $F_1$ and $F_2$ and the cost of the system using a Stackelberg strategy optimization method. The optimal cost for the system analyzed was found

$$J_1^* = 0.289793$$

Output and state responses of the system are shown in Figure 5.7 and Figure 5.8 respectively.

**TABLE 5.4**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>Close loop system cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[0.00 0.00]</td>
<td>[0.00 0.00]</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>[0.31 -0.17]</td>
<td>[0.00 -0.33]</td>
<td>0.29081194</td>
</tr>
<tr>
<td>2</td>
<td>[0.33 -0.17]</td>
<td>[0.00 -0.34]</td>
<td>0.28996933</td>
</tr>
<tr>
<td>3</td>
<td>[0.33 -0.17]</td>
<td>[0.00 -0.34]</td>
<td>0.28982460</td>
</tr>
<tr>
<td>4</td>
<td>[0.33 -0.17]</td>
<td>[0.00 -0.34]</td>
<td>0.28979296</td>
</tr>
</tbody>
</table>
Figure 5.7 Output response for output feedback controller with Stackelberg strategy.

Figure 5.8 States responses for output feedback controller with Stackelberg game
5.3 Example of Optimal Control for Induction Motor Model

Field oriented control (FOC), also known as vector control, is a well-known and the most popular control method for induction motors [37]. In this example, the fast variables are the stator’s currents $z_1$ and $z_2$ and the slow variables are the rotor fluxes [38]. The controller representation shown in Figure 5.9 [39] is assumed to be able to measure the stator’s current $i_{\alpha s}$ and $i_{\beta s}$ to use it for the control process to control the slow variable (the corresponding flux). The resulting feedback gain should be applied to generate the control input $v_{\alpha s}$. For the induction motor model shown in (5.73), the optimal control design is to optimize the performance index (5.74). The parameters in the induction motor model used in this example were presented in [39] with constant nominal speed to demonstrate the application of decentralized control over singularly perturbed systems.

![Figure 5.9 Induction motor representation of a 2x2 control system.](image)

$$
\begin{align*}
\dot{z}_1 &= A_{10}\ddot{x} + A_{11}z_1 + B_1 u_1 \\
\dot{z}_2 &= A_{20}\ddot{x} + A_{22}z_2 + B_2 u_2 \\
\dot{x} &= A_{00}x + A_{01}z_1 + A_{02}z_2 
\end{align*}
$$

(5.73)
\[
J(t) = \frac{1}{2} x^T(t_r) S x(t_r) + \frac{1}{2} \int_t^{t_f} \{ x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) \} d\tau \quad (5.74)
\]

where

\[
A_{10} = \begin{bmatrix} 141.8 & 3323.96 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} -359.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 9.7 \end{bmatrix}
\]

\[
A_{20} = \begin{bmatrix} -3323.96 & 141.8 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -359.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 9.7 \end{bmatrix}
\]

\[
A_{00} = \begin{bmatrix} -15.4 & -361.3 \\ 361.3 & -15.4 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 21.9 \\ 0 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} 0 \\ 21.9 \end{bmatrix}
\]

The reduced order model of the singularly perturbed system found

\[
\dot{x} = \begin{bmatrix} -6.75 & -158.64 \\ 158.64 & -6.75 \end{bmatrix} x + \begin{bmatrix} 59.14 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 59.14 \end{bmatrix} u_2 \quad (5.75)
\]

The system’s optimal feedback gains using the state feedback controller found

\[
F_1^* = \begin{bmatrix} 3.97 & 1.35 \end{bmatrix}
\]

\[
F_2^* = \begin{bmatrix} -1.35 & 3.97 \end{bmatrix}
\]

To apply these feedback gains to the full order system using following zeros they will have the following form

\[
F^*_{full,1} = \begin{bmatrix} 3.97 & 1.35 & 0 & 0 \end{bmatrix}
\]

\[
F^*_{full,2} = \begin{bmatrix} -1.35 & 3.97 & 0 & 0 \end{bmatrix}
\]

When applied to the system, the total cost is

\[
J^* = 0.2762906411
\]

with the following matrix as the solution of the Riccati equation

\[
P(0) = \begin{bmatrix} 0.30 & 0.00 & 0.02 & -0.01 \\ 0.00 & 0.30 & 0.01 & 0.02 \\ 0.02 & 0.01 & 0.00 & 0.00 \\ -0.01 & 0.02 & 0.00 & 0.00 \end{bmatrix}
\]

and the closed loop system poles are
\[ p = \begin{bmatrix} -1025.8 \\ -143.9 + j296.7 \\ -143.9 - j296.7 \\ -182.1 \end{bmatrix} \]

By using the output feedback controller, the total cost of the system was

\[ J^* = 0.4580856634 \]

with the feedback gains

\[ F^*_{full,1} = \begin{bmatrix} +0.00 & 0.11 & 0 & 0 \end{bmatrix}, \quad F^*_{full,2} = \begin{bmatrix} -0.12 & 0.01 & 0 & 0 \end{bmatrix} \]

This cost is considered very high compared with the state feedback results. A better result is expected by performing the game theoretic optimization procedure. For the state feedback controller, neither the Nash or Stackelberg strategy improved results as it is assumed to be already optimal and additional conditioning didn’t help lower the cost. On the other hand, the output feedback controller showed an excellent cost optimization using both strategies. For the Stackelberg strategy optimization method, the output feedback gains were found to be

\[ F^*_{full,1} = \begin{bmatrix} -0.06 & -1.45 & 0 & 0 \end{bmatrix}, \quad F^*_{full,2} = \begin{bmatrix} +1.58 & -0.07 & 0 & 0 \end{bmatrix} \]

When applied to the system, the total cost is

\[ J^* = 0.1913860766 \]

with the following matrix as the solution of the Riccati equation

\[ P(0) = \begin{bmatrix} 0.1713 & 0.0005 & 0.0106 & 0.0001 \\ 0.0005 & 0.1645 & 0.0002 & 0.0100 \\ 0.0106 & 0.0002 & 0.0020 & 0.0000 \\ 0.0001 & 0.0100 & 0.0000 & 0.0020 \end{bmatrix} \]

and the closed loop system poles are

\[ p = \begin{bmatrix} -86.74 + j287.76 \\ -86.74 - j287.76 \\ -287.86 + j73.44 \\ -287.86 - j73.44 \end{bmatrix} \]
Figure 5.10 Output response for OFB controller.

Figure 5.11 Output response for SFB controller.
Figure 5.12 States responses for OFB controller.

Figure 5.13 States responses for SFB controller.
CHAPTER 6
CONCLUSION AND FUTURE WORK

6.1 Summary

Large scale decentralized systems are usually complicated and computation extensive. Hence, the two methods proposed in this research assume the ability to apply singular perturbation techniques to large scale systems. This includes the ability to separate the time scale for subsystems from that of the main system. For the studied singularly perturbed large scale decentralized system, two methods were proposed to analyze and optimize the performance of the system. The first one was to solve the basic problem using LQG approach and Kalman filter design procedure to find a robust optimal solution for the problem. The second method involves a game theoretic approach used together with a Stackelberg strategy on a leader-follower basis to obtain the optimal solution for the system.

The singularly perturbed decentralized system was found to have an optimum control with optimum feedback gain satisfying the Riccati equation in the quasi-steady state model derived in section 2.5. When the system is exposed to either a measurement noise, input disturbance or both, a solution exists using LQG approach and Kalman filter with certain assumptions in addition to basic knowledge of the white noise distribution. The game theoretic approach would help attain much more realistic and practical results in the sense of cooperation and compromise among conflicts in subsystems’ control constraints and needs. Although it is not guaranteed in some game theoretic techniques to end up with a global minimum, in Stackelberg a set of equations were derived to form the final solution of the optimal value.
6.2 Conclusion

To analyze and optimize a singularly perturbed large scale system, a standard singular perturbation procedure was performed. In the model formulated in this research, the decentralized nature of the system (in which neither the main system states nor the output is available for the controllers) was taken in consideration. Due to the unavailability of system states, two control strategies were used in this research to design the controller. The first used state feedback controller assuming the availability of states estimates from the Kalman filter. The second was the static output feedback controller where the feedback gain was calculated directly from the output signal. LQG approach with the Kalman filter design procedures was used to develop an optimal solution.

Another way to perform analysis and control design for a decentralized system is to consider the two subsystems’ players in a two-player game. A Nash equilibrium game strategy was used in section 5.1. This strategy doesn’t guarantee the global optimal value as it might converge to a local minimum. So, the cooperative Stackelberg game setup was used in section 5.1. The Stackelberg game assumes that there is a leader player and a follower. The leader’s decision is not affected by the follower’s decisions, while the follower takes into consideration the leader’s action and consequences. Using this methodology, a final and optimal solution for the quadratic performance index was found.

6.3 Future Work

There is a lot of research that still can be investigated for decentralized singularly perturbed systems. The large scale and cascading approximation and simplification procedures performed in this research have many areas of investigation. Uncertainties of system parameters can be included as further system uncertainty. $H_\infty$ control design can be considered for system stability analysis. In singular perturbation, investigating multi-time scale systems can be a valuable extension. In
addition, further research can consider system and communication delays and study their effect on system stability and optimal design. Discretizing the system can help in the application for discrete systems. Furthermore, more than two players may be considered for the game theoretic optimal design in chapter 5.
REFERENCES
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