INCREASING STABILITY IN THE INVERSE PROBLEM FOR THE SCHRÖDINGER EQUATION

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ABSTRACT

The Schrödinger equation is a partial differential equation that describes how the quantum state of a physical system changes with time. It was formulated in late 1925 by the Austrian physicist Erwin Schrödinger. The study of the inverse problem for the Schrödinger equation focuses on finding the potential c from the prescribed boundary condition, which is generally given as Cauchy data containing both solution on the boundary and its normal derivative, or the Dirichlet-to-Neumann operator which maps the solution on the boundary to its normal derivative.

The result of research has direct application to optical tomography, which is an inverse problem of reconstructing medical images through transmission of light. More precisely, one can detect cancer by recovering the absorption and scattering coefficients in the transport equation. The paper [30] discussed the simplification of the transport equation into the Schödinger equation. Optical tomography with partial data is considered extremely valuable, since we do not have access to the full boundary in real application. The research in the dissertation assumes partial data, which can be applied to breast cancer detection.

The main result of this dissertation demonstrates the increasing stability phenomenon in the inverse problem for the Schrödinger equation with partial data. We establish the theorem which contains the stability estimate bound for c. The bound decays as the energy k grows in a certain interval, and hence shows a better stability of recovering c there. In addition, we found a numerical algorithm for the linearized (simplified) inverse problem by using the Neumann-to-Dirichlet boundary map. The algorithm gives numerical evidence of increasing stability, which confirmed the theoretical prediction.

The proof of uniqueness for this inverse problem was established before. The proof used almost exponential solution for the Schrödinger equation and the Fourier transform of c. A similar technique will be used in this dissertation to obtain the stability bound, but our choice of ζ in the almost exponential solutions is new.

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CHAPTER 1

INTRODUCTION

The study of inverse problems allows us to calculate parameters which can't be directly observed. In particular, inverse problems of partial differential equations are to recover coefficients from boundary measurements. The coefficients often represent important physical objects contained in a domain which we can't get in. Uniqueness and stability play important roles in these problems, since the first one implies that we have enough data to determine the coefficients, and the later one describes how errors on the measurements impact the accuracy of the recovered solution. In this Dissertation, we will show the increasing stability of recovering the coefficient $c \in C^1(\Omega)$ in the Schrödinger equation when the Dirichlet-to-Neumann map is given on part of the boundary. The proof uses almost exponential solutions of the Schrödinger equation as the primary technique for achieving the result. The idea of these solutions was first introduced by Faddeev [14], then rediscovered by Sylvester and Uhlmann [1] in the proof of the global uniqueness of c in the three dimensional case. Alessandrini in the paper [5] has obtained a logarithmic stability estimate for c from the Dirichlet-to-Neumann map. Mandache [6] demonstrated the optimality of log-type stability. However, the logarithmic stability was disappointing in applications, since small errors in the data of the inverse problem result in large errors in numerical reconstruction of physical properties of the medium. Consequently, it restricts resolution in the electrical impedance tomography. Isakov proved the uniqueness of c in the Schrödinger equation with partial data in the paper [7], where he used Riemann-Lebesgue Lemma and the methods of reflection for almost exponential solutions. He also derived the increasing stability bound for c in different ranges of frequency k for full boundary data in [8], where both complex- and real-valued geometrical optics solutions were used in the proof. The proof was simplified in [9] where only complex-valued geometrical optics solution were used. Similar results were obtained by Isaev and Novikov [10] by less explicit and more complicated methods of scattering theory.

Since the access to full boundary data can be very expensive in real life, inverse problems with local data generally have valuable applications. This thesis will show increasing stability of c in the Schrödinger equation when the Dirichlet-to-Neumann map is given on an arbitrary part of the boundary, assuming that the remaining part is contained in a plane in 3 dimension. The result has applications to situations where the part of the boundary of the domain is inaccessible . For instance, when detecting breast cancer, one can assume the domain is a half unit ball. The measurements can be made on the top of the half ball by putting sensors along the top surface. However the bottom surface is inside human body so it is impossible to take measurements from there. The result of this paper allows us to better recover potential c inside the half ball from prescribed partial data for large frequencies k.

We use methods of even reflection to build almost complex exponential solutions which have vanishing Neumann data on the partial boundary contained in a plane, then give bounds on these solutions by using sharp bounds on regular fundamental solutions of some linear partial differential operators. In the proof, we encountered some new difficulties compared to [7], [8]. More precisely, this paper uses the type of almost exponential solutions introduced in [7]. The advantage is that, in contrast to [8], they will work in both cases of high and low frequencies. However the product of these solutions will produce an extra term which decays in an uncontrollable way, so in addition to [8], we use integration by parts for this extra term, then use the stability of analytic continuation to obtain the bound for c.

Moreover, the dissertation contains both theoretical and numerical evidence of increasing stability phenomenon in the linearized inverse problem for the Schrödinger equation with full/partial Neumann-to-Dirichlet map. The phenomenon shows a better stability in recovering potential c as the energy frequency k grows. We will first justify the linearization by assuming that c is a small perturbation, then obtain an increasing stability bound for c with explicit constants as the theoretical evidence. In the case of partial data, the methods of reflection and analytic continuation are also needed to derive such bound. For numerical experiments, we take measurements of Dirichlet data along the boundary surface, then compute c for different choices of frequency k to confirm the prediction of the theory. Currently, there are many researches regarding the linearized inverse conductivity problem with zero energy frequency. These works can be traced back to the 1980s when Calderón [17] proposed the idea of determining electrical conductivity of a medium by making voltage and current measurements along its boundary. Inspired by his idea, Isaacson [18]-[20] made many contributions to the research in Electrical Impedance Tomography, which has great potential for the medical application. He [21] also introduced the concepts of distinguish-ability as a means of characterizing unrecoverable information in the presence of measurement errors. In the paper [22], Isaacson and Cheney have made analysis on the effects of measurements' precision on the linearized inverse problem for the conductivity equation. Dobson and Santosa [23] computed conductivities by prescribing a set of dipole current patterns along the circular boundary and examined the resolution limit. Their analysis shows severe limitations on the image of the recovered solutions when $k = 0$. This paper will focus on the linearized Schrödinger equation, which is an alternative form of the conductivity equation, and will show that the stability can be improved if one increases the energy k.

The thesis is organized as follows. In next chapter, we present readers with the inverse problem for the Schrödinger equation and give proofs for uniqueness and stability. In chapter 3, we justify the linearization for the Schödinger equation, then derive the increasing stability bound for c from the prescribed Neumann-to-Dirichlet map on the full boundary. chapter 4 contains the numerical result in the full data case, which is found by forming a system of equations, and solve the system for different values of k. Then, we obtain similar results when the partial data is given. The methods of reflection and analytic continuation are used here. Finally, in the last section, we outline future developments and challenges.

1.1 NOTATION

The notation introduced here will be used throughout the entire thesis.

$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)e^{i\xi\cdot x}$ denotes the Fourier transform of f.

$H^s(\Omega) = \{f \in L^2(\Omega), \int_{\mathbb{R}^3}(1+|\xi|^2)^s\hat{f}^2(\xi) < +\infty\}$ denotes Sobolev space with order $s \in \mathbb{R}$. In the case of $s \in \mathbb{Z}^+$, the definition is equivalent to $H^s(\Omega) = \{\sum_{|\alpha|\leq s}\int_\Omega |D^\alpha f|^2 < +\infty\}$.

$\overset{\circ}{H}{}^s(\Omega)$ denotes the space of functions $f \in H^s(\Omega)$ and supp $f \subset \Omega$.

$\|f\|_{(s)}(\Omega) = [\int_{\mathbb{R}^3}(1+|\xi|^2)^s|\hat{f}^2(\xi)|]^{\frac{1}{2}}$ denotes the norm for Sobolev space.

$\|f\|_p(\Omega) = (\int_\Omega |f|^p)^{\frac{1}{p}}$ denotes the norm for L_p space for $1 \leq p < \infty$.

C, $C(M,\Omega)$, C_0, C_Ω denote the generic constants whose values change from line to line, and only depend on the indicated arguments.

1.2 PRELIMINARY

Theorem 1.2.1 (Green's theorem) *Let $v, u \in H^2(\Omega)$, where Ω is a Lipschitz domain, we have*

$$\int_\Omega v\Delta u = -\int_\Omega \nabla u \nabla v + \int_{\partial\Omega} (\partial_\nu u)v,$$
$$\int_\Omega v\Delta u = \int_\Omega v\Delta u + \int_{\partial\Omega} v\partial_\nu u - u\partial_\nu v.$$

Theorem 1.2.2 (Trace's theorem) *Let Ω be a bounded Lipschitz domain. We define a linear operator $T : C^\infty(\bar{\Omega}) \longrightarrow L^2(\partial\Omega)$ as*

$$Tf = f|_{\partial\Omega},$$

then there exist a unique extension of T such that

$$T : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\partial\Omega),$$

for $\frac{1}{2} < s \leq 1$, and T is a bounded linear operator.

Theorem 1.2.3 (Sobolev embedding theorem) *Let $H^{s,p}(\Omega)$ denote the Sobolev space consisting of all functions on Ω whose first k weak derivatives are functions in $L^p(\Omega)$ for $1 \leq p < \infty$. If $k > l$ and $1 \leq p < q < \infty$ are real numbers such that $(k-l)p < n$ and*

$$\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n},$$

then

$$H^{s,p}(\Omega) \subseteq H^{l,q}(\Omega)$$

Definition An open set $\Omega \subset \mathbb{R}^n$ is called **Lipschitz domain** if for every point $p \in \partial\Omega$, there exits a open ball $B(p, r)$ centered at p with radius $r > 0$, and a map $V_p : B(p, r) \longrightarrow B$, where B denotes the unit ball, such that

1. V_p is bijection,

2. V_p and V_p^{-1} are both Lipschitz continuous functions,

3. $V_p(\partial\Omega \cap B(p,r)) = B_0$,

4. $V_p(\Omega \cap B(p,r)) = B_+$,

where

$$B_0 = \{x \in B | x_n = 0\},$$

$$B_+ = \{x \in B | x_n > 0\}.$$

Definition The domain $\Omega \in R^n$ is said to satisfy **cone condition** if each point $x \in \Omega$ is vertex of a finite, right-spherical cone C_x contained Ω. (C_x is the union of all points on line segments from x to points of a ball not containing x.)

Theorem 1.2.4 (the Riemann-Lebesgue Lemma) *If f is L_1 integrable on $\mathbb{R}^n$, then*

$$\lim_{|\xi|\to 0} \int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x} = 0$$

CHAPTER 2

UNIQUENESS OF THE INVERSE PROBLEM

In this chapter, we will introduce the inverse problem for the Schrödinger equation with partial data, then prove uniqueness by the method of reflection and Fourier transform.

2.1 THE SCHRÖDINGER EQUATION

Let Ω be a domain in $\mathbb{R}^3$ with Lipschitz boundary. We assume $\Omega \subset \{|x_i| < 1,\ i = 1, 2, 3,\ x_3 > 0\}$ and a part Γ of $\partial\Omega$ lies inside the plane $\{x \in \mathbb{R}^3 |\, x_3 = 0\}$. We consider the Schrödinger equation

$$-\Delta u - k^2 u + cu = 0 \text{ in } \Omega \tag{2.1.1}$$

with the boundary data

$$\begin{aligned} u &= g_0 \in H^{\frac{1}{2}}(\partial\Omega_+), \\ \partial_\nu u &= 0 \text{ on } \Gamma, \end{aligned} \tag{2.1.2}$$

where $\partial\Omega_+ = \partial\Omega \cap \{x_3 > 0\}$ and the potential $c \in C^1(\Omega)$. Suppose that $k^2 \notin \sigma(-\Delta + c)$, where $\sigma(-\Delta + c)$ is the set of eigenvalues of Dirichlet problem (2.1.1)(2.1.2) for the operator $-\Delta + c$. By [29], $\sigma(-\Delta + c)$ forms a discrete set. Thus we can define the Dirichlet-to-Neumann operator which maps $g_0 \in H^{\frac{1}{2}}(\partial\Omega_+)$ into $\partial_\nu u \in H^{-\frac{1}{2}}(\partial\Omega_+)$ as

$$\Lambda_c g_0 = \partial_\nu u \text{ on } \partial\Omega_+. \tag{2.1.3}$$

The trace $\partial_v u$ is understood as follows. One can take a sequence u_m convergent to u in the space $H^1(\Omega)$. From trace theorem, it follows that $\|\partial_v u_m\|_{(-\frac{1}{2})}(\partial\Omega)$ is bounded by $\|u_m\|_{(1)}(\Omega)$. Since u_m converges in $H^1(\Omega)$, their boundary trace $\partial_\nu u_m$ must converge in $H^{-\frac{1}{2}}(\Omega)$ and the limit of $\partial_\nu u_m$ is the desired trace of u. In the practical situation, one takes measurements on $\partial_\nu u$ for each prescribed function g_0. Thus, in order for Λ maps g_0 into $\partial_\nu u$, there must exist a solution to the boundary value problem (2.1.1)(2.1.2).

The inverse problem for the Schrödinger equation is to find the potential c in (2.1.1) from the boundary conditions (2.1.2) and (2.1.3). It is known that these conditions can determine the coefficient c uniquely. The proof for this claim will be given in the next section.

2.2 THE PROOF OF UNIQUENESS

The following theory establishes the uniqueness of recovering c from the Dirichlet-to-Neumann map given as (2.1.3).

Theorem 2.2.1 *If $\Lambda_{c_1} = \Lambda_{c_2}$, then $c_1 = c_2$.*

Proof: Let u_1, u_2 solve the boundary value problems (2.1.1) (2.1.2) corresponding to c_1 and c_2 respectively. By the application of the Green's theorem and $\Lambda_{c_1} = \Lambda_{c_2}$, we have the orthogonality relation

$$\int_\Omega (c_2 - c_1) u_1 u_2 = 0. \tag{2.2.1}$$

Now, suppose that $\xi \in \mathbb{R}^3$ and $\xi^* = (\xi_1, \xi_2, -\xi_3)$. We introduce vectors

$$e(1) = (\xi_1^2 + \xi_2^2)^{-\frac{1}{2}}(\xi_1, \xi_2, 0), \quad e(3) = (0, 0, 1)$$

and $e(2)$ to form orthonormal basis $e(1), e(2), e(3)$ in $\mathbb{R}^3$. Denote the coordinate system in this basis by $(x_{1e}, x_{2e}, x_{3e})_e$. Observe it preserves inner product in the sense that

$$xy = x_1 y_1 + x_2 y_2 + x_3 y_3 = x_{1e} y_{1e} + x_{2e} y_{2e} + x_{3e} y_{3e}$$

Let

$$\begin{aligned}
\zeta(1) &= (\frac{\xi_{1e}}{2} - \tau\xi_3, i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, \frac{\xi_3}{2} + \tau\xi_{1e})_e, \\
\zeta^*(1) &= (\frac{\xi_{1e}}{2} - \tau\xi_3, i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, -\frac{\xi_3}{2} - \tau\xi_{1e})_e, \\
\zeta(2) &= (\frac{\xi_{1e}}{2} + \tau\xi_3, -i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, \frac{\xi_3}{2} - \tau\xi_{1e})_e, \\
\zeta^*(2) &= (\frac{\xi_{1e}}{2} + \tau\xi_3, -i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, -\frac{\xi_3}{2} + \tau\xi_{1e})_e,
\end{aligned} \tag{2.2.2}$$

where $\xi_{1e} = \sqrt{\xi_1^2 + \xi_2^2}$ and τ is a positive real number. One can check that $\zeta(1) \cdot \zeta(1) = \zeta^*(1) \cdot \zeta^*(1) = \zeta(2) \cdot \zeta(2) = \zeta^*(2) \cdot \zeta^*(2) = -k^2$.

Assume $c = 0$ on $\{x_3 > 0\} \setminus \Omega$, then extend c onto $\mathbb{R}^3$ as even function of x_3. Denote $V^*(x_1, x_2, x_3) = V(x_1, x_2, -x_3)$ for a given function V. By [16], section 5.3, there are almost exponential solutions

$$u(x; j) = e^{i\zeta(j)\cdot x}(1 + V(x; j)) + e^{i\zeta^*(j)\cdot x}(1 + V^*(x; j)), \; j = 1, 2, \tag{2.2.3}$$

to (2.1.1) with $c = c_1$, c_2 respectively, and

$$\|V(; j)\|_{(0)}(\Omega) \to 0, \; \|V^*(; j)\|_{(0)}(\Omega) \to 0 \text{ as } \tau \to \infty. \tag{2.2.4}$$

Then it is obvious that

$$\partial_\nu u(x; 1) = \partial_\nu u(x; 2) = 0 \text{ on } \Gamma_0 \tag{2.2.5}$$

Now, let $c(x) = c_1(x) - c_2(x)$, by (2.2.1), (2.2.3), (2.2.5), we have that

$$\begin{aligned} 0 = \int_\Omega cu(x; 1)u(x; 2) = \int_\Omega c(x)(e^{i(\zeta(1)+\zeta(2))x}&(1 + V(; 1))(1 + V(; 2) \\ +e^{i(\zeta^*(1)+\zeta(2))x}&(1 + V^*(; 1))(1 + V(; 2)) \\ +e^{i(\zeta(1)+\zeta^*(2))x}&(1 + V(; 1))(1 + V^*(; 2)) \\ +e^{i(\zeta^*(1)+\zeta^*(2))x}&(1 + V^*(; 1))(1 + V^*(; 2)))dx. \end{aligned} \tag{2.2.6}$$

Due to (2.2.2), the above equation implies that

$$\begin{aligned} &\int_\Omega c(x)(e^{i\xi x}(1 + V(x; 1))(1 + V(x; 2)) + e^{i\xi^* x}(1 + V^*(x; 1))(1 + V^*(x; 2)) \\ &+ e^{i\xi_{1e}(x_{1e}-2\tau x_3)}(1 + V^*(x; 1))(1 + V(x; 2)) \\ &+ e^{i\xi_{1e}(x_{1e}+2\tau x_3)}(1 + V(x; 1))(1 + V^*(; 2))) = 0. \end{aligned} \tag{2.2.7}$$

If we send $\tau \to \infty$, since all moduli of exponents are bounded by 1, then the limits of all terms containing V and V^* are 0 due to (2.2.4). By the Riemann-Lebesgue Lemma, limits of

$$\int_\Omega e^{i\xi_{1e}(x_{1e}-2\tau x_3)}, \; \int_\Omega e^{i\xi_{1e}(x_{1e}+2\tau x_3)} \tag{2.2.8}$$

are 0 as $\tau \to \infty$ provided $\xi_{1e} \neq 0$. Thus from (2.2.7) we have that

$$\int_{\Omega} c(x)(e^{i\xi x} + e^{i\xi^* x}) = 0 \tag{2.2.9}$$

for $\xi_{1e} \neq 0$. Since c is a compactly supported function, the left side of (2.2.9) is analytic with respect to ξ_{1e}. Thus (2.2.9) holds for all $\xi \in \mathbb{R}^3$. Because c is an even function of x_3,

$$\int_{\Omega} c(x)(e^{i\xi x} + e^{i\xi^* x}) = \int_{\mathbb{R}^3} c(x) e^{i\xi x}. \tag{2.2.10}$$

By (2.2.7) and the definition of Fourier transform,

$$\hat{c}(\xi) = \int_{\mathbb{R}^3} c(x) e^{i\xi x} = 0, \tag{2.2.11}$$

which implies $c = 0$ or $c_1 = c_2$ by the uniqueness of inverse Fourier transform. □

CHAPTER 3

THE INCREASING STABILITY PHENOMENON

The increasing stability phenomenon shows that the error for recovered solution c decreases when k grows in a certain interval. In this chapter, we will obtain some bound for c which can be viewed as an evidence of the phenomenon. The proof uses almost exponential solutions and methods of reflection. The following lemma is needed for establishing the main result.

Lemma 3.0.2 *The Dirichlet-to-Neumann map defined in (2.1.3) is a linear bounded operator from $L^2(\Omega_+)$ into $L^2(\Omega_+)$.*

Proof: To prove $\Lambda_{c_1} - \Lambda_{c_2}$ is a linear bounded operator from $L^2(\partial\Omega_+)$ to $L^2(\partial\Omega_+)$, we will firstly show the same result for $\Lambda_{c_1} - \Lambda_0$. Λ_0 denotes the Dirichlet-to-Neumann map for Laplace equation. Throughout the proof, we will use the generic constant C which depends on Ω. Let u_0 and u_{c_1} be the solutions for the following equations.

$$\Delta u_0 = 0 \text{ in } \Omega, \tag{3.0.1}$$

$$u_0 = g_0 \text{ on } \partial\Omega_+, \tag{3.0.2}$$

and

$$\Delta u_{c_1} + c_1 u_{c_1} = 0 \text{ in } \Omega, \tag{3.0.3}$$

$$u_{c_1} = g_0 \text{ on } \partial\Omega_+, \tag{3.0.4}$$

We define the operator $(\Lambda_{c_1} - \Lambda_0)g_0 = \partial_\nu u_{c_1} - \partial_\nu u_0$ on $\partial\Omega_+$. Denote $\Gamma = \partial\Omega \cap \{x_3 = 0\}$. Now let us reflect functions u_0, u_{c_1}, and c_1 across Γ as even functions of x_3. Denote Ω^* the domain after reflection. Assume Ω_1 is the support of c_1 contained in Ω^*, so $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega^*$. Then by the theorem 2.2.22, p.56 [15],

$$\|u_0\|_{(0)}(\Omega_1) \leq C\,\|g_0\|_{(0)}(\partial\Omega^*). \tag{3.0.5}$$

Now let $u = u_{c_1} - u_0$, then subtracting equation (3.0.3) from (3.0.1) results in

$$\Delta u + c_1 u = -c_1 u_0 \text{ in } \Omega^*, \tag{3.0.6}$$

$$u = 0 \text{ on } \partial\Omega^*. \tag{3.0.7}$$

By the theorem 4.1, p.90 [16] for elliptic equations, it follows that

$$\|u\|_{(1)}(\Omega^*) \leq C\,\|c_1 u_0\|_{(0)}(\Omega^*) \leq C\,\|u_0\|_{(0)}(\Omega_1). \tag{3.0.8}$$

Introduce a larger domain $B \supset \Omega^*$ with the smooth boundary and a function U satisfying

$$\Delta U = -c_1 u - c_1 u_0 \text{ in } B, \tag{3.0.9}$$

$$U = 0 \text{ on } \partial B, \tag{3.0.10}$$

where right side of (3.0.9) equals 0 in $B \setminus \Omega^*$. Since B is a smooth domain,

$$\|U\|_{(2)}(B) \leq C(\|u\|_{(0)}(\Omega_1) + \|u_0\|_{(0)}(\Omega_1)) \leq C\,\|g_0\|_{(0)}(\partial\Omega^*), \tag{3.0.11}$$

where the second inequality is due to (3.0.8) and (3.0.5). Let $U_1 = U - u$, then subtracting equation (3.0.9) from (3.0.6) gives us

$$\Delta U_1 = 0 \text{ in } \Omega^*, \tag{3.0.12}$$

$$U_1 = U \text{ on } \partial\Omega^*. \tag{3.0.13}$$

It follows that

$$\|\partial_\nu U_1\|_{(0)}(\partial\Omega^*) \leq C\,\|U\|_{(1)}(\partial\Omega^*) \leq C(\|\nabla U\|_{(0)}(\partial\Omega^*) + \|U\|_{(0)}(\partial\Omega^*)), \tag{3.0.14}$$

where the first inequality is due to the theorem 2.2.13 (a), p.54 [15] together with the theorem 2.2.22, p.56, and the second one is by the definition of sobolev norm.

Now, by applying the trace theorem for the partial derivative $\partial_j U$, $j = 1, 2, 3$, we have $\|\partial_j U\|_{(0)}(\partial\Omega^*) \leq \|\partial_j U\|_{(1)}(\Omega^*) \leq \|U\|_{(2)}(\Omega^*)$. By using the partition of unity, we can assume $\partial\Omega^* = \{x \in \mathbb{R}^3 |\ x_3 = \varphi(x_1, x_2)\}$, where φ is Lipschitz. Let $x' = (x_1, x_2) \in D\varphi$, then for $j = 1, 2$,

$$\|\partial_j U(x', \varphi(x'))\|_{(0)}(D\varphi) \leq \|(\partial_j U)(x', \varphi(x'))\|_{(0)}(D\varphi) +$$
$$\|(\partial_3 U)(x', \varphi(x'))\partial_j \varphi(x')\|_{(0)}(D\varphi)$$
$$\leq \|\partial_j U\|_{(0)}(\partial\Omega^*) + C\|\partial_3 U\|_{(0)}(\partial\Omega^*) \leq C\|U\|_{(2)}(\Omega^*), \tag{3.0.15}$$

due to the chain rule, bounds of $\|\partial_j U\|_{(0)}$ by the trace theorem, and the Lipschitz conditions of φ. The inequality (3.0.15) gives us a bound for $\|\nabla U\|_{(0)}(\partial\Omega^*)$, which is the first term of right side of (3.0.14). The second term $\|U\|_{(0)}(\partial\Omega^*)$ can be bounded by trace theorem, $\|U\|_{(0)}(\partial\Omega^*) \leq \|U\|_{(\frac{1}{2})}(\Omega^*) \leq \|U\|_{(2)}(\Omega^*)$. Combining these results with (3.0.11)(3.0.14), we conclude that $\|\partial_\nu U_1\|_{(0)}(\partial\Omega^*) \leq C\|U\|_{(2)}(\Omega^*) \leq C\|g_0\|_{(0)}(\partial\Omega^*)$. Then since $u = U - U_1$, it follows that

$$\|\partial_\nu u\|_{(0)}(\partial\Omega^*) \leq \|\partial_\nu U\|_{(0)}(\partial\Omega^*) + \|\partial_\nu U_1\|_{(0)}(\partial\Omega^*)$$
$$\leq C(\|\nabla U\|_{(0)}(\partial\Omega^*) + \|U\|_{(0)}(\partial\Omega^*)) \leq \|g_0\|(\partial\Omega^*) \tag{3.0.16}$$

The above inequality shows $\Lambda_{c_1} - \Lambda_0$ is a bounded operator from $L^2(\partial\Omega_+)$ to $L^2(\partial\Omega_+)$. Hence the same result holds for $\Lambda_{c_1} - \Lambda_{c_2}$. □

3.1 THE STABILITY ESTIMATE BOUND

In this section, we will give the main theorem which contains the stability estimate bound for c. Consider the boundary value problem given in (2.1.1)(2.1.2)(2.1.3).

Theorem 3.1.1 *Let*

$$\|c_j\|_\infty(\Omega) \leq M_0,\ \|\nabla c_j\|_\infty(\Omega) \leq M_1,\ j = 1, 2,\ c = 0 \ \textit{near}\ \partial\Omega_+, \tag{3.1.1}$$

and $\varepsilon = \|\Lambda_{c_2} - \Lambda_{c_1}\|$, $E = -log\varepsilon$, *then there are constants* $C(\Omega, M)$, $0 < \lambda < 1$ *and* $0 < \lambda_1 < 1$ *such that*

$$\|c_1 - c_2\|^2_{(0)}(\Omega) \leq C(\Omega, M)\varepsilon^\lambda (E + k)^{\lambda_1} + \frac{C(\Omega, M)}{(E + k)^{\frac{2}{3}\lambda_1}}, \tag{3.1.2}$$

where $M = \sqrt{M_0^2 + M_1^2}$.

For a given ε, one can minimize the bound of (3.1.2) with respect to k. We found that, at the minimum point $k = (\frac{3}{2}\varepsilon^{-\lambda})^{\frac{3}{5\lambda_1}} - E$, the right hand side $C(\Omega, M)[\varepsilon^{\lambda}(E+k)^{\lambda_1} + \frac{1}{(E+k)^{\frac{2}{3}\lambda_1}}] = C(\Omega, M)\varepsilon^{\frac{2}{5}\lambda}$, which is Hölder continuous in ε. Thus, as k grows in the zone $k < (\frac{3}{2}\varepsilon^{-\lambda})^{\frac{3}{5\lambda_1}} - E$, the bound decreases and become more like Hölder type when k approaches to the minimum point. This fact shows increasing stability phenomenon in the above zone.

We need some lemmas to prove the theorem. The following inequality follows from the application of the Green's theorem.

Lemma 3.1.2 *For all solutions $u_j \in H^1(\Omega)$ satisfying (2.1.1))(2.1.2)(2.1.3) with potential c_j, j=1,2, we have*

$$\int_{\Omega}(c_2 - c_1)u_1 u_2 = \int_{\partial\Omega_+}((\Lambda_{c_2} - \Lambda_{c_1})u_1)u_2. \tag{3.1.3}$$

Proof: Subtract equations $-\Delta u_1 u_2 - k^2 u_1 u_2 + c_1 u_1 u_2 = 0$ and $-\Delta u_2 u_1 - k^2 u_2 u_1 + c_2 u_2 u_1 = 0$. Then integrate by parts to obtain

$$\begin{aligned}\int_{\Omega}(c_2 - c_1)u_1 u_2 &= \int_{\Omega} -\Delta u_1 u_2 + \Delta u_2 u_1 = \int_{\partial\Omega_+} -\partial_\nu u_1 u_2 + \partial_\nu u_2 u_1 \\ &= \int_{\partial\Omega_+} -\partial_\nu u_1 u_2 + \partial_\nu u_2 u_2 - \partial_\nu u_2 u_2 + \partial_\nu u_2 u_1 \\ &= \int_{\partial\Omega_+} (\partial_\nu u_2 - \partial_\nu u_1)u_2 + \partial_\nu u_2(u_2 - u_1) = \int_{\partial\Omega_+}((\Lambda_{c_2} - \Lambda_{c_1})u_1)u_2,\end{aligned}$$

by the definition of Λ_{c_j}, (2.1.2), and $u_2 = u_1 = g_0$ on $\partial\Omega_+$. □

3.2 THE ALMOST EXPONENTIAL SOLUTIONS

Now, assume $c(x) = 0$ on $\{x \in \mathbb{R}^3 | x_3 > 0\} \setminus \Omega$. We extend c onto the whole space $\mathbb{R}^3$ as a even function of x_3. Denote $V^*(x_1, x_2, x_3) = V(x_1, x_2, -x_3)$ and $\xi^* = (\xi_1, \xi_2, -\xi_3)$. Under these abbreviations, we introduce the almost exponential solutions for the Schrödinger equation.

Lemma 3.2.1 *Let $\xi \in \mathbb{R}^3$, $(1 + 4\tau^2) \geq \frac{4k^2}{|\xi|^2}$, and $(1 + 4\tau^2) \geq \frac{4C_0^2 M^2 + 2k^2 - 4}{|\xi|^2}$ for some constant $C_0 > 0$, then there are almost exponential solutions*

$$u(x; j) = e^{i\zeta(j)\cdot x}(1 + V(x; j)) + e^{i\zeta^*(j)\cdot x}(1 + V^*(x; j)), \ j = 1, 2, \tag{3.2.1}$$

to equation

$$-\Delta u(x;j) - k^2 u(x;j) + c_j u(x;j) = 0 \text{ in } \Omega, \tag{3.2.2}$$

with

$$\zeta(1) + \zeta(2) = \xi,\ |Im\zeta(j)| = |Im\zeta(j)^*| = \sqrt{|\xi|^2(\frac{1}{4} + \tau^2) - k^2}, \tag{3.2.3}$$

$$\|V(x;j)\|_{(1)}(\Omega) \le \frac{2C_0 M}{\sqrt{|\xi|^2(1 + 4\tau^2) - 2k^2 + 4}}, \tag{3.2.4}$$

$$\|V^*(x;j)\|_{(1)}(\Omega) \le \frac{2C_0 M}{\sqrt{|\xi|^2(1 + 4\tau^2) - 2k^2 + 4}}, \tag{3.2.5}$$

$$\|V(x;j)\|_{(2)}(\Omega) \le C_0 M(1 + \sqrt{2}),\ \|V^*(x;j)\|_{(2)}(\Omega) \le C_0 M(1 + \sqrt{2}) \tag{3.2.6}$$

and

$$\partial_\nu u(x;j) = 0 \text{ on } \{x_3 = 0\}. \tag{3.2.7}$$

Proof: Suppose $\xi \in \mathbb{R}^3$, $\xi \neq 0$. We define a coordinate system $(x_{1e}, x_{2e}, x_{3e})_e$ in the following way. $e(1) = (\xi_1^2 + \xi_2^2)^{-\frac{1}{2}}(\xi_1, \xi_2, 0)$, $e(3) = (0, 0, 1)$, $e(2)$ is chosen to form an orthonormal basis $e(1), e(2), e(3)$ in $\mathbb{R}^3$. Denote $\xi_{1e} = \sqrt{\xi_1^2 + \xi_2^2}$. Now, let

$$\zeta(1) = (\frac{\xi_{1e}}{2} - \tau\xi_3, i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, \frac{\xi_3}{2} + \tau\xi_{1e})_e, \tag{3.2.8}$$

$$\zeta(2) = (\frac{\xi_{1e}}{2} + \tau\xi_3, -i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, \frac{\xi_3}{2} - \tau\xi_{1e})_e, \tag{3.2.9}$$

where τ is a positive real number. Then (3.2.3) is satisfied for the above $\zeta(1)$, $\zeta(2)$. We will show the first term $e^{i\zeta(j)\cdot x}(1 + V(x;j))$ on right side of (3.2.1) is a solution for (3.2.2) in set $A = \{|x_i| < 1, i = 1, 2, 3\}$, then the second term is obtained by doing even reflection to $e^{i\zeta(j)\cdot x}(1 + V(x;j))$ across $\{x_3 = 0\}$, so it must be also a solution since $-k^2 + c$ is even with respect to x_3.

Indeed, $e^{i\zeta(j)\cdot x}(1 + V(x;j))$ solves (3.2.2) in A if and only if

$$-\Delta V(;j) - 2i\zeta(j) \cdot \nabla V(;j) = c_j(1 + V(;j)). \tag{3.2.10}$$

Let $P(\zeta; j) = \zeta \cdot \zeta + 2\zeta(j) \cdot \zeta$. By known results [2],[3], there is a regular fundamental solution $E(j)$ of $P(; j)$ such that for any linear partial differential operator Q with constant coefficients

$$\|QE(j)f\|_{(0)}(A) \leq C_0 sup \frac{\hat{Q}(\xi^*)}{\hat{P}(\xi^*)} \|f\|_{(0)}(A) \ (sup \text{ over } \xi^* \in \mathbb{R}^3) \tag{3.2.11}$$

for any $f \in L^2(A)$, where

$$\hat{P}(\xi) = (\sum_{|\alpha| \leq 2} |\partial_\xi^\alpha P(\xi)|^2)^{\frac{1}{2}}.$$

In our particular case, by letting $\zeta = \xi(j) + i\eta(j), \xi(j), \eta(j) \in \mathbb{R}^3$, for any $\xi^* \in \mathbb{R}^3$, we have

$$\begin{aligned}\hat{P}^2(\xi^*; j) &= (|\xi^*|^2 + 2\xi(j) \cdot \xi^*)^2 + 4(\eta(j) \cdot \xi^*)^2 + 4(|\xi^* + \xi(j)|^2 + |\eta(j)|^2) + 12 \\ &= (|\xi^*|^2 + 2\xi^* \cdot \xi(j) + 2)^2 + 4(\eta(j) \cdot \xi^*)^2 + 4(|\xi(j)|^2 + |\eta(j)|^2) + 8 \\ &\geq 4(|\xi(j)|^2 + |\eta(j)|^2) + 8 = 2(|\xi|^2(1 + 4\tau^2) - 2k^2) + 8,\end{aligned} \tag{3.2.12}$$

due to the choice of $\zeta(j)$ in (3.2.8) and (3.2.9).

Similarly,

$$\begin{aligned}\hat{P}^2(\xi^*; j) &\geq (|\xi^* + \xi(j)|^2 - |\xi(j)|^2)^2 + 4|\xi^* + \xi(j)|^2 + 12 \\ &\geq 2(|\xi^* + \xi(j)|^2 + |\xi(j)|^2) + 11 \geq |\xi^*|^2 + 1,\end{aligned} \tag{3.2.13}$$

due to the elementary inequalities $(a-b)^2 + 2(a-b) + 1 \geq 0$, or $(a-b)^2 + 4a + 2 \geq 2(a+b) + 1$, with $a = |\xi^* + \xi(j)|^2$, $b = |\xi(j)|^2$.

The regular fundamental solution in [2] is a convolution operator, so it commutes with differentiations, and hence from (3.2.11) it follows that

$$\|QE(j)f\|_{(1)}(A) \leq C_0 sup \frac{\hat{Q}(\xi^*)}{\hat{P}(\xi^*)} \|f\|_{(1)}(A) \ (sup \text{ over } \xi^* \in \mathbb{R}^3). \tag{3.2.14}$$

Since $E(j)$ is a fundamental solution, any solution $V(; j)$ to the equation

$$V(; j) = E(j)(c_j(1 + V(; j))) \text{ on } A \tag{3.2.15}$$

solves (3.2.10). From (3.2.14) with $Q = 1$ and (3.2.12), it follows that

$$\|E(j)f\|_{(1)}(A) \leq C_0(|\xi|^2(1 + 4\tau^2) - 2k^2 + 4)^{-\frac{1}{2}} \|f\|_{(1)}(A), \tag{3.2.16}$$

or

$$\|E(j)f\|_{(1)}(A) \leq \theta \|f\|_{(1)}(A), \ \theta = \frac{C_0}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}}. \tag{3.2.17}$$

Observe that

$$\begin{aligned} \|c_j(1+V)\|_{(1)}(A) &\leq \|c_j\|_{(1)}(A) + \|c_j V\|_{(1)}(A) \\ &\leq M Vol^{\frac{1}{2}}(A) + \sqrt{2}M \|V\|_{(1)}(A), \end{aligned} \tag{3.2.18}$$

where we used the bound (3.1.1) and $M = \sqrt{M_0^2 + M_1^2}$. So the operator $F(V(;j))$ in the right side of (3.2.15) maps the ball $B(\rho) = \{V : \|V\|_{(1)}(A) \leq \rho\}$ into the ball $B(\theta M Vol^{\frac{1}{2}}(A) + \theta\sqrt{2}M\rho)$, and hence into $B(\rho)$ when

$$\theta M Vol^{\frac{1}{2}}(A) \leq (1 - \theta\sqrt{2}M)\rho. \tag{3.2.19}$$

The condition $(1+4\tau^2) \geq \frac{4C_0^2M^2+2k^2-4}{|\xi|^2}$ and (3.2.17) imply that $\sqrt{2}\theta M \leq \frac{1}{\sqrt{2}}$, and hence (3.2.19) holds with

$$\rho = \frac{2C_0 M}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}}, \tag{3.2.20}$$

which implies (3.2.4), since $\Omega \subset A$. Likewise, using (3.2.13) and (3.2.11) with $Q(\xi^*) = \xi_k^*$, $k = 1, 2, 3$, from (3.2.15), we have that for $V(;j) \in B(\rho)$

$$\begin{aligned} \|V(;j)\|_{(2)}(A) &\leq C_0 \|c_j(1+V(;j))\|_{(1)}(A) \leq C_0 M(1 + \sqrt{2}\|V(;j)\|_{(1)}(A)) \\ &\leq C_0 M(1 + \frac{2\sqrt{2}C_0 M}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}}) \leq C_0 M(1+\sqrt{2}), \end{aligned} \tag{3.2.21}$$

due to the bound (3.2.4) given by (3.2.20) and the inequality $(1+4\tau^2) \geq \frac{4C_0^2M^2+2k^2-4}{|\xi|^2}$. Therefore the operator F is continuous from $H^1(A)$ into $H^2(A)$ and therefore compact from $H^1(A)$ into itself. Now, this operator maps convex closed set $B(\rho) \subset H^1(A)$ into itself and is compact, hence by Schauder-Tikhonov Theorem, it has a fixed point $V(;j) \in B(\rho)$. In view of (3.2.20) we have the bound (3.2.4) and due to (3.2.21) we have the first bound in (3.2.6). Similarly, one can apply the same argument for reflected solution $e^{i\zeta^*(j)\cdot x}(1 + V^*(x;j))$ to achieve the bound (3.2.5) and the second bound in (3.2.6). The proof is completed. □

Lemma 3.2.2 *Let $u(x;j)$ be the solutions (3.2.1) to the Schrödinger equation $-\Delta u_j - k^2 u_j + c_j u_j = 0$ in Ω from Lemma 2, then we have*

$$\|u(x;j)\|_{(0)}(\partial\Omega_+) \leq 2C(\Omega)e^{\sqrt{|\xi|^2(\frac{1}{4}+\tau^2)-k^2}}. \tag{3.2.22}$$

Proof: Let $e(x;j) = e^{i\zeta(j)\cdot x}$, and $e^*(x;j) = e^{i\zeta^*(j)\cdot x}$. Clearly, we have

$$\|e(x;j)\|_{(0)}(\partial\Omega_+) \leq |\partial\Omega_+|^{\frac{1}{2}} e^{|Im\zeta(j)|},$$
$$\|e^*(x;j)\|_{(0)}(\partial\Omega_+) \leq |\partial\Omega_+|^{\frac{1}{2}} e^{|Im\zeta^*(j)|}.$$

Moreover, from trace theorems for Sobolev spaces

$$\|V(;j)\|_{(0)}(\partial\Omega_+) \leq C(\Omega)\,\|V(;j)\|_{(1)}(\Omega) \leq C(\Omega),$$

by (3.2.4) and the condition $(1+4\tau^2) \geq \frac{4C_0^2M^2+2k^2-4}{|\xi|^2}$. Similarly, for the same reason

$$\|V^*(;j)\|_{(0)}(\partial\Omega_+) \leq C(\Omega)\,\|V^*(;j)\|_{(1)}(\Omega) \leq C(\Omega).$$

Therefore we have that

$$\begin{aligned}
\|u(;j)\|_{(0)}(\partial\Omega_+) &\leq \|e(;j)\|_{(0)}(\partial\Omega_+) + e^{|Im\zeta(j)|}\,\|V(;j)\|_{(0)}(\partial\Omega_+) \\
&\quad + \|e^*(;j)\|_{(0)}(\partial\Omega_+) + e^{|Im\zeta^*(j)|}\,\|V^*(;j)\|_{(0)}(\partial\Omega_+) \\
&\leq |\partial\Omega_+|^{\frac{1}{2}} e^{|Im\zeta(j)|} + C(\Omega)e^{|Im\zeta(j)|} \\
&\quad + |\partial\Omega_+|^{\frac{1}{2}} e^{|Im\zeta^*(j)|} + C(\Omega)e^{|Im\zeta^*(j)|} \\
&= 2(|\partial\Omega_+|^{\frac{1}{2}} + C(\Omega))e^{\sqrt{|\xi|^2(\frac{1}{4}+\tau^2)-k^2}} \\
&= 2C(\Omega)e^{\sqrt{|\xi|^2(\frac{1}{4}+\tau^2)-k^2}}
\end{aligned} \tag{3.2.23}$$

due to (3.2.3). □

3.3 THE PROOF OF INCREASING STABILITY

In this section, we prove the Theorem (3.1.1) which yields stability estimates for the potential c.

Proof of Theorem 3.1.1. Let $c(x) = c_2(x) - c_1(x)$, then we find

$$\begin{aligned} u(x;1)u(x;2) = e^{i(\zeta(1)+\zeta(2))x}&(1+V(;1))(1+V(;2) \\ +e^{i(\zeta^*(1)+\zeta(2))x}&(1+V^*(;1))(1+V(;2)) \\ +e^{i(\zeta(1)+\zeta^*(2))x}&(1+V(;1))(1+V^*(;2)) \\ +e^{i(\zeta^*(1)+\zeta^*(2))x}&(1+V^*(;1))(1+V^*(;2)), \end{aligned} \tag{3.3.1}$$

Substituting $u(x;1)u(x;2)$ into identity (3.1.3) and using triangle inequality give us

$$\begin{aligned} |\int_\Omega c(x)(e^{i\xi x} + e^{i\xi^* x})| \leq |\int_\Omega c(x)(V(;1) + V(;2) + V(;1)V(;2)e^{i\xi x})| \\ + |\int_\Omega c(x)(V^*(;1) + V^*(;2) + V^*(;1)V^*(;2)e^{i\xi x})| \\ +|\int_\Omega c(x)e^{i\xi_{1e}(x_{1e}-2\tau x_3)}(1 + V^*(;1) + V(;2) + V^*(;1)V(;2))| \\ +|\int_\Omega c(x)e^{i\xi_{1e}(x_{1e}+2\tau x_3)}(1 + V(;1) + V^*(;2) + V(;1)V^*(;2))| \\ + \varepsilon \|u(;1)\|_{(0)}(\partial\Omega_+) \|u(;2)\|_{(0)}(\partial\Omega_+), \end{aligned} \tag{3.3.2}$$

where $\varepsilon = \|\Lambda_{c_2} - \Lambda_{c_1}\|$. Sobolev Embedding Theorems and (3.2.6) imply that

$$\|V(;j)\|_\infty(\Omega) \leq C_e \|V(;j)\|_{(2)}(\Omega) \leq C_e C_0(1+\sqrt{2})M. \tag{3.3.3}$$

Therefore, by **Lemma 3.2.1**, (3.3.3) and Hölder's inequality

$$\begin{aligned} &|\int_\Omega c(x)(V(;1) + V(;2) + V(;1)V(;2)e^{i\xi x})| \\ &\leq \int_\Omega |c(x)||V(;1)| + \int_\Omega |c(x)||V(;2)| + \int_\Omega |c(x)||V(;1)||V(;2)| \\ &\leq \|c\|_{(0)}(\Omega)(\|V(;1)\|_{(0)} + \|V(;2)\|_{(0)}) + C_e C_0(1+\sqrt{2})M \int_\Omega |c||V(;1)| \\ &\leq \|c\|_{(0)}(\Omega)(2 + C_e C_0(1+\sqrt{2})M)\frac{2C_0 M}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}}. \end{aligned} \tag{3.3.4}$$

For the same reason

$$\begin{aligned} &|\int_\Omega c(x)(V^*(;1) + V^*(;2) + V^*(;1)V^*(;2)e^{i\xi^* x})| \\ &\leq \|c\|_{(0)}(\Omega)(2 + C_e C_0(1+\sqrt{2})M)\frac{2C_0 M}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}}. \end{aligned} \tag{3.3.5}$$

For the third term on right side of (3.3.2), let $\xi_{1e} = \sqrt{\xi_1^2 + \xi_2^2} \geq 1$, since $c(x) = 0$ near the boundary $\partial\Omega$, we use integration by parts with respect to x_3,

$$\begin{aligned}
&|\int_\Omega c(x)e^{i\xi_{1e}(x_{1e}-2\tau x_3)}(1 + V^*(;1) + V(;2) + V^*(;1)V(;2))| \\
\leq& \int_\Omega |c(x)\frac{\partial}{\partial x_3}(\frac{e^{i\xi_{1e}(x_{1e}-2\tau x_3)}}{2\tau\xi_{1e}})| + \int_\Omega |c(x)(V^*(;1) + V(;2) + V^*(;1)V(;2))| \\
\leq& \frac{1}{2\tau}\int_\Omega |\frac{\partial(c(x))}{\partial x_3}| + \int_\Omega |c(x)V^*(;1)| + \int_\Omega |c(x)V(;2)| + \int_\Omega |c(x)V^*(;1)V(;2)| \\
\leq& \frac{M_1 Vol(\Omega)}{2\tau} + \|c\|_{(0)}(2 + C_e C_0(1+\sqrt{2})M)\frac{2C_0 M}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}}.
\end{aligned} \tag{3.3.6}$$

For the same reason, the forth term is bounded by

$$\begin{aligned}
&|\int_\Omega c(x)e^{i\xi_{1e}(x_{1e}-2\tau x_3)}(1 + V(;1) + V^*(;2) + V(;1)V^*(;2))| \\
\leq& \frac{M_1 Vol(\Omega)}{2\tau} + \|c\|_{(0)}(2 + C_e C_0(1+\sqrt{2})M)\frac{2C_0 M}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}}.
\end{aligned} \tag{3.3.7}$$

Because $c(x)$ is an even function of x_3, by Fourier transform, the left side of (3.3.2) gives us

$$|\int_\Omega c(x)(e^{i\xi x} + e^{i\xi^* x})| = |\int_{\mathbb{R}^3} c(x)e^{i\xi x}| = |\hat{c}|(\xi). \tag{3.3.8}$$

Combining the above results (3.3.3)-(3.3.8), with help of (3.2.22), we obtain the bound for Fourier transform $|\hat{c}|$ in $\xi_{1e} > 1$,

$$\begin{aligned}
|\hat{c}|(\xi) \leq& \|c\|_{(0)}(\Omega)(2 + C_e C_0(1+\sqrt{2})M)\frac{8C_0 M}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}} \\
&+ \frac{M_1 Vol(\Omega)}{\tau} + \varepsilon C_\Omega^2 e^{\sqrt{|\xi|^2(1+4\tau^2)-4k^2}} \\
\leq& C(\Omega, M)(\frac{1}{\sqrt{|\xi|^2(1+4\tau^2) - 2k^2 + 4}} + \frac{1}{\tau} + \varepsilon e^{\sqrt{|\xi|^2(1+4\tau^2)-4k^2}})
\end{aligned} \tag{3.3.9}$$

for some constant $C(\Omega, M)$. Taking square on both sides of (3.3.9), and using elementary inequality $(a+b+c)^2 \leq 4a^2 + 4b^2 + 2c^2$, then choose $1 + 4\tau^2 = \frac{4k^2 + (\frac{E}{2})^2}{|\xi|^2}$, we obtain

$$\begin{aligned}
|\hat{c}|^2(\xi) \leq& C(\Omega, M)(\frac{1}{2k^2 + (\frac{E}{2})^2 + 4} + \frac{|\xi|^2}{4k^2 + (\frac{E}{2})^2 - |\xi|^2} + \varepsilon) \\
\leq& C(\Omega, M)(\frac{1}{2k^2 + (\frac{E}{2})^2 + 4} + \frac{\rho^2}{4k^2 + (\frac{E}{2})^2 - 2\rho^2} + \varepsilon)
\end{aligned} \tag{3.3.10}$$

for $1 < |\xi_{1e}| < \rho$ and $|\xi_3| < \rho$.

Now, writing $f(\xi_{1e}) = \hat{c}^2(\xi_{1e}, \xi_3) = (\int_{\mathbb{R}^3} c(x) e^{i(x_{1e}\xi_{1e}+x_3\xi_3)})^2$, one can check that $f(\xi_{1e})$ is an entire function if we extend $\xi_{1e} = r_1 + ir_2$ onto $\mathbb{C}$. Assume $\Gamma_1 = \{0 < r_1 < 1, r_2 = 0\}$, $\Omega_1 = \{|r_i| < 2, i = 1, 2\}$, and $\Gamma_2 = \{1 < r_1 < 2, r_2 = 0\}$ in $\mathbb{C}$. Applying the corollary 1.2.2 [4] for analytic function $f(\xi_{1e})$, $\xi_{1e} \in \mathbb{C}$, we obtain the inequality

$$\sup_{\Gamma_1} |f(\xi_{1e})| \leq C(\sup_{\Omega_1} |f(\xi_{1e})|)^{1-\lambda}(\sup_{\Gamma_2} |f(\xi_{1e})|)^{\lambda} \tag{3.3.11}$$

for some $0 < \lambda < 1$, and constant C. The bound of $\sup_{\Gamma_2} |f(\xi_{1e})|$ is given by (3.3.10). Also by the definition of f, we have $\sup_{\Omega_1} |f(\xi_{1e})| \leq Me^2$. Therefore, (3.3.11) implies, for $\xi_{1e} < 1$ and $|\xi_3| < \rho$

$$\begin{aligned} |\hat{c}|^2(\xi) \leq& \sup_{\Gamma_1} |f(\xi_{1e})| \leq CMe^2(\sup_{\Gamma_2} |f(\xi_{1e})|)^{\lambda} \\ &\leq C(\Omega, M)[\frac{1}{(2k^2 + (\frac{E}{2})^2 + 4)^{\lambda}} + \frac{\rho^{2\lambda}}{(4k^2 + (\frac{E}{2})^2 - 2\rho^2)^{\lambda}} + \varepsilon^{\lambda}]. \end{aligned} \tag{3.3.12}$$

Adding the bounds of (3.3.10) and (3.3.12) gives us the bound for $|\hat{c}|^2$ in the domain $|\xi| < \rho$

$$|\hat{c}|^2(\xi) \leq C(\Omega, M)[\frac{1}{(2k^2 + (\frac{E}{2})^2 + 4)^{\lambda}} + \frac{\rho^{2\lambda}}{(4k^2 + (\frac{E}{2})^2 - 2\rho^2)^{\lambda}} + \varepsilon^{\lambda}], \tag{3.3.13}$$

where we have dropped all the terms in (3.3.10) due to their lower orders.

Now we take integral on both sides of (3.3.13) over $|\xi| < \rho$. Let $|\xi| = r$ and choose $\rho = (E + k)^{\frac{\lambda}{2\lambda+3}}$. It follows that

$$\begin{aligned} \int_{|\xi|<\rho} |\hat{c}|^2(\xi) &\leq C(\Omega, M)[\int_0^{\rho} \frac{r^2 dr}{(2k^2 + (\frac{E}{2})^2 + 4)^{\lambda}} + \int_0^{\rho} \frac{\rho^{2\lambda} r^2 dr}{(4k^2 + (\frac{E}{2})^2 - 2\rho^2)^{\lambda}} + \int_0^{\rho} \varepsilon^{\lambda} r^2 dr] \\ &\leq C(\Omega, M)[\frac{\rho^3}{(2k^2 + (\frac{E}{2})^2 + 4)^{\lambda}} + \frac{\rho^{2\lambda+3}}{(4k^2 + (\frac{E}{2})^2 - 2\rho^2)^{\lambda}} + \varepsilon^{\lambda}\rho^3] \\ &\leq C(\Omega, M)[\frac{(E + k)^{\lambda}}{(4k^2 + (\frac{E}{2})^2)^{\lambda}} + \varepsilon^{\lambda}(E + k)^{\frac{3\lambda}{2\lambda+3}}], \end{aligned} \tag{3.3.14}$$

where the first term on right side gets absorbed by the second one due to its lower order.

Finally, in the domain $|\xi| > \rho$, we use properties of Fourier transform,

$$\begin{aligned} \int_{\rho<|\xi|} |\hat{c}|^2(\xi) \leq& \frac{1}{1 + \rho^2} \int_{\mathbb{R}^3} (1 + |\xi|^2)|\hat{c}|^2(\xi) d\xi = \frac{1}{1 + \rho^2} \|c\|^2_{(1)}(\Omega) \\ &\leq \frac{4}{1 + \rho^2} M^2 = \frac{4M^2}{1 + (E + k)^{\frac{2\lambda}{2\lambda+3}}}. \end{aligned} \tag{3.3.15}$$

Adding inequalities (3.3.14) and (3.3.15), then using parseval's identity, we obtain the bound for $\|c\|_{(0)}(\Omega)$. Note that the first term in the bound (3.3.14) is lower order comparing to the bound (3.3.15) due to $0 < \lambda < 1$, hence it gets absorbed. Therefore

$$\|c_1 - c_2\|^2_{(0)}(\Omega) \leq C(\Omega, M)\varepsilon^{\lambda}(E+k)^{\frac{3\lambda}{2\lambda+3}} + \frac{C(\Omega, M)}{(E+k)^{\frac{2\lambda}{2\lambda+3}}} \tag{3.3.16}$$

for some constant $C(\Omega, M)$. Let $\lambda_1 = \frac{3\lambda}{2\lambda+3}$, then (3.3.16) implies (3.1.2). This proves the theorem 3.1.1.

3.4 THE CAUCHY BOUNDARY DATA

Some people argue that, since the eigenvalues for the problems (2.1.1)(2.1.3) are unknown, as $k \to \infty$, it hits infinitely many of them and we do not know when this happens. Thus the assumption that k^2 is not an eigenvalue is quite inconvenient. To avoid the spectral issue, one can replace the boundary condition (2.1.2)(2.1.3) by the Cauchy data,

$$\begin{gathered} S_c = \{(u, \partial_\nu u) \in H^{\frac{1}{2}}(\partial\Omega_+) \times H^{-\frac{1}{2}}(\partial\Omega_+)\}, \\ \partial_\nu u = 0 \text{ on } \Gamma, \end{gathered} \tag{3.4.1}$$

where $\partial\Omega_+ = \partial\Omega \cap \{x_3 > 0\}$ and the potential $c \in C^1(\Omega)$. Denote $\|.\|_{(s)}(\Omega)$ the norm for the Sobolev space $H^s(\Omega)$. Let S_{c_1} and S_{c_2} be two Cauchy data corresponding to c_1 and c_2 respectively. To measure the distance between two Cauchy data, we define

$$\begin{aligned} \text{dist}(S_{c_1}, S_{c_2}) = \max\{ &\max_{(f,g)\in S_{c_1}} \min_{(\widetilde{f},\widetilde{g})\in S_{c_2}} \frac{\left\|(f,g) - (\widetilde{f},\widetilde{g})\right\|_{(\frac{1}{2},-\frac{1}{2})}}{\|(f,g)\|_{(\frac{1}{2},-\frac{1}{2})}}, \\ &\max_{(f,g)\in S_{c_2}} \min_{(\widetilde{f},\widetilde{g})\in S_{c_1}} \frac{\left\|(f,g) - (\widetilde{f},\widetilde{g})\right\|_{(\frac{1}{2},-\frac{1}{2})}}{\|(f,g)\|_{(\frac{1}{2},-\frac{1}{2})}}\} \end{aligned}$$

where

$$\|(f,g)\|_{(\frac{1}{2},-\frac{1}{2})} = (\|f\|^2_{(\frac{1}{2})}(\partial\Omega_+) + \|g\|^2_{(-\frac{1}{2})}(\partial\Omega_+))^{\frac{1}{2}}. \tag{3.4.2}$$

The main result (3.1.2) still holds under above assumptions with different choices of λ_1 and λ_2. The following theorem gives details.

Theorem 3.4.1 *Let*

$$\|c_j\|_\infty(\Omega) \le M_0,\ \|\nabla c_j\|_\infty(\Omega) \le M_1,\ j=1,2,\ c=0 \ near\ \partial\Omega_+, \tag{3.4.3}$$

and $\mathrm{dist}(S_{c_1}, S_{c_2}) = \varepsilon$, $E = -log\varepsilon$, $E > 0$, *then there are constants* $C(\Omega, M)$, $0 < \lambda < 1$ *and* $0 < \lambda_i < 5$, $i = 1,2$ *such that*

$$\|c_1 - c_2\|^2_{(0)}(\Omega) \le C(\Omega, M)\varepsilon^\lambda (E+k)^{\lambda_1} + \frac{C(\Omega, M)}{(E+k)^{\lambda_2}}, \tag{3.4.4}$$

where $M = \sqrt{M_0^2 + M_1^2}$.

For a given ε, one can minimize the bound of (3.4.4) with respect to k. We found that, at the minimum point $k = (\frac{\lambda_2}{\lambda_1}\varepsilon^{-\lambda})^{\frac{1}{\lambda_1+\lambda_2}} - E$, the right hand side $C(\Omega, M)[\varepsilon^\lambda (E+k)^{\lambda_1} + \frac{1}{(E+k)^{\lambda_2}}] = C(\Omega, M)\varepsilon^{\frac{\lambda\lambda_2}{\lambda_1+\lambda_2}}$, which is Hölder continuous in ε. Thus, as k grows in the zone $k < (\frac{\lambda_2}{\lambda_1}\varepsilon^{-\lambda})^{\frac{1}{\lambda_1+\lambda_2}} - E$, the bound decreases and become more like Hölder type when k approaches to the minimum point. This fact shows increasing stability phenomenon in the above zone.

The following lemmas will be used in the proof of above theorem.

Lemma 3.4.2 *For all solutions* $u_j \in H^1(\Omega)$ *satisfying*

$$-\Delta u_j - k^2 u_j + c_j u_j = 0 \ in\ \Omega,\ j = 1,2$$

with the the Cauchy boundary condition (3.4.1), we have

$$|\int_\Omega (c_2 - c_1) u_1 u_2| \le \|(u_1, \partial_\nu u_1)\|_{(\frac{1}{2}, -\frac{1}{2})} \|(u_2, \partial_\nu u_2)\|_{(\frac{1}{2}, -\frac{1}{2})} \mathrm{dist}(S_{c_1}, S_{c_2}). \tag{3.4.5}$$

Proof: Define $\nu(u_j)$ on $\partial\Omega_+$ in the weak form as

$$< \nu(u_j) \mid u_k >= \int_\Omega \nabla u_j \cdot \nabla u_k + c_j u_j u_k,$$

for $j, k \in \{1, 2\}$, where $< \mid >$ denotes the duality between $H_{-\frac{1}{2}}(\partial\Omega_+)$ and $H_{\frac{1}{2}}(\partial\Omega_+)$. It is clear that

$$< \nu(u_1) \mid u_2 > - < \nu(u_2) \mid u_1 >= \int_\Omega (c_2 - c_1) u_2 u_1.$$

Now, since $< \partial_\nu u_2 \mid u_2 > - < \nu(u_2) \mid u_2 >= 0$ for any $(u_2, \partial_\nu u_2) \in S_{c_2}$, we have that

$$< \nu(u_1) - \partial_\nu u_2 \mid u_2 > - < \nu(u_2) \mid u_1 - u_2 >= \int_\Omega (c_2 - c_1) u_2 u_1. \tag{3.4.6}$$

By the definition of $\text{dist}(S_{c_1}, S_{c_2})$, this gives the desired result. □

Lemma 3.4.3 *Let $u(x; j)$ be the solutions (3.2.1) to the Schrödinger equation $-\Delta u_j - k^2 u_j + c_j u_j = 0$ in Ω, then we have*

$$\|(u(;j), \partial_\nu u(;j)\|_{(\frac{1}{2},-\frac{1}{2})} \leq C(\Omega)(\sqrt{5 + |\xi|^2(1 + 4\tau^2) - 2k^2} e^{\sqrt{|\xi|^2(\frac{1}{4}+\tau^2)-k^2}}). \tag{3.4.7}$$

Proof: Let $e(x; j) = e^{i\zeta(j)\cdot x}$, and $e^*(x; j) = e^{i\zeta^*(j)\cdot x}$. Due to (**??**) and the condition $(1 + 4\tau^2) \geq \frac{4C_0^2 M^2 + 2k^2 - 4}{|\xi|^2}$, we have $\|V(;j)\|_{(1)}(\Omega) \leq 1$. Hence by the definition of Sobolev norm,

$$\begin{aligned}
&\|e(;j)(1 + V(;j))\|_{(1)}^2(\Omega) \\
=&(\int_\Omega |e(;j)|^2 |1 + V(;j)|^2 + \sum_{i=1}^3 \int_\Omega |\partial_i e(;j)|^2 |1 + V(;j)|^2 \\
&+ \sum_{i=1}^3 \int_\Omega |e(;j)|^2 |\partial_i V(;j)|^2) \leq C(\Omega)(e^{2|Im\zeta(j)|}(1 + |\zeta(j)|^2)).
\end{aligned}$$

Similarly, for the same reason

$$\|e^*(;j)(1 + V^*(;j))\|_{(1)}^2(\Omega) \leq C(\Omega)(e^{2|Im\zeta(j)|}(1 + |\zeta(j)|^2)).$$

Therefore, by (3.4.2) and the trace theorem

$$\begin{aligned}
&\|(u(;j), \partial_\nu u(;j)\|_{(\frac{1}{2},-\frac{1}{2})} \leq C(\Omega) \|u(;j)\|_{(1)}(\Omega) \\
&\leq C(\Omega)(\|e(;j)(1 + V(;j))\|_{(1)}(\Omega) + \|e^*(;j)(1 + V^*(;j))\|_{(1)}(\Omega)) \\
&\leq C(\Omega)(\sqrt{5 + |\xi|^2(1 + 4\tau^2) - 2k^2} e^{\sqrt{|\xi|^2(\frac{1}{4}+\tau^2)-k^2}}).
\end{aligned} \tag{3.4.8}$$

The proof is completed. □

Proof (Theorem 3.4.1): Let $c(x) = c_2(x) - c_1(x)$, then we find

$$\begin{aligned}
u(x;1)u(x;2) =& e^{i(\zeta(1)+\zeta(2))x}(1 + V(;1))(1 + V(;2) \\
&+ e^{i(\zeta^*(1)+\zeta(2))x}(1 + V^*(;1))(1 + V(;2)) \\
&+ e^{i(\zeta(1)+\zeta^*(2))x}(1 + V(;1))(1 + V^*(;2)) \\
&+ e^{i(\zeta^*(1)+\zeta^*(2))x}(1 + V^*(;1))(1 + V^*(;2)),
\end{aligned}$$

Substituting $u(x;1)u(x;2)$ into identity (3.4.7) and using triangle inequality give us

$$\begin{aligned}
&|\int_\Omega c(x)(e^{i\xi x}+e^{i\xi^* x})| \\
\le &|\int_\Omega c(x)(V(;1)+V(;2)+V(;1)V(;2)e^{i\xi x})| \\
&+|\int_\Omega c(x)(V^*(;1)+V^*(;2)+V^*(;1)V^*(;2)e^{i\xi x})| \\
&+|\int_\Omega c(x)e^{i\xi_{1e}(x_{1e}-2\tau x_3)}(1+V^*(;1)+V(;2)+V^*(;1)V(;2))| \\
&+|\int_\Omega c(x)e^{i\xi_{1e}(x_{1e}+2\tau x_3)}(1+V(;1)+V^*(;2)+V(;1)V^*(;2))| \\
&+\varepsilon\left\|(u_1,\partial_\nu u_1)\right\|_{(\frac{1}{2},-\frac{1}{2})}\left\|(u_2,\partial_\nu u_2)\right\|_{(\frac{1}{2},-\frac{1}{2})}.
\end{aligned} \tag{3.4.9}$$

Sobolev Embedding Theorems and (3.2.6) imply that

$$\|V(;j)\|_\infty(\Omega)\le C_e\|V(;j)\|_{(2)}(\Omega)\le C_eC_0(1+\sqrt{2})M. \tag{3.4.10}$$

Therefore, by the bounds of $\|V(;j)\|_{(0)}$ given in **Lemma** 3.2.1

$$\begin{aligned}
&|\int_\Omega c(x)(V(;1)+V(;2)+V(;1)V(;2)e^{i\xi x})| \\
\le &\int_\Omega |c(x)||V(;1)|+\int_\Omega |c(x)||V(;2)|+\int_\Omega |c(x)||V(;1)||V(;2)| \\
\le &\|c\|_{(0)}(\Omega)(\|V(;1)\|_{(0)}+\|V(;2)\|_{(0)})+C_eC_0(1+\sqrt{2})M\int_\Omega |c||V(;1)| \\
\le &\|c\|_{(0)}(\Omega)(2+C_eC_0(1+\sqrt{2})M)\frac{2C_0M}{\sqrt{|\xi|^2(1+4\tau^2)-2k^2+4}}.
\end{aligned} \tag{3.4.11}$$

For the same reason

$$\begin{aligned}
&|\int_\Omega c(x)(V^*(;1)+V^*(;2)+V^*(;1)V^*(;2)e^{i\xi^* x})| \\
\le &\|c\|_{(0)}(\Omega)(2+C_eC_0(1+\sqrt{2})M)\frac{2C_0M}{\sqrt{|\xi|^2(1+4\tau^2)-2k^2+4}}.
\end{aligned} \tag{3.4.12}$$

For the third term on right side of (3.4.9), let $\xi_{1e}=\sqrt{\xi_1^2+\xi_2^2}\ge 1$, since $c(x)=0$ near the boundary $\partial\Omega$, we use integration by parts with respect to x_3,

$$
\begin{aligned}
&|\int_\Omega c(x)e^{i\xi_{1e}(x_{1e}-2\tau x_3)}(1+V^*(;1)+V(;2)+V^*(;1)V(;2))|\\
\leq&\int_\Omega |c(x)\frac{\partial}{\partial x_3}(\frac{e^{i\xi_{1e}(x_{1e}-2\tau x_3)}}{2\tau\xi_{1e}})|+\int_\Omega |c(x)(V^*(;1)+V(;2)+V^*(;1)V(;2))|\\
\leq&\frac{1}{2\tau}\int_\Omega|\frac{\partial(c(x))}{\partial x_3}|+\int_\Omega|c(x)V^*(;1)|+\int_\Omega|c(x)V(;2)|+\int_\Omega|c(x)V^*(;1)V(;2)|\\
\leq&\frac{M_1Vol(\Omega)}{2\tau}+\|c\|_{(0)}(2+C_eC_0(1+\sqrt{2})M)\frac{2C_0M}{\sqrt{|\xi|^2(1+4\tau^2)-2k^2+4}}.
\end{aligned}
\tag{3.4.13}
$$

For the same reason, the forth term is bounded by

$$
\begin{aligned}
&|\int_\Omega c(x)e^{i\xi_{1e}(x_{1e}-2\tau x_3)}(1+V(;1)+V^*(;2)+V(;1)V^*(;2))|\\
\leq&\frac{M_1Vol(\Omega)}{2\tau}+\|c\|_{(0)}(2+C_eC_0(1+\sqrt{2})M)\frac{2C_0M}{\sqrt{|\xi|^2(1+4\tau^2)-2k^2+4}}.
\end{aligned}
\tag{3.4.14}
$$

Because $c(x)$ is an even function of x_3, by Fourier transform, the left side of (3.4.9) gives us

$$
|\int_\Omega c(x)(e^{i\xi x}+e^{i\xi^* x})|=|\int_{\mathbb{R}^3}c(x)e^{i\xi x}|=|\hat{c}|(\xi).
\tag{3.4.15}
$$

Combining the above results (3.4.11)-(3.4.15), with help of (3.4.7), we obtain the bound for Fourier transform $|\hat{c}|$ in $\xi_{1e}>1$,

$$
\begin{aligned}
|\hat{c}|(\xi)\leq&\|c\|_{(0)}(\Omega)(2+C_eC_0(1+\sqrt{2})M)\frac{8C_0M}{\sqrt{|\xi|^2(1+4\tau^2)-2k^2+4}}\\
&+\frac{M_1Vol(\Omega)}{\tau}+\varepsilon C(\Omega)(5+|\xi|^2(1+4\tau^2)-2k^2)e^{\sqrt{|\xi|^2(1+4\tau^2)-4k^2}}\\
\leq&C(\Omega,M)(\frac{1}{\sqrt{|\xi|^2(1+4\tau^2)-2k^2+4}}+\frac{1}{\tau}\\
&+\varepsilon(5+|\xi|^2(1+4\tau^2)-2k^2)e^{\sqrt{|\xi|^2(1+4\tau^2)-4k^2}})
\end{aligned}
\tag{3.4.16}
$$

for some constant $C(\Omega,M)$. Taking square on both sides of (3.4.16), and using elementary inequality $(a+b+c)^2\leq 4a^2+4b^2+2c^2$, then choose $1+4\tau^2=\frac{4k^2+(\frac{E}{2})^2}{|\xi|^2}$, we obtain

$$
\begin{aligned}
|\hat{c}|^2(\xi)\leq&C(\Omega,M)[\frac{1}{2k^2+(\frac{E}{2})^2+4}+\frac{|\xi|^2}{4k^2+(\frac{E}{2})^2-|\xi|^2}+\varepsilon(5+2k^2+\frac{E^2}{4})^2]\\
\leq&C(\Omega,M)[\frac{1}{2k^2+(\frac{E}{2})^2+4}+\frac{\rho^2}{4k^2+(\frac{E}{2})^2-2\rho^2}+\varepsilon(5+2k^2+\frac{E^2}{4})^2]
\end{aligned}
\tag{3.4.17}
$$

for $1<|\xi_{1e}|<\rho$ and $|\xi_3|<\rho$.

Now, writing $f(\xi_{1e}) = \hat{c}^2(\xi_{1e}, \xi_3) = (\int_{\mathbb{R}^3} c(x)e^{i(x_{1e}\xi_{1e}+x_3\xi_3)})^2$, one can check that $f(\xi_{1e})$ is an entire function if we extend $\xi_{1e} = r_1 + ir_2$ onto $\mathbb{C}$. Assume $\Gamma_1 = \{0 < r_1 < 1, r_2 = 0\}$, $\Omega_1 = \{|r_i| < 2, i = 1, 2\}$, and $\Gamma_2 = \{1 < r_1 < 2, r_2 = 0\}$ in $\mathbb{C}$. Applying the corollary 1.2.2 [4] for analytic function $f(\xi_{1e})$, $\xi_{1e} \in \mathbb{C}$, we obtain the inequality

$$\sup_{\Gamma_1} |f(\xi_{1e})| \leq C(\sup_{\Omega_1} |f(\xi_{1e})|)^{1-\lambda}(\sup_{\Gamma_2} |f(\xi_{1e})|)^{\lambda} \tag{3.4.18}$$

for some $0 < \lambda < 1$, and constant C. The bound of $\sup_{\Gamma_2}|f(\xi_{1e})|$ is given by (3.4.17). Also by the definition of f, we have $\sup_{\Omega_1} |f(\xi_{1e})| \leq Me^2$. Therefore, (3.4.18) implies, for $\xi_{1e} < 1$ and $|\xi_3| < \rho$

$$\begin{aligned} |\hat{c}|^2(\xi) &\leq \sup_{\Gamma_1} |f(\xi_{1e})| \leq CMe^2(\sup_{\Gamma_2} |f(\xi_{1e})|)^{\lambda} \\ &\leq C(\Omega, M)[\frac{1}{(2k^2 + (\frac{E}{2})^2 + 4)^{\lambda}} + \frac{\rho^{2\lambda}}{(4k^2 + (\frac{E}{2})^2 - 2\rho^2)^{\lambda}} + \varepsilon^{\lambda}(5 + 2k^2 + \frac{E^2}{4})^{2\lambda}]. \end{aligned} \tag{3.4.19}$$

Adding the bounds of (3.4.17) and (3.4.19) gives us the bound for $|\hat{c}|^2$ in the domain $|\xi| < \rho$

$$\begin{aligned} |\hat{c}|^2(\xi) \leq &C(\Omega, M)[\frac{1}{(2k^2 + (\frac{E}{2})^2 + 4)^{\lambda}} + \frac{\rho^{2\lambda}}{(4k^2 + (\frac{E}{2})^2 - 2\rho^2)^{\lambda}} \\ &+ \varepsilon^{\lambda}(5 + 2k^2 + \frac{E^2}{4})^{2\lambda}], \end{aligned} \tag{3.4.20}$$

where we have dropped all terms in (3.4.17) due to their lower orders.

Now we take integral on both sides of (3.4.20) over $|\xi| < \rho$. Let $|\xi| = r$ and choose $\rho = (E + k)^{\frac{\lambda}{2\lambda+3}}$. It follows that

$$\begin{aligned} \int_{|\xi|<\rho} |\hat{c}|^2(\xi) \leq &C(\Omega, M)[\int_0^{\rho} \frac{r^2 dr}{(2k^2 + (\frac{E}{2})^2 + 4)^{\lambda}} + \int_0^{\rho} \frac{\rho^{2\lambda} r^2 dr}{(4k^2 + (\frac{E}{2})^2 - 2\rho^2)^{\lambda}} \\ &+ \int_0^{\rho} \varepsilon^{\lambda}(5 + 2k^2 + \frac{E^2}{4})^{2\lambda} r^2 dr] \leq C(\Omega, M)[\frac{\rho^3}{(2k^2 + (\frac{E}{2})^2 + 4)^{\lambda}} \\ &+ \frac{\rho^{2\lambda+3}}{(4k^2 + (\frac{E}{2})^2 - 2\rho^2)^{\lambda}} + \varepsilon^{\lambda}(5 + 2k^2 + \frac{E^2}{4})^{2\lambda}\rho^3] \\ \leq &C(\Omega, M)[\frac{(E + k)^{\lambda}}{(4k^2 + (\frac{E}{2})^2)^{\lambda}} + \varepsilon^{\lambda}(E + k)^{\frac{8\lambda^2+15\lambda}{2\lambda+3}}], \end{aligned} \tag{3.4.21}$$

where the first term on right side gets absorbed by the second one due to its lower order.

Finally, in the domain $|\xi| > \rho$, we use properties of Fourier transform,

$$\begin{aligned}\int_{\rho<|\xi|} |\hat{c}|^2(\xi) &\leq \frac{1}{1+\rho^2}\int_{\mathbb{R}^3}(1+|\xi|^2)|\hat{c}|^2(\xi)d\xi = \frac{1}{1+\rho^2}\left\|c\right\|^2_{(1)}(\Omega) \\ &\leq \frac{4}{1+\rho^2}M^2 = \frac{4M^2}{1+(E+k)^{\frac{2\lambda}{2\lambda+3}}}.\end{aligned} \tag{3.4.22}$$

Adding inequalities (3.4.21) and (3.4.22), then using Parseval's identity, we obtain the bound for $\|c\|_{(0)}(\Omega)$. Note that the first term in the bound (3.4.21) is lower order comparing to the bound (3.4.22) due to $0 < \lambda < 1$, hence it gets absorbed. Therefore

$$\|c_1 - c_2\|^2_{(0)}(\Omega) \leq C(\Omega, M)\varepsilon^{\lambda}(E+k)^{\frac{8\lambda^2+15\lambda}{2\lambda+3}} + \frac{C(\Omega, M)}{(E+k)^{\frac{2\lambda}{2\lambda+3}}} \tag{3.4.23}$$

for some constant $C(\Omega, M)$. Let $\lambda_1 = \frac{8\lambda^2+15\lambda}{2\lambda+3}$ and $\lambda_2 = \frac{3\lambda}{2\lambda+3}$, then (3.4.23) implies (4.0.8). This proves Theorem 3.4.1. □

CHAPTER 4

THE LINEARIZED INVERSE PROBLEM FOR THE SCHRÖDINGER EQUATION

In the view of equation (2.1.1), we notice that, since u depends on c, the inverse problem of solving c is non-linear and not convex. Thus a good approach is to consider linearized inverse problem. Such approach is useful from two point of view. First, it often occurs that a coefficient of a differential equation differs little from a known function. The problem linearized in the neighbourhood of this function may give satisfactory results for application. On the other hand, the algorithm solving full, non-linearized inverse problem would be too difficult and time consuming in real life. Therefore, for numerical computation, it is sufficient to use linearize inverse problem for the Schrödinger equation. The following proof justifies the linearization by assuming that c is a small perturbation.

Let $\Omega = \{x \in \mathbb{R}^3 | 0 < x_i < \pi,\ i = 1, 2, 3\}$. Consider the Schrödinger equation

$$-\Delta u - k^2 u + cu = 0 \text{ in } \Omega, \tag{4.0.1}$$

with the boundary data

$$\partial_\nu u = g \in H^{-\frac{1}{2}}(\partial\Omega). \tag{4.0.2}$$

Introduce a function u_0 which satisfies

$$\begin{aligned} -\Delta u_0 - k^2 u_0 &= 0 \text{ in } \Omega, \\ \partial_\nu u_0 &= g \text{ on } \partial\Omega. \end{aligned} \tag{4.0.3}$$

By [26], pp. 160-162,

$$\|u_0\|_{(1)}(\Omega) < C\, \|g\|_{(-\frac{1}{2})}(\partial\Omega), \tag{4.0.4}$$

where C is a constant depending on k and Ω. Let $u_2 = u - u_0$. Subtracting (4.0.3) from (4.0.1) results in

$$\begin{aligned} -\Delta u_2 - k^2 u_2 &= -cu \text{ in } \Omega, \\ \partial_\nu u_2 &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.0.5}$$

We are interested in comparing u_2 with the function u_1 which solves the linearised Schrödinger equation

$$
\begin{aligned}
-\Delta u_1 - k^2 u_1 &= -c u_0 \text{ in } \Omega, \\
\partial_\nu u_1 &= 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{4.0.6}
$$

If $\|c\|_\infty < \delta$, then

$$
\|u_1\|_{(1)}(\Omega) \leq C\delta \|g\|_{(-\frac{1}{2})}(\partial\Omega) \tag{4.0.7}
$$

due to [26], pp. 160-162 and (4.0.4). Now, subtracting (4.0.6) from (4.0.5) gives

$$
\begin{aligned}
-\Delta(u_2 - u_1) - k^2(u_2 - u_1) + c(u_2 - u_1) &= -c u_1 \text{ in } \Omega, \\
\partial_\nu (u_2 - u_1) &= 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{4.0.8}
$$

By [26], pp. 160-162 and (4.0.7), we have $\|u_2 - u_1\|_{(1)}(\Omega) \leq C\delta^2 \|g\|_{(-\frac{1}{2})}(\partial\Omega)$. This justifies the linearisation (replacement of u_2 by u_1). From now on, we will use the linearised Schrödinger equation (4.0.6).

Suppose that k^2 is not eigenvalue for (4.0.6). Define the Neumann-to-Dirichlet map

$$
\Lambda : g \longrightarrow u_1 \text{ on } \partial\Omega, \tag{4.0.9}
$$

which is a bounded operator from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$ due to (4.0.7).

Theorem 4.0.4 *Let*

$$
\|c\|_\infty(\Omega) \leq M_0, \ \|\nabla c\|_\infty(\Omega) \leq M_1 \tag{4.0.10}
$$

and $\varepsilon = \|\Lambda\|$, $E = -ln\varepsilon$, $E > 0$, *then*

$$
\|c\|^2_{(0)}(\Omega) \leq \frac{4}{3} k^4 \pi^4 \varepsilon (E + k)^3 + \frac{4M^2}{(E + k)^2}, \tag{4.0.11}
$$

where $M = \sqrt{M_0 + M_1}$.

The bound in right hand side of (4.0.11) suggests that there is a optimal choice of k for given ε. Indeed, by taking derivative, we found that, at the approximate minimum point

$k_0 = (\frac{21M^2}{20\pi^4\varepsilon})^{\frac{1}{9}} - E$, the bound $\frac{4}{3}k_0^4\pi^4\varepsilon(E+k_0)^3 + \frac{4M^2}{(E+k_0)^2} < C(\Omega, M)\varepsilon^{\frac{4}{9}}$, which is Hölder continuous in ε. Thus the bound decreases in the zone $k \in (0, k_0)$ and shows increasing stability phenomenon there. If $(\frac{21M^2}{20\pi^4\varepsilon})^{\frac{1}{9}} < E$, then $\varepsilon > \alpha > 0$ for some constant α, and Hölder stability still holds in this case.

We need some lemmas to prove Theorem 4.0.4. First introduce a function v that solves the equations

$$\begin{aligned} -\Delta v - k^2 v = 0 \text{ in } \Omega, \\ \partial_\nu v = g_1 \text{ on } \partial\Omega. \end{aligned} \tag{4.0.12}$$

The following lemma follows from the simple application of Green's theorem.

Lemma 4.0.5 *For solutions $u_1, v \in H^1(\Omega)$ satisfying (4.0.6),(4.0.12), we have*

$$\int_\Omega c u_0 v = \int_{\partial\Omega} (\Lambda(\partial_\nu u_0))\partial_\nu v. \tag{4.0.13}$$

Lemma 4.0.6 *Let $\xi \in \mathbb{R}^3$, $(1+4\tau^2) \geq \frac{4k^2}{|\xi|^2}$, then there are exponential solutions*

$$u_0 = e^{i\zeta(1)\cdot x},\ v = e^{i\zeta(2)\cdot x} \tag{4.0.14}$$

to the equations

$$-\Delta u_0 - k^2 u_0 = 0,\ -\Delta v - k^2 v = 0 \ \textit{in}\ \Omega \tag{4.0.15}$$

with

$$\begin{aligned} \zeta(1) + \zeta(2) = \xi,\ \zeta(1)\cdot\zeta(1) = \zeta(2)\cdot\zeta(2) = k^2 \\ |Im\zeta(1)| = |Im\zeta(2)| = \sqrt{|\xi|^2(\frac{1}{4}+\tau^2) - k^2} \end{aligned} \tag{4.0.16}$$

in addition,

$$\|\partial_\nu u_0\|_{(0)}(\partial\Omega) \leq \sqrt{2}k\pi e^{|Im\zeta(1)|},\ \|\partial_\nu v\|_{(0)}(\partial\Omega) \leq \sqrt{2}k\pi e^{|Im\zeta(2)|} \tag{4.0.17}$$

Proof: Suppose $\xi \in \mathbb{R}^3$, $\xi \neq 0$. We define a coordinate system $(x_{1e}, x_{2e}, x_{3e})_e$ in the following way. $e(1) = (\xi_1^2+\xi_2^2)^{-\frac{1}{2}}(\xi_1, \xi_2, 0)$, $e(3) = (0,0,1)$, $e(2)$ is chosen to form an orthonormal basis

$e(1), e(2), e(3)$ in $\mathbb{R}^3$. Denote $\xi_{1e} = \sqrt{\xi_1^2 + \xi_2^2}$. Now, let

$$\zeta(1) = (\frac{\xi_{1e}}{2} - \tau\xi_3, i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, \frac{\xi_3}{2} + \tau\xi_{1e})_e, \tag{4.0.18}$$

$$\zeta(2) = (\frac{\xi_{1e}}{2} + \tau\xi_3, -i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, \frac{\xi_3}{2} - \tau\xi_{1e})_e, \tag{4.0.19}$$

where τ is a positive real number. Then one can check that (4.0.15)(4.0.16) are satisfied for the above choices of $\zeta(1)$, $\zeta(2)$. (4.0.17) follows by direct computation of L^2 norm of $\partial_\nu u_0$, $\partial_\nu v$ in the domain $\Omega = \{x \in \mathbb{R}^3 | 0 < x_i < \pi,\ i = 1, 2, 3\}$. □

Now, we are ready to prove Theorem 4.0.4, which gives the stability estimate for potential c with full data.

Proof (Theorem 4.0.4): For u_0, v defined in (4.0.14), substituting them into identity (4.0.13) and using Cauchy Schwarz inequality results in

$$|\int_\Omega c(x)e^{i\xi x}| < \varepsilon \left\|\partial_\nu u_0\right\|_{(0)} \left\|\partial_\nu v\right\|_{(0)} < 2\varepsilon k^2\pi^2 e^{\sqrt{|\xi|^2(1+4\tau)^2-4k^2}}, \tag{4.0.20}$$

where $\varepsilon = \|\Lambda\|$, and the second inequality is due to (4.0.17). Assume $c(x) = 0$ outside of Ω, by the definition of Fourier transform, the left side of (4.0.20) implies

$$|\int_\Omega c(x)e^{i\xi x}| = |\int_{\mathbb{R}^3} c(x)e^{i\xi x}| = |\hat{c}|(\xi). \tag{4.0.21}$$

Let $1 + 4\tau^2 = \frac{4k^2+(\frac{E}{2})^2}{|\xi|^2}$, where $E = -ln\varepsilon$. Then (4.0.20)(4.0.21) gives

$$|\hat{c}|(\xi) < 2k^2\pi^2\sqrt{\varepsilon}. \tag{4.0.22}$$

Taking square on both sides of (4.0.22), then integral it over $|\xi| < \rho$. Let $|\xi| = r$, we obtain

$$\int_{|\xi|<\rho} |\hat{c}|^2(\xi) < \int_0^\rho 4k^4\pi^4\varepsilon r^2 dr < \frac{4}{3}k^4\pi^4\varepsilon\rho^3. \tag{4.0.23}$$

To bound the value of integral in the domain $|\xi| > \rho$, we use properties of Fourier transform,

$$\int_{\rho<|\xi|} |\hat{c}|^2(\xi) \le \frac{1}{1+\rho^2}\int_{\mathbb{R}^3}(1+|\xi|^2)|\hat{c}|^2(\xi)d\xi = \frac{1}{1+\rho^2}\left\|c\right\|^2_{(1)}(\Omega) \le \frac{4}{1+\rho^2}M^2. \tag{4.0.24}$$

Adding inequalities (4.0.23) and (4.0.24), let $\rho = (E + k)$, then using Parseval's identity, we obtain the bound for $\|c\|_{(0)}(\Omega)$,

$$\|c\|^2_{(0)}(\Omega) \leq \frac{4}{3}k^4\pi^4\varepsilon(E+k)^3 + \frac{4M^2}{(E+k)^2}. \tag{4.0.25}$$

The above inequality proves Theorem 4.0.4. □

4.1 NUMERICAL SIMULATION

For numerical simulation, we will choose the trigonometric representation for the potential,

$$c(x) = \sum_{n_1,n_2,n_3=1}^{5} C_{n_1n_2n_3} cos(n_1x_1)cos(n_2x_2)cos(n_3x_3), \tag{4.1.1}$$

where n_1, n_2, n_3 are integers from 1 to 5. The goal is to recover all coefficients $C_{n_1n_2n_3}$. Let $u_0 = e^{i\zeta(1)\cdot x}$, $v = e^{i\zeta(2)\cdot x}$ with

$$\zeta(1) = (\frac{\xi_{1e}}{2}, (k^2 - \frac{|\xi|^2}{4})^{\frac{1}{2}}, \frac{\xi_3}{2})_e,\ \zeta(2) = (\frac{\xi_{1e}}{2}, -(k^2 - \frac{|\xi|^2}{4})^{\frac{1}{2}}, \frac{\xi_3}{2})_e. \tag{4.1.2}$$

Substituting above u_0, v, and (4.1.1) into the equation (4.0.13) ,with help of (4.0.9), results in

$$\sum_{n_1,n_2,n_3=1}^{5} C_{n_1n_2n_3} \int_\Omega e^{i\xi\cdot x} cos(n_1x_1)cos(n_2x_2)cos(n_3x_3) = \int_{\partial\Omega} u_1(x)\partial_\nu(e^{i\zeta(2)\cdot x}), \tag{4.1.3}$$

for u_1 is the solution to (4.0.6) and $\Omega = \{x \in \mathbb{R}^3 | 0 < x_i < \pi,\ i = 1, 2, 3\}$. To solve all $C_{n_1n_2n_3}$ uniquely, it is necessary to select vector ξ multiple times to form a system of equations. Let

$$a_{n_1n_2n_3}(\xi) = \int_\Omega e^{i\xi\cdot x} cos(n_1x_1)cos(n_2x_2)cos(n_3x_3),\ b(k,\xi) = \int_{\partial\Omega} u_1\partial_\nu(e^{i\zeta(2)\cdot x}), \tag{4.1.4}$$

then the system of equations generated by (4.1.3) will be

$$\begin{pmatrix} a_{111}(\xi^1) & a_{112}(\xi^1) & \cdots & a_{555}(\xi^1) \\ a_{111}(\xi^2) & a_{112}(\xi^2) & \cdots & a_{555}(\xi^2) \\ \vdots & \vdots & \ddots & \vdots \\ a_{111}(\xi^{125}) & a_{112}(\xi^{125}) & \cdots & a_{555}(\xi^{125}) \end{pmatrix} \begin{pmatrix} C_{111} \\ C_{112} \\ \vdots \\ C_{555} \end{pmatrix} = \begin{pmatrix} b(k,\xi^1) \\ b(k,\xi^2) \\ \vdots \\ b(k,\xi^{125}) \end{pmatrix}, \tag{4.1.5}$$

where $\xi^i,\ i = 1, 2..125$ denotes the multiple choices of ξ.

The function u_1 contained in $b(k, \xi)$ is the data in practical situation. Here we will design it by solving the direct problem (4.0.6). For this purpose, choose c in the first equation of (4.0.6) as

$$c(x) = \frac{1}{2} cos(5x_1) cos(2x_2) cos(4x_3) + cos(4x_1) cos(5x_2) cos(2x_3). \tag{4.1.6}$$

The solution u_1 is given by the Fourier series

$$u_1(x) = \sum_{m_1, m_2, m_3 = 0}^{\infty} f_{m_1 m_2 m_3} cos(m_1 x_1) cos(m_2 x_2) cos(m_3 x_3) \tag{4.1.7}$$

with

$$f_{m_1 m_2 m_3} = \frac{C_f^{m_1 m_2 m_3} \int_\Omega c u_0 cos(m_1 x_1) cos(m_2 x_2) cos(m_3 x_3)}{-(m_1^2 + m_2^2 + m_3^2) + k^2}, \tag{4.1.8}$$

where $C_f^{m_1 m_2 m_3}$ represents the constant for normalizing the orthogonal basis $\{cos(m_i x_i)\}$. Their values are given as follows

$$C_f^{m_1 m_2 m_3} = \frac{1}{\pi^3} \text{ if } m_1 = m_2 = m_3 = 0 \qquad C_f^{m_1 m_2 m_3} = \frac{4}{\pi^3} \text{ if } m_i = 0, m_j m_k \neq 0$$
$$C_f^{m_1 m_2 m_3} = \frac{2}{\pi^3} \text{ if } m_i \neq 0, m_j = m_k = 0 \qquad C_f^{m_1 m_2 m_3} = \frac{8}{\pi^3} \text{ if } m_1 m_2 m_3 \neq 0,$$

where $i, j, k \in \{1, 2, 3\}$ and i, j, k do not take the same value. By computing right side of (4.1.8) carefully, we find that

$$\begin{aligned} f_{m_1 m_2 m_3} = \{ & C_f^{m_1 m_2 m_3} [\frac{1}{2} P(\zeta_1(1), m_1, 5) P(\zeta_1(2), m_2, 2) P(\zeta_1(3), m_3, 4) \\ & + P(\zeta_1^*(1), m_1, 5) P(\zeta_1^*(2), m_2, 2) P(\zeta_1^*(3), m_3, 4) \\ & + P(\zeta_1(1), m_1, 4) P(\zeta_1(2), m_2, 5) P(\zeta_1(3), m_3, 2) \\ & + P(\zeta_1^*(1), m_1, 4) P(\zeta_1^*(2), m_2, 5) P(\zeta_1^*(3), m_3, 2)]\} / -(m_1^2 + m_2^2 + m_3^2) + k^2, \end{aligned}$$

where $P(x, y, z)$ is a function of three variables defined as

$$P(x, y, z) = \frac{1}{4} (\frac{e^{i(x+y+z)\pi} - 1}{i(x+y+z)} + \frac{e^{i(x-y+z)\pi} - 1}{i(x-y+z)} + \frac{e^{i(x+y-z)\pi} - 1}{i(x+y-z)} + \frac{e^{i(x-y-z)\pi} - 1}{i(x-y-z)}),$$

and $\zeta_i(1)$ is the ith component of vector $\zeta(1)$.

Now, to find $C_{n_1n_2n_3}$, we solve the system of equations (4.1.5) by finding the inverse of the matrix, then compare the result with the designed solution (4.1.6). The prediction of the theory suggests that there is an increasing stability zone and a optimal choice of k. Thus we will confirm this prediction by solving (4.1.5) multiple times for different k. The following algorithm is used for computation.

Algorithm 1

input: k, ξ^i, for $i = 1, 2...125$, $f_{m_1m_2m_3}$.

output: $C_{n_1n_2n_3}$, for $n_1, n_2, n_3 = 1, 2..5$.

$A = \text{zeros}(125);$

for $i = 1 : 125$

  $j = 1;$

  for $n_1 = 1 : 5$

    for $n_2 = 1 : 5$

      for $n_3 = 1 : 5$

      $A(i, j) = \int_\Omega e^{i\xi^i \cdot x} cos(n_1x_1)cos(n_2x_2)cos(n_3x_3);$

      $j = j + 1$

      end

    end

  end

end

$b = \text{zeros}(1, 125);$

for $i = 1 : 125$

$h = 0;$

  for $m_1 = 0 : N$, (N is the truncation number for Fourier series (4.1.7))

    for $m_2 = 0 : N$

      for $m_3 = 0 : N$

      $h = h + \int_{\partial\Omega} f_{m_1m_2m_3} cos(m_1x_1)cos(m_2x_2)cos(m_3x_3)\partial_\nu(e^{i\varsigma^i(2)\cdot x});$

(computation of $b(k,\xi^i)$)

end

end

end

$b(i) = h$;

end

$c = A/b$;

In the above algorithm, the choice of ξ^i is quite crucial. They need to be selected so that the condition number of the first matrix in (4.1.6) is as small as possible. A good choice is given by the following code

$i = 1$;

for $n_1 = 1:5$

for $n_2 = 1:5$

for $n_3 = 1:5$

$\xi^i = ((-1)^{i+1}(\frac{1}{2}+n_1), (-1)^i(\frac{1}{2}+n_2), (-1)^{i+1}(\frac{1}{2}+n_3))$;

$i = i+1$

end

end

end,

more precisely, in the above code, for $i = 1, 2..5$, $n_1 = 1$, $n_2 = 2$, $n_3 = 1, 2..5$, for $i = 6, 7..10$, $n_1 = 1$, $n_2 = 2$, $n_3 = 1, 2..5$, and so forth. The condition number roughly equals 200 for the above ξ^i. The following figures illustrate the outcome of the algorithm.

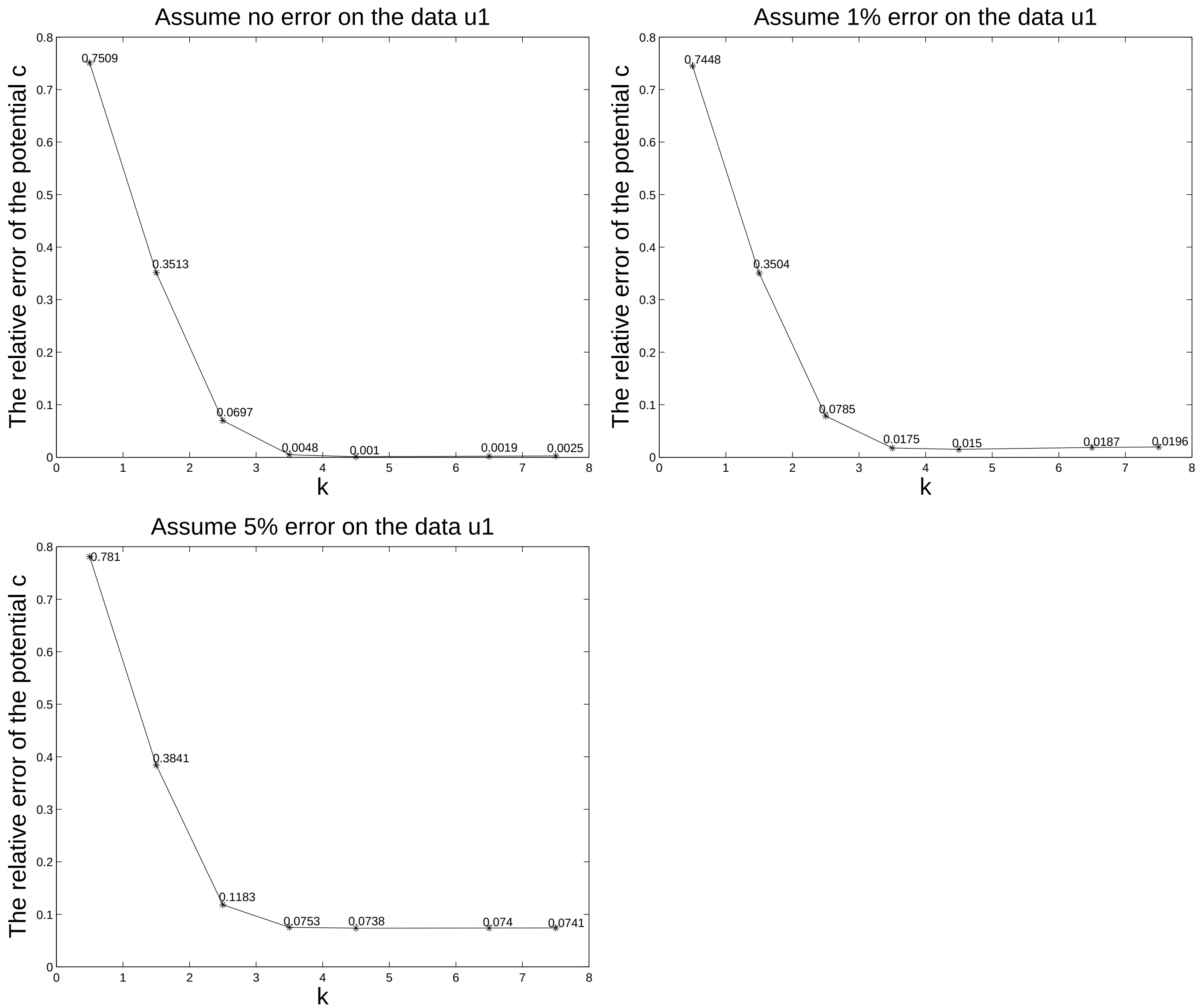

Figure 4.1: the plot of relative error $\frac{\|\hat{c}-c\|}{\|c\|}$ versus k. The labels on each data point indicates the value of relative error at corresponding k. In the second and third picture, we have artificially added some errors to the data u_1 in (4.1.3).

The outcome shows that the relative error for the potential c decays fast in the interval $k \in (0, 4.5)$, and achieve its minimum around $k = 4.5$. This observation confirms the theoretical prediction as promised. When k close to 0, the relative error becomes intolerable. This is not only true for the particular choice of u_0, v as (4.1.2), but more generally, any u_0, v solves (4.0.3) and (4.0.12). We make the reference [23] to verify this claim. The analysis in [23] shows that the computation for 125 terms is numerically impossible if $k = 0$.

4.2 STABILITY ESTIMATE FOR PARTIAL DATA

In this section, we will obtain the stability bound for c when the partial data is given. Let $\Omega = \{x \in \mathbb{R}^3 | 0 < x_i < \pi,\ i = 1, 2, 3\}$, $\Gamma = \partial\Omega \cap \{x_3 = 0\}$, and $\partial\Omega_+ = \partial\Omega \cap \{x_3 > 0\}$. Consider the Schrd̈inger equation

$$-\Delta u - k^2 u + cu = 0 \text{ in } \Omega, \tag{4.2.1}$$

with the partial boundary data

$$\begin{aligned} \partial_\nu u &= h \in H^{-\frac{1}{2}}(\partial\Omega_+). \\ \partial_\nu u &= 0 \text{ on } \Gamma. \end{aligned} \tag{4.2.2}$$

By using the same argument as in section 1, one can obtain linearised Schrödinger equation

$$\begin{aligned} -\Delta u_1 - k^2 u_1 &= -cu_0 \text{ in } \Omega, \\ \partial_\nu u_1 &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.2.3}$$

in which u_0 solves

$$\begin{aligned} -\Delta u_0 - k^2 u_0 &= 0 \text{ in } \Omega, \\ \partial_\nu u_0 &= h \text{ on } \partial\Omega_+, \\ \partial_\nu u_0 &= 0 \text{ on } \Gamma. \end{aligned} \tag{4.2.4}$$

Assume k is not eigenvalue to equations (4.2.3), then we can define the Neumann-to-Dirichlet map

$$\Lambda_p : h \longrightarrow u_1 \text{ on } \partial\Omega_+. \tag{4.2.5}$$

Λ_p is a bounded operator from $L^2(\partial\Omega)$ into itself due to the same reason stated in section 1. Denote its operator norm by $\|\Lambda_p\|$.

Theorem 4.2.1 *Let*

$$\|c\|_\infty(\Omega) \le M_0,\ \|\nabla c\|_\infty(\Omega) \le M_1 \tag{4.2.6}$$

and $\varepsilon = \|\Lambda_p\|$, $E = -ln\varepsilon$, $E > 0$, *then there are constants* $C(\Omega, M)$, $0 < \lambda < 1$ *and* $0 < \lambda_i < 2$, $i = 1, 2$ *such that*

$$\|c\|^2_{(0)}(\Omega) \le C(\Omega, M)(k^{4\lambda}\varepsilon^\lambda(E + k)^{\lambda_1} + \frac{1}{(E + k)^{\frac{2\lambda_1}{3}}}), \tag{4.2.7}$$

where $M = \sqrt{M_0 + M_1}$.

Similar to theorem 4.0.4, one can minimize the bound of (4.2.7) with respect to k. Since the the powers λ, λ_1, and λ_2 are unknown, this minimum value is only an approximation. At $k_0 = (\frac{\lambda_2}{(4\lambda+\lambda_1)\varepsilon^\lambda})^p - E$, for $p = \frac{1}{4\lambda+\lambda_1+\lambda_2}$, the bound achieve its minimum value, and $k_0^{4\lambda}\varepsilon^\lambda(E+k_0)^{\lambda_1} + \frac{1}{(E+k_0)^{\lambda_2}} < C(\Omega, M)\varepsilon^{p_1}$ for some power $0 < p_1 < 1$. Clearly the bound decays in the zone $k \in (0, k_0)$ and behave more like Hölder stable bound as k approaches k_0. This demonstrate the increasing stability zone and the optical choice of k. In the case $k_0 < 0$, then $\varepsilon > \alpha > 0$, and Hölder stability is still true.

Now, let the function v satisfies

$$\begin{aligned} -\Delta v - k^2 v = 0 \text{ in } \Omega, \\ \partial_\nu v = h_1 \text{ on } \partial\Omega_+, \\ \partial_\nu v = 0 \text{ on } \Gamma. \end{aligned} \tag{4.2.8}$$

Similar to lemma 4.0.5, the next lemma follows by application of Green's theorem and boundary conditions in (4.2.7) and (4.2.8)

Lemma 4.2.2 *For solutions $u_1, v \in H^1(\Omega)$ satisfying (4.2.3),(4.2.8), we have*

$$\int_\Omega c u_0 v = -\int_{\partial\Omega_+} (\Lambda(\partial_\nu u_0))\partial_\nu v. \tag{4.2.9}$$

Lemma 4.2.3 *Let $\xi \in \mathbb{R}^3$, $(1+4\tau^2) \geq \frac{4k^2}{|\xi|^2}$, and $\xi^* = (\xi_1, \xi_2, -\xi_3)$. then there are exponential solutions*

$$u_0(x) = e^{i\zeta(1)\cdot x} + e^{i\zeta^*(1)\cdot x},\ v(x) = e^{i\zeta(2)\cdot x} + e^{i\zeta^*(2)\cdot x} \tag{4.2.10}$$

to the boundary value problems (4.2.4) and (4.2.8) with

$$\begin{aligned} \zeta(1) + \zeta(2) = \xi,\ \zeta(j)\cdot\zeta(j) = \zeta^*(j)\cdot\zeta^*(j) = k^2, \\ |Im\zeta(j)| = |Im\zeta^*(j)| = \sqrt{|\xi|^2(\frac{1}{4}+\tau^2) - k^2},\ j = 1, 2, \end{aligned} \tag{4.2.11}$$

in addition,

$$\|\partial_\nu u_0\|_{(0)}(\partial\Omega_+) \leq 2k\pi e^{|Im\zeta(1)|},\ \|\partial_\nu v\|_{(0)}(\partial\Omega_+) \leq 2k\pi e^{|Im\zeta(2)|} \tag{4.2.12}$$

Proof: Suppose $\xi \in \mathbb{R}^3$, $\xi \neq 0$. We define a coordinate system $(x_{1e}, x_{2e}, x_{3e})_e$ in the following way. $e(1) = (\xi_1^2+\xi_2^2)^{-\frac{1}{2}}(\xi_1, \xi_2, 0)$, $e(3) = (0, 0, 1)$, $e(2)$ is chosen to form an orthonormal basis $e(1), e(2), e(3)$ in $\mathbb{R}^3$. Denote $\xi_{1e} = \sqrt{\xi_1^2 + \xi_2^2}$. Now, let

$$\zeta(1) = (\frac{\xi_{1e}}{2} - \tau\xi_3, i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, \frac{\xi_3}{2} + \tau\xi_{1e})_e, \tag{4.2.13}$$

$$\zeta(2) = (\frac{\xi_{1e}}{2} + \tau\xi_3, -i(|\xi|^2(\frac{1}{4} + \tau^2) - k^2)^{\frac{1}{2}}, \frac{\xi_3}{2} - \tau\xi_{1e})_e, \tag{4.2.14}$$

where τ is a positive real number. Then (4.2.11) is satisfied for the above $\zeta(1)$, $\zeta(2)$. By substituting (4.2.10) into (4.2.4) and (4.2.8), we can verify u_0, v solves the boundary value problems. (4.2.12) is obtained by direct computation of L^2 norm on the boundary $\partial\Omega_+$ □

4.3 NUMERICAL SIMULATION FOR PARTIAL DATA

In the case of partial data, we choose the same representation for the potential as (4.1.1)

$$c(x) = \sum_{n_1,n_2,n_3=1}^{5} C_{n_1n_2n_3} cos(n_1x_1)cos(n_2x_2)cos(n_3x_3), \tag{4.3.1}$$

where n_1, n_2, n_3 are integers from 1 to 5. Let $u_0 = e^{i\zeta(1)\cdot x} + e^{i\zeta^*(1)\cdot x}$, $v = e^{i\zeta(2)\cdot x} + e^{i\zeta^*(2)\cdot x}$ with

$$\zeta(1) = (\frac{\xi_{1e}}{2}, (k^2 - \frac{|\xi|^2}{4})^{\frac{1}{2}}, \frac{\xi_3}{2})_e,\ \zeta(2) = (\frac{\xi_{1e}}{2}, -(k^2 - \frac{|\xi|^2}{4})^{\frac{1}{2}}, \frac{\xi_3}{2})_e. \tag{4.3.2}$$

$\zeta^*(j) = (\zeta_1(j), \zeta_2(j), -\zeta_3(j)),\ j = 1, 2$, then u_0 and v solves (4.2.4) and (4.2.8) respectively. Substituting them and (4.3.1) into the equation (4.2.9) gives

$$\begin{aligned} \sum_{n_1,n_2,n_3=1}^{5} \{C_{n_1n_2n_3}[\int_\Omega (e^{i\xi\cdot x} + e^{i\xi^*\cdot x}) cos(n_1x_1)cos(n_2x_2)cos(n_3x_3) \\ + \int_\Omega (e^{i(\zeta(1)+\zeta^*(2))\cdot x} + e^{i(\zeta(2)+\zeta^*(1))\cdot x}) cos(n_1x_1)cos(n_2x_2)cos(n_3x_3)]\} \\ = \int_{\partial\Omega} u_1(x)\partial_\nu(e^{i\zeta(2)\cdot x} + e^{i\zeta^*(2)\cdot x}), \end{aligned} \tag{4.3.3}$$

where u_1 solves (4.2.3). In the above equation, the value of the second integral containing $e^{i(\zeta(1)+\zeta^*(2))\cdot x} + e^{i(\zeta(2)+\zeta^*(1))\cdot x}$ equals 0 due to the choice of (4.3.2) and the domain $\Omega = \{x \in \mathbb{R}^3 | 0 < x_i < \pi,\ i = 1, 2, 3\}$.

Similar to the case of full data, now we select many vectors $\xi^i,\ i = 1, 2, ..125$ in (4.3.3) to form a system of equations. Let

$$\begin{aligned} a_{n_1n_2n_3}(\xi) &= \int_\Omega (e^{i\xi\cdot x} + e^{i\xi^*\cdot x}) cos(n_1x_1)cos(n_2x_2)cos(n_3x_3), \\ b(k,\xi) &= \int_{\partial\Omega} u_1 \partial_\nu (e^{i\zeta(2)\cdot x} + e^{i\zeta^*(2)\cdot x}), \end{aligned} \tag{4.3.4}$$

then the system of equations is obtained as follows

$$\begin{pmatrix} a_{111}(\xi^1) & a_{112}(\xi^1) & \cdots & a_{555}(\xi^1) \\ a_{111}(\xi^2) & a_{112}(\xi^2) & \cdots & a_{555}(\xi^2) \\ \vdots & \vdots & \ddots & \vdots \\ a_{111}(\xi^{125}) & a_{112}(\xi^{125}) & \cdots & a_{555}(\xi^{125}) \end{pmatrix} \begin{pmatrix} C_{111} \\ C_{112} \\ \vdots \\ C_{555} \end{pmatrix} = \begin{pmatrix} b(k,\xi^1) \\ b(k,\xi^2) \\ \vdots \\ b(k,\xi^{125}) \end{pmatrix} \tag{4.3.5}$$

Next, we design the data u_1 by solving the direct problem (4.2.3). The potential c in right side of (4.2.3) is chosen the same way as in section 3,

$$c(x) = \frac{1}{2} cos(5x_1)cos(2x_2)cos(4x_3) + cos(4x_1)cos(5x_2)cos(2x_3). \tag{4.3.6}$$

The solution is given by the Fourier series

$$u_1(x) = \sum_{m_1,m_2,m_3=0}^{\infty} f_{m_1m_2m_3} cos(m_1x_1)cos(m_2x_2)cos(m_3x_3) \tag{4.3.7}$$

with

$$f_{m_1m_2m_3} = \frac{C_f \int_\Omega c u_0 cos(m_1x_1)cos(m_2x_2)cos(m_3x_3)}{-(m_1^2+m_2^2+m_3^2)+k^2}, \tag{4.3.8}$$

where u_0 is given as above and C_f represents the constant for normalizing the orthogonal basis $\{cos(m_ix_i)\}$. Their values are given as follows

$$\begin{aligned} C_f^{m_1m_2m_3} &= \frac{1}{\pi^3} \text{ if } m_1 = m_2 = m_3 = 0 \qquad & C_f^{m_1m_2m_3} &= \frac{4}{\pi^3} \text{ if } m_i = 0, m_jm_k \neq 0 \\ C_f^{m_1m_2m_3} &= \frac{2}{\pi^3} \text{ if } m_i \neq 0, m_j = m_k = 0 \qquad & C_f^{m_1m_2m_3} &= \frac{8}{\pi^3} \text{ if } m_1m_2m_3 \neq 0, \end{aligned}$$

where $i, j, k \in \{1, 2, 3\}$ and i, j, k do not take the same value. By computing right side of (4.1.8) carefully, one find that

$$\begin{aligned} f_{m_1m_2m_3} = \{C_f^{m_1m_2m_3}[&\frac{1}{2}P(\zeta_1(1), m_1, 5)P(\zeta_1(2), m_2, 2)P(\zeta_1(3), m_3, 4) \\ &+ P(\zeta_1(1), m_1, 4)P(\zeta_1(2), m_2, 5)P(\zeta_1(3), m_3, 2)]\} / -(m_1^2+m_2^2+m_3^2)+k^2, \end{aligned}$$

where $P(x,y,z)$ is a function of three variables defined as

$$P(x,y,z) = \frac{1}{4}(\frac{e^{i(x+y+z)\pi}-1}{i(x+y+z)} + \frac{e^{i(x-y+z)\pi}-1}{i(x-y+z)} + \frac{e^{i(x+y-z)\pi}-1}{i(x+y-z)} + \frac{e^{i(x-y-z)\pi}-1}{i(x-y-z)}),$$

and $\zeta_i(1)$ is the ith component of vector $\zeta(1)$.

A slightly modified version of algorithm 1 is used for solving (4.3.5). The code is written as follows.

Algorithm 2

input: k, ξ^i, for $i = 1, 2...125$, $f_{m_1 m_2 m_3}$.

output: $C_{n_1 n_2 n_3}$, for $n_1, n_2, n_3 = 1, 2..5$.

$A = \text{zeros}(125)$;

for $i = 1 : 125$

$j = 1$;

for $n_1 = 1 : 5$

for $n_2 = 1 : 5$

for $n_3 = 1 : 5$

$A(i,j) = \int_\Omega (e^{i\xi^i \cdot x} + e^{i\xi^{*i} \cdot x}) cos(n_1 x_1) cos(n_2 x_2) cos(n_3 x_3)$;

$j = j + 1$

end

end

end

end

$b = \text{zeros}(1, 125)$;

for $i = 1 : 125$

$h = 0$;

for $m_1 = 0 : N$, (N is the truncation number for Fourier series (4.3.7))

for $m_2 = 0 : N$

for $m_3 = 0 : N$

$h = h + \int_{\partial\Omega_+} f_{m_1 m_2 m_3} cos(m_1 x_1) cos(m_2 x_2) cos(m_3 x_3) \partial_\nu (e^{i\zeta^i(2) \cdot x})$;

(computation of $b(k, \xi^i)$)

end

end

end

$b(i) = h$;

end

$c = A/b$;

We have chose ξ^i as following

$i = 1$;

for $n_1 = 1 : 5$

for $n_2 = 1 : 5$

for $n_3 = 1 : 5$

$\xi^i = ((-1)^{i+1}(\frac{1}{2} + n_1), (-1)^i(\frac{1}{2} + n_2), (\frac{1}{2} + n_3))$;

$i = i + 1$

end

end

end

Under this choice, the condition number of the first matrix in (4.3.5) roughly equals 180. The following figures illustrate the result of the algorithm.

In the view of following pictures, they confirm the increasing stability zone and the optimal choice of k as the theory suggests. Surprisingly, the result is slightly better than that in the case of full data. Currently We do not fully understand the reason behind this fact. Our guess would be that the particular choice of c as (4.3.1) causes the second integral in (4.3.3) to vanish.

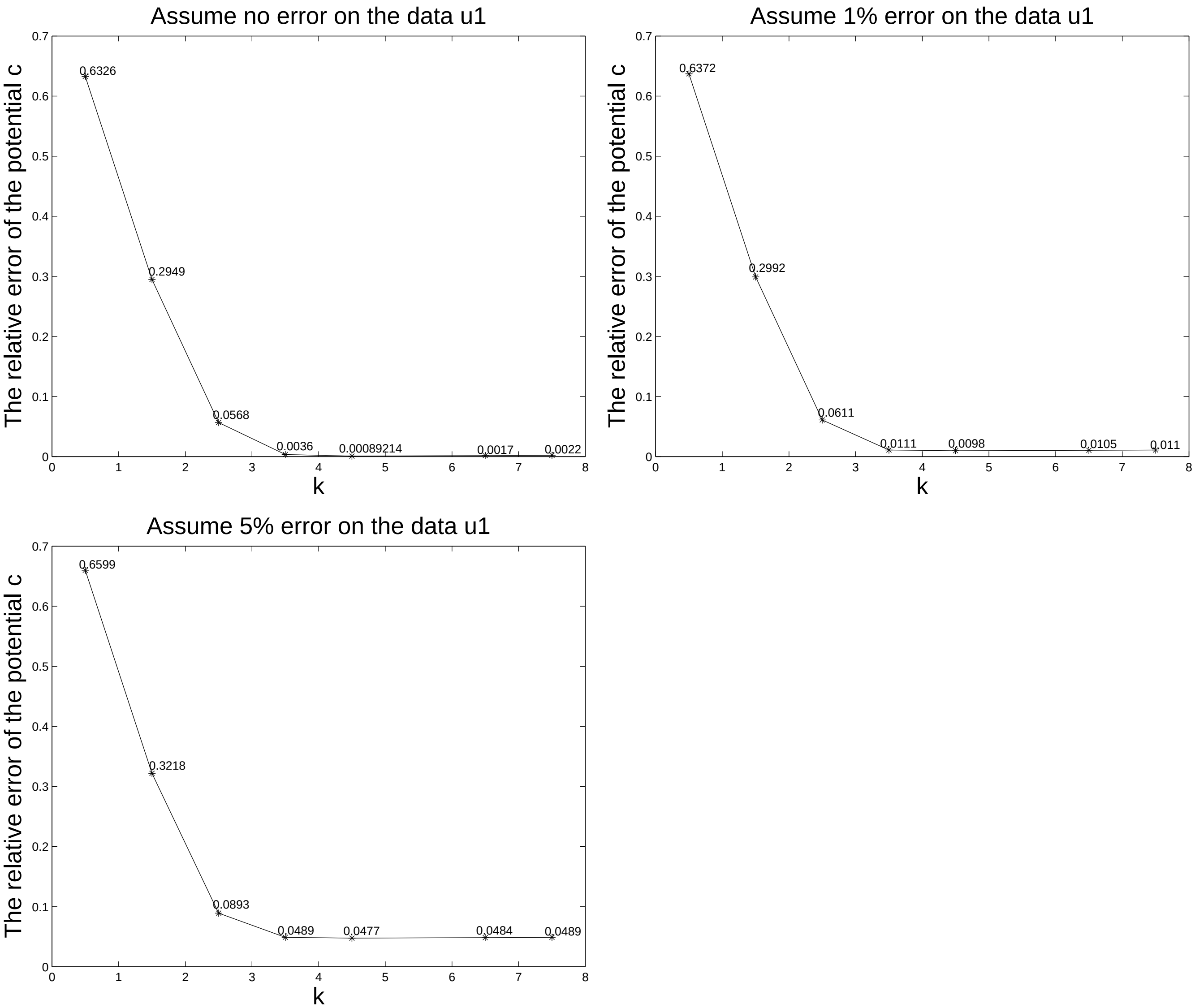

Figure 4.2: The result for partial data

CHAPTER 5
THE CONCLUSION AND FUTURE CHALLENGES

In this thesis, we have shown the increasing stability of recovering potential $c \in C^1(\Omega)$ for the Schrödinger equation with the given partial Dirichlet-to-Neumann map. We used the methods of even reflection to build almost complex exponential solutions which have vanishing Neumann data on the boundary contained in $\{x_3 = 0\}$. This boundary surface can be viewed as the surface of a human body in practical case of detecting breast cancer. The bound we achieved in (3.1.2) shows increasing stability for c, and it involves an unknown constant $C(\Omega, M)$ which can be further evaluated by using periodic Faddeev-type solutions [11] or choosing better regular fundamental solutions based on the work of Hörmander [2] if Ω is a half unit ball. The power λ can be also found by using the minimum value of harmonic measure defined on some rectangular domain on $\mathbb{C}$. Moreover, we have studied the linearized inverse problem for the Schrödinger equation with full/partial Neumann-to-Dirichlet map. We derived the bound with explicit constants in the full data case. This bound demonstrated increasing stability phenomenon as the energy k grew and suggested the optimal choice of k. When partial data was given, a similar bound was found but with not explicit constants. The numerical experiments confirmed the increasing stability phenomenon in the both cases. The results showed that the relative error for c was quite large for k close to 0, and decays dramatically as k increases in the interval $(0, 4.5)$.

There are some problems left as challenges. Firstly, we would like to show increasing stability for the coefficient a in the conductivity equation $div(a\nabla u) + k^2 u = 0$. In fact, we observe that the conductivity equation can be transformed into a Helmholtz-type equation $-\Delta u - k^2 a^{-1} u + cu = 0$, where $c = a^{-\frac{1}{2}} \Delta a^{-\frac{1}{2}}$. At present, there are only some results [12] for the simpler equation $(-\Delta - k^2 a_0^2)u = 0$ under some non-trapping conditions on a_0 and bounds with constants exponentially growing with k [13]. Secondly, we wish to prove increasing stability for c in equation (4.0.1) in more general domains. More specifically, we would like to show the same result when the partial boundary where $\partial_\nu u = 0$ is contained in a sphere. According to [7], the uniqueness of c still holds in this case, but increasing stability remains as a challenge.

Also, we would like to design an algorithm to solve the full, non-linearized inverse problem. The goal is quite difficult to achieve since the problem is no longer convex. Also, we wish to recover the inclusion for c in a smaller domain $\{0 < x_i < \pi - \delta,\ i = 1, 2, 3\}$ and analyze the relationship between stability and the choice of δ. Moreover, it is desirable to have increasing stability for the inverse problem for the conductivity equation, since it means a big breakthrough for improving the resolution for Electrical Impedance Tomography, which currently has severe stability issues. Finally, we are interested in solving linearized inverse problem for the Maxwell system. In paper [25], the authors estimated reconstruction error for this type of inverse problem by using a boundary mapping regarding tangential components of the electric and magnetic field. Isakov with other two authors [28] have obtained some good results of improving stability as k increases for the same problem.

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