THE SPACE OF POSITIVE SCALAR CURVATURE METRICS ON A MANIFOLD WITH BOUNDARY

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Abstract. We study the space of Riemannian metrics with positive scalar curvature on a compact manifold with boundary. These metrics extend a fixed boundary metric and take a product structure on a collar neighbourhood of the boundary. We show that the weak homotopy type of this space is preserved by certain surgeries on the boundary in codimension at least three. Thus, there is a weak homotopy equivalence between the space of such metrics on a simply connected spin manifold W, of dimension $n \geq 6$ and with simply connected boundary, and the corresponding space of metrics of positive scalar curvature on the standard disk D^n. Indeed, for certain boundary metrics, this space is weakly homotopy equivalent to the space of all metrics of positive scalar curvature on the standard sphere S^n. Finally, we prove analogous results for the more general space where the boundary metric is left unfixed.

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Key words and phrases. space of Riemannian metrics of positive scalar curvature, manifold with boundary, surgery, bordism, spin, Gromov-Lawson construction, weak homotopy equivalence.

Contents

1. Introduction

Recently, much progress has occurred in understanding the topology of the space $\mathcal{R}^+(X)$, of Riemannian metrics of positive scalar curvature (psc-metrics) on a smooth closed manifold X; see [8] and [12]. In this paper, we examine the related problem when the manifold has a boundary. Here, it is necessary to impose certain boundary conditions on metrics. More precisely, let W be a smooth compact manifold with $\dim W = n+1$, and boundary $\partial W = X$, a closed manifold with $\dim X = n$. We specify a collar $c : (-1, 0] \times X \hookrightarrow W$ and denote by $\mathcal{R}^+(W)$, the space of all psc-metrics on W which take a product structure on the image $c\big((-\frac{1}{2}, 0] \times X\big)$. Thus, $g \in \mathcal{R}^+(W)$ if $c^*g = dt^2 + h$ on $(-\frac{1}{2}, 0] \times X$ for some $h \in \mathcal{R}^+(X)$. Henceforth, we will assume that $\mathcal{R}^+(W)$ (and hence $\mathcal{R}^+(X)$) is non-empty. A further boundary condition we impose is to fix a psc-metric $h \in \mathcal{R}^+(X)$ with the property that h extends to an element $g \in \mathcal{R}^+(W)$. We then define the subspace $\mathcal{R}^+(W)_h \subset \mathcal{R}^+(W)$ of all psc-metrics $g \in \mathcal{R}^+(W)$ where $c^*g|_{\partial W} = h$.

To formulate our main theorem, we consider a smooth compact $(n+1)$-manifold Z whose boundary $\partial Z = X_0 \sqcup X_1$, is a disjoint union of closed n-manifolds. Here we specify a pair of disjoint collars $c_0 : (-1, 0] \times X_0 \hookrightarrow W$, $c_1 : (-1, 0] \times X_1 \hookrightarrow W$ around X_0 and X_1 respectively. We fix a pair of psc-metrics $h_0 \in \mathcal{R}^+(X_0)$ and $h_1 \in \mathcal{R}^+(X_1)$ and denote by $\mathcal{R}^+(Z)_{h_0,h_1}$, the space of psc-metrics m on Z so that $c_i^*m = dt^2 + h_i$ on $(-\frac{1}{2}, 0] \times X$ for $i = 0, 1$. Again, we assume that $\mathcal{R}^+(Z)_{h_0,h_1}$ is non-empty. Suppose $\partial W = X = X_0$, where X_0 is one of the boundary components of Z. Let $W \cup Z$ denote the manifold obtained by gluing Z to W along this boundary component. Denoting by c the collar c_1, we consider the subspace $\mathcal{R}^+(W \cup Z)_{h_1}$ of $\mathcal{R}^+(W \cup Z)$, consisting of psc-metrics which restrict as $dt^2 + h_1$ on $c((0, \frac{-1}{2}] \times X_1)$. For any element $m \in \mathcal{R}^+(Z)_{h_0,h_1}$, there is a map

$$
\begin{aligned}
\mu_{Z,m} : \mathcal{R}^+(W)_{h_0} &\longrightarrow \mathcal{R}^+(W \cup Z)_{h_1} \\
g &\longmapsto g \cup m,
\end{aligned}
\tag{1.1}
$$

where $g \cup m$ is the metric obtained on $W \cup Z$ by the obvious gluing depicted in Fig.1.

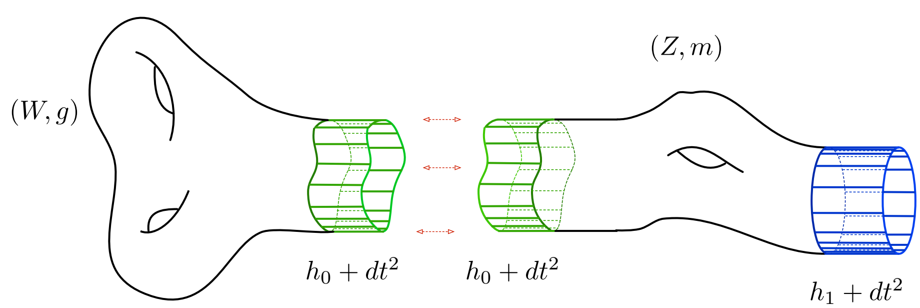

Figure 1. Attaching (W, g) to (Z, m) along a common boundary

1.1. Main Results.

Suppose $\phi : S^p \times D^{q+1} \hookrightarrow X$ is an embedding with $p + q + 1 = n$. Let $Z = T_\phi$, be the trace of the surgery on X with respect to ϕ. Thus $\partial T_\phi = X \sqcup X'$ where X' is the manifold obtained from X by surgery. In the case when $q \geq 2$, the Surgery Theorem of Gromov and Lawson [11], describes a technique for constructing, from any psc-metric $h \in \mathcal{R}^+(X)$, a psc-metric $h' \in \mathcal{R}^+(X')$. A useful strengthening of this technique allows for the determination of a particular psc-metric $\bar{h} \in \mathcal{R}(T_\phi)_{h,h'}$, known as a *Gromov-Lawson trace* (or, more generally, a *Gromov-Lawson cobordism*); see [9], [22]. Metrics which are accessible from each other by a sequence of Gromov-Lawson surgeries are said to be *Gromov-Lawson cobordant.* We now have a map $\mu_{T_{\phi,\bar{h}}} : \mathcal{R}^+(W) \to \mathcal{R}^+(W')$, where $W' = W \cup T_\phi$. Our results are as follows.

Theorem A. *Suppose $p, q \geq 2$. For any $h \in \mathcal{R}^+(X)$, there exist psc-metrics $h' \in \mathcal{R}^+(X')$ and $\bar{h} \in \mathcal{R}^+(T_\phi)_{h,h'}$ so that the map $\mu_{T_{\phi},\bar{h}}$ (1.1) is a weak homotopy equivalence:*

$$\mathcal{R}^+(W)_h \simeq \mathcal{R}^+(W')_{h'}.$$

Theorem B. *Suppose W is a smooth compact spin manifold with closed boundary X. We further assume that W and X are both simply connected and that* $\dim W = n + 1 \geq 6$.

(i.) For any $h \in \mathcal{R}^+(X)$ where $\mathcal{R}^+(W)_h$ is non-empty, there is a psc-metric $h' \in \mathcal{R}^+(S^n)$ and a weak homotopy equivalence:

$$\mathcal{R}^+(W)_h \simeq \mathcal{R}^+(D^{n+1})_{h'}.$$

(ii.) Furthermore, if h is Gromov-Lawson cobordant to the standard round metric ds_n^2, there is a weak homotopy equivalence:

$$\mathcal{R}^+(W)_h \simeq \mathcal{R}^+(S^{n+1}).$$

Theorem C. *When $p, q \geq 2$, the spaces $\mathcal{R}^+(W)$ and $\mathcal{R}^+(W')$ are weakly homotopy equivalent.*

Corollary D. *When W and X satisfy the hypotheses of Theorem B, there is a weak homotopy equivalence:*

$$\mathcal{R}^+(W) \simeq \mathcal{R}^+(D^{n+1}).$$

1.2. **Background.** We begin with a brief discussion of the original problem for a closed n-manifold X. The space $\mathcal{R}^+(X)$ is an open subspace of the space of all Riemannian metrics on X, denoted $\mathcal{R}(X)$, under its usual C^∞-topology. An old question in this subject is whether or not X admits any psc-metrics, i.e. whether or not $\mathcal{R}^+(X)$ is non-empty. Although work continues on this problem, in the case when X is simply connected and $n \geq 5$, necessary and sufficient conditions are known: $\mathcal{R}^+(X) = \emptyset$ if and only if X is both spin and the Dirac index $\alpha(X) \in KO_n$ is non-zero. This result is due to Stolz [21], following important work by Gromov, Lawson [11] and others. For a survey of this problem, see [19].

In the case when the space $\mathcal{R}^+(X)$ is non-empty, one may inquire about its topology. Up until recently, very little was known about this space beyond the level of path-connectivity. Hitchin showed for example, in [15], that if X is spin, $\pi_0(\mathcal{R}^+(X)) \neq 0$ when $n \equiv 0, 1(\text{mod}8)$ and that $\pi_1(\mathcal{R}^+(X)) \neq 0$ when $n \equiv 0, -1(\text{mod}8)$. It is worth noting that all of these non-trivial elements disappear once one descends to the moduli space $\mathcal{M}^+(X) := \mathcal{R}^+(X)/\text{Diff}(X)$ where $\text{Diff}(X)$ is the group of self-diffeomorphisms on X and acts on $\mathcal{R}^+(X)$ by pulling back metrics. Later, Carr showed in [7], that when X is the sphere S^n, $\pi_0(\mathcal{R}^+(S^{4k-1}))$ is infinite for all $k \geq 2$ and all but finitely many of these non-trivial elements survive in the moduli space. Various generalisations of this result have been achieved. In particular, Botvinnik and Gilkey showed that $\pi_0(\mathcal{R}^+(X)) \neq 0$ in the case when X is spin and $\pi_1(X)$ is finite; see

[4]. More recently, there have been a number of significant results which exhibit the non-triviality of higher homotopy groups of both $\mathcal{R}^+(X)$ and $\mathcal{M}^+(X)$ for a variety of manifolds X; see [3], [8] and [12]. Most of these results ([15], [8], [12]) involve showing that for certain closed spin manifolds X and certain psc-metrics $h \in \mathcal{R}^+(X)$, a particular variation of the Dirac index, introduced by Hitchin in [15], often induces non-trivial homomorphisms

$$A_k(X,h) : \pi_k(\mathcal{R}^+(X), h) \longrightarrow KO_{k+n+1}.$$

Most recently of all, Botvinnik, Ebert and Randal-Williams in [2], show that this map is always non-trivial when the codomain is non-trivial. Their methods are new and make use of work done by Randal-Williams and Galatius on moduli spaces of manifolds; see [10].

One result which the authors in [2] make use of and which is of particular relevance here, is the following theorem which utilises a family version of the Gromov-Lawson construction. This theorem was originally proved by Chernysh in [5]. Later, we provided a simpler proof in [24].

Theorem 1.2.1. (Chernysh [5]**, W.** [24]**)** *Let X be a smooth compact manifold of dimension n. Suppose X' is obtained from X by surgery on a sphere $i : S^p \hookrightarrow X$ with $p+q+1 = n$ and $p, q \geq 2$. Then the spaces $\mathcal{R}^+(X)$ and $\mathcal{R}^+(X')$ are homotopy equivalent.*

Theorems A and C in this paper are of course generalisations of Theorem 1.2.1 for manifolds with boundary. Generalising Theorem 1.2.1 is trivial in the case when the surgery takes place on the interior of such a manifold. We focus on the more challenging case when the surgery takes place on the boundary itself. We close by recalling some fundamental questions which motivate this work.

(1.) Given some $h \in \mathcal{R}^+(X)$, is the space $\mathcal{R}^+(W)_h$ non-empty?
(2.) If $\mathcal{R}^+(W)_h \neq \emptyset$, what can we say about its topology?
(3.) More generally, what can we say about the topology of the space $\mathcal{R}^+(W)$?

Although strictly not the focus of this work, Question (1.) is relevant here as its answer is often negative. For example, the methods used by Carr in [7] give rise to psc-metrics on S^{4k-1} which do not extend to elements of $\mathcal{R}^+(D^{4k})$, for all $k \geq 2$. Questions (2.) and (3.) are posed in problem 3, section 2.1 of the survey article [19], and our results are a contribution to answering these questions.

1.3. **Structure of the paper.** Theorems A, B, C and Corollary D will be proved in section 6, drawing of course on the results of previous sections. In section 2, we review the proof of Theorem of 1.2.1. Much of the ideology behind the proof of Theorem 1.2.1 carries over to the proof of Theorem A. In particular, just as the manifolds X and X' in Theorem 1.2.1 are mutually obtainable by complementary surgeries, so are the manifolds W and W' in Theorem A. In the latter case however, these surgeries are of two different types. This is detailed in section 3. The surgery on the embedded sphere $S^p \hookrightarrow \partial W \subset W$ which turns W into W' is defined to be a surgery of type 1. The complementary surgery in this case, which turns W' back into W, is actually a surgery on an embedded disk $D^{q+1} \hookrightarrow W'$. This embedding restricts on the boundary of the disk to a certain embedding $S^q \to \partial W$. This second type of surgery is defined to be a surgery of type 2.

In the next section, section 4, we consider various subspaces of the space of $\mathcal{R}^+(W)$, in which the metrics are standardised near embedded surgery spheres and disks to prepare them for Gromov-Lawson style *geometric* surgeries of types 1 and 2. Importantly, the various standardized subspaces we construct are shown over this and the next section, section 5, to be weakly homotopy equivalent to the appropriate ambient space of psc-metrics. The idea, as in Theorem 1.2.1, is to replace $\mathcal{R}^+(W)$ and $\mathcal{R}^+(W')$ (or the appropriate subspaces which fix a boundary metric) with weakly homotopy equivalent spaces of metrics which take a standard form near the embedded surgery sphere or disk. Such spaces are demonstrably homeomorphic. The difficult part is in demonstrating the weak homotopy equivalences between these standardized spaces and their original counterparts. In particular, the most challenging case of this problem is demonstrated in Lemma 5.2.1, which is in some ways, the technical heart of the paper. An important tool in this "standardising process" is a variation of the torpedo metric called a "boot metric". We devote an appendix, section 7, to the construction of these metrics and the demonstration of their relevant properties. Finally, in section 6, we put together these various pieces to prove Theorem A. Theorems B and C, and Corollary D, follow without too much difficulty by utilising some standard topological arguments and certain results of Chernysh in [6].

1.4. **Acknowledgements.** This work was originally intended as a short appendix to the paper, [2], by B. Botvinnik, J. Ebert and O. Randal-Williams. As is so often the case, it quickly grew into a larger project of independent interest. This author would like to express gratitude to all three authors for their various comments and suggestions as this project developed, and of course for the original invitation to submit this work as an appendix. A special thanks is due to B. Botvinnik, for originally suggesting this project to me and for some significant contributions to its development. The author is also grateful to the Simons Foundation, as much of this work took place with the aid of Simons Foundation Collaboration Grant for Mathematicians No. 280310. Finally, as a number of important insights relating to this work were gained by attendance of the workshop "Analysis and Topology in Interaction" in Cortona, June 2014, the author wishes to convey his gratitude to the organisers, epecially W. Lück, P. Piazza and T. Schick, for their kind invitation.

2. Surgery, the Gromov-Lawson construction and Theorem 1.2.1

Before focussing on manifolds with boundary, it is important to review some aspects of the earlier work on closed manifolds. Here, we draw on work done in [22] and [24]. Throughout this paper X is always a smooth compact n-dimensional manifold with empty boundary, while W is a smooth compact $(n+1)$-dimensional manifold with boundary $\partial W = X$. As usual, S^n and D^n denote the standard Euclidean unit sphere and disk while I denotes the unit interval $[0, 1]$. We begin with some technical preliminaries concerning standard metrics on the sphere and disk.

We denote by ds_n^2, the standard round metric of radius 1 on S^n. Recall that a *δ-torpedo metric* on the disk D^n, denoted $g_{\text{torp}}^n(\delta)$, is an $O(n)$-symmetric metric which is the round metric of radius δ, $\delta^2 ds_n^2$, near the centre of the disk but transitions to a standard round Riemannian cylinder metric of radius δ, $dr^2 + \delta^2 ds_{n-1}^2$, on a collar neighbourhood of the boundary diffeomorphic to $[0, \epsilon] \times S^{n-1}$. This collar is known as the *neck* of the torpedo. Here, r denotes the radial distance coordinate. Away from the origin, the metric takes the form of a warped product $dr^2 + \eta(r)^2 ds_{n-1}^2$ for some appropriate smooth warping function η. Later, in the appendix, we describe in more detail, the construction of these metrics and some of their properties. The important point to grasp here is simply that, when $n \geq 3$, the scalar curvature of such a metric may be bounded below by an arbitrarily large positive constant, provided δ is chosen to be sufficiently small. Typically, we will suppress the δ parameter in the notation, referring to the metric simply as a torpedo metric g_{torp}; we assume that δ is small enough for our purposes.

2.1. **Surgery.** Suppose $\phi : S^p \times D^{q+1} \to X$ is an embedding, where $\dim X = n = p+q+1$. Recall that a surgery on a smooth n-dimensional manifold X with respect to the embedding ϕ is the construction of a manifold X' by removing the image of ϕ from X and using the restricted map $\phi|_{S^p \times S^q}$ to attach $D^{p+1} \times S^q$ along the common boundary. The resulting manifold X', depicted in Fig. 2 is therefore defined

$$X' := (X \setminus \phi(S^p \times \overset{\circ}{D}{}^{q+1})) \cup_\phi D^{p+1} \times S^q.$$

The *trace of the surgery on* ϕ is the manifold T_ϕ obtained by gluing the cylinder $X \times [0,1]$

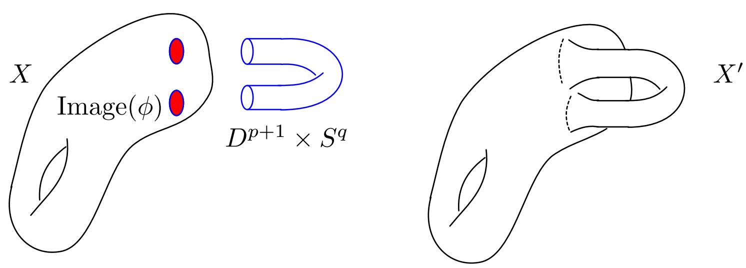

FIGURE 2. Performing a surgery on the embedding ϕ

to the disk product $D^{p+1} \times D^{q+1}$ via the embedding ϕ. This is done by attaching $X \times \{1\}$ to the boundary component $S^p \times D^{q+1}$ via $\phi : S^p \times D^{q+1} \hookrightarrow X \hookrightarrow X \times \{1\}$. Finally, we denote by W', the manifold obtained by gluing W to T_ϕ by identification of ∂W with $X \times \{0\}$; see Fig. 3.

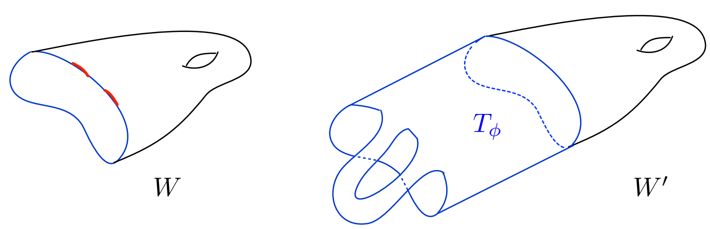

FIGURE 3. The manifold W' obtained by attaching W to the trace of the surgery on ϕ, T_ϕ

2.2. **The Gromov-Lawson Construction.** Let $S^p \hookrightarrow X$ be an embedding with trivial normal bundle, where $n - p \geq 3$. We fix a reference metric on X, and denote by N the total space of the normal bundle over the sphere $S^p \hookrightarrow X$. We use the reference metric to choose an orthonormal frame over S^p; this specifies a bundle isomorphism $S^p \times \mathbb{R}^{q+1} \to N$, where $p+q+1 = n$. Then we consider the following composition

$$\phi_\rho : S^p \times D^{q+1}_\rho \qquad S^p \qquad \mathbb{R}^{q+1} \to N \xrightarrow{\exp} X.$$

Here $D^{q+1}_\rho \subset \mathbb{R}^{q+1}$ is the Euclidean disk of radius ρ, and exp is the exponential map with respect to the reference metric. We fix a constant $\bar{\rho} > 0$ such that the map ϕ_ρ is an embedding for all $\rho \in (0, \bar{\rho}]$. We denote by $N_\rho \cong S^p \times D^{q+1}_\rho$ the tubular neighborhood of radius ρ. We call the embedding

$$\phi := \phi_\rho : N_\rho \cong S^p \times D^{q+1}_\rho \longrightarrow X, \tag{2.1}$$

a *surgery datum.*

Now we consider the standard metric $ds_p^2 + g^{q+1}_{\text{torp}}$ on N_ρ. The Gromov-Lawson Surgery Theorem provides a technique for replacing the psc-metric $g \in \mathcal{R}^+(X)$ with a new psc-metric $g_{\text{std}} \in \mathcal{R}^+(X)$. The metric g_{std} satisfies the condition that, for some $\rho \in (0, \bar{\rho})$, $g_{\text{std}}|_{N_\rho} = ds_p^2 + g^{q+1}_{\text{torp}}$ while outside $N_{\bar{\rho}}$, $g_{\text{std}} = g$. The metric g_{std} is thus prepared for surgery. By removing part of the standard piece $(S^p \times D^{q+1}, ds_p^2 + g^{q+1}_{\text{torp}})$ and replacing it with $(D^{p+1} \times S^q, g^{p+1}_{\text{torp}} + \delta^2 ds_q^2)$, for some appropriately small $\delta > 0$, we obtain a psc-metric $g' \in \mathcal{R}^+(X')$. This is depicted a little later on in Fig. 9. Thus, the most important part of the Gromov-Lawson construction is standardising the metric g.

2.3. **Isotopy and Concordance.** In a moment we will describe an additional observation about the Gromo-Lawson construction. First, we recall two important equivalence relations on the space of psc-metrics: isotopy and concordance. Two psc-metrics $h_0, h_1 \in \mathcal{R}^+(X)$ are *isotopic* if there exists a path $\gamma : I \longrightarrow \mathcal{R}^+(X)$ so that $\gamma(0) = h_0$ and $\gamma(1) = h_1$. The path γ is called an isotopy. Two psc-metrics $h_0, h_1 \in \mathcal{R}^+(X)$ are said to be *concordant* if the space $\mathcal{R}^+(X \times I)_{h_0,h_1}$ is non-empty. Elements of this space are called *concordances.* It is a well-known fact that isotopic metrics are concordant; see Lemma 1.3 of [22]. In particular, suppose $\gamma(t) = h_t \in \mathcal{R}^+(X)$, is an isotopy where $t \in I$. Consider the warped product metric $h_t + dt^2$ on $X \times I$. This psc-metric does not necessarily have positive scalar curvature as negative curvature may arise in the t-direction. However, by appropriately "slowing down" change in this direction we can minimize negative curvature and use the slices to obtain overall positivity.

It will be quite useful later on to have a well-defined way of doing this. In other words, we would like to specify a map

$$\begin{aligned} C^\infty(I, \mathcal{R}^+(X)) &\longrightarrow \mathcal{R}^+(X \times I) \\ h_t &\longmapsto \bar{h}, \end{aligned} \tag{2.2}$$

which turns an isotopy h_t into a concordance $\bar{h}$ between h_0 and h_1. We do this as follows. Fix some family of appropriate cut-off functions $\nu_\lambda : [0, \lambda] \to [0, 1]$ as shown on the left image in Fig. 4 defined so that $\nu_\lambda(s) = \lambda \nu_1(\frac{s}{\lambda})$. Note that $\nu_\lambda = 0$ on $[0, \frac{\lambda}{3}]$, $\nu = 1$ on $[\frac{2\lambda}{3}, \lambda]$ and $\lambda > 0$. For each λ we obtain a metric $h'_{\nu(s)} + ds^2$ on $X \times [0, \lambda]$. In Lemma 1.3 of [22] we show that, provided λ is sufficiently large, this metric has positive scalar curvature. The resulting psc-metric on $X \times [0, \lambda]$ then pulls back via the obvious rescaling to a psc-concordance $\bar{h}$ on $X \times I$. One can easily describe a formula for obtaining the quantity λ based on the infimum of the scalar curvature of the metric h_t over all t. This determines a well-defined concordance.

2.4. **An isotopy form of the Gromov-Lawson construction.** Regarding the Gromov-Lawson construction, we now have the following lemma.

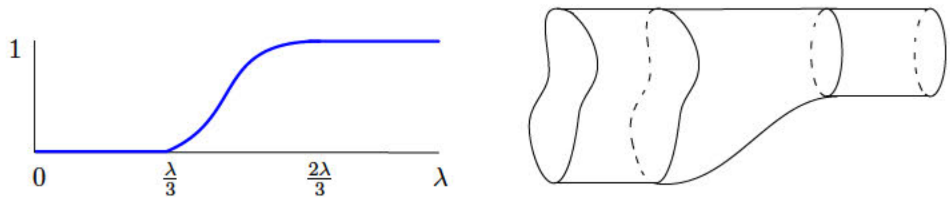

FIGURE 4. The cutoff function ν_1 left and the concordance obtained from h_t after appropriate rescaling.

Lemma 2.4.1. (cf. [22]) *Let $g \in \mathcal{R}^+(X)$. There exists an isotopy $\gamma : I \to \mathcal{R}^+(X)$ such that $\gamma(0) = g$ and $\gamma(1) = g_{\text{std}}$, a psc-metric which satisfies $g_{\text{std}}|_{N_\rho} = ds_p^2 + g_{\text{torp}}^{q+1}$ for some $\rho < \bar{\rho}$.*

We note that the isotopy γ depends continuosly on the metric g and on the choices of the reference metric and the bundle isomorphism ϕ. Such an isotopy is called a *Gromov-Lawson isotopy.* A detailed account of the construction of such an isotopy is given in [22] and so we will not recall it all here. It is, however, worth providing the following very brief summary of the construction.

Part 1: By specifying a curve Γ of the type shown on the left of Fig. 5, it is possible to adjust the metric g inside N_ρ. This is done by pushing out geodesic spheres along disk fibres in the space $N_{\bar{\rho}} \times [0, \infty)$ in a way that is determined by Γ. More precisely, for each point $x \in S^p$, a geodesic sphere of radius r is pushed out to lie in the slice $\{x\} \times D_\rho^{q+1} \times \{t\}$ where $(t, r) \in \Gamma$. We note that the curve Γ begins on the r-axis as a vertical line segment. Hence, the induced metric on the resulting hypersurface, which we denote S_Γ and depict in the right image of Fig. 5, extends smoothly onto the rest of X as the original metric g. We denote this new metric g_Γ. Crucially, Gromov and Lawson show that the curve Γ may be chosen so that g_Γ has positive scalar curvature.

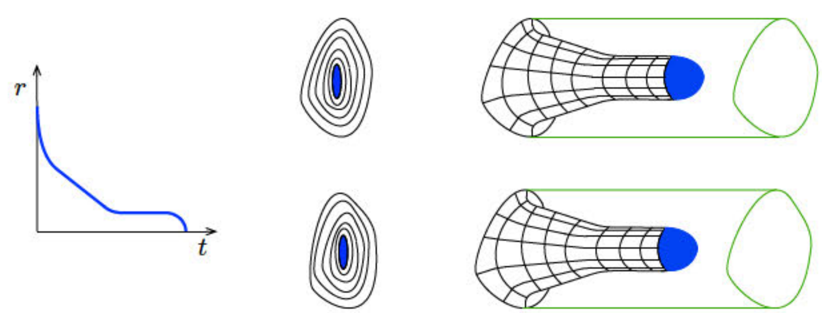

FIGURE 5. The curve Γ (left), geodesic spheres on the fibres of neighbourhood N_ρ (middle) and the hypersurface $S_\Gamma \subset N_\rho \times [0, \infty)$ obtained by pushing out the geodesic spheres with respect to Γ

A curve Γ of this type, which induces a positive scalar curvature metric g_Γ, is known as a *Gromov-Lawson curve.* Essentially, this is a smooth curve which consists of 5 regions. Roughly speaking, the first is a vertical line segment along the r-axis, the second and fourth form part of the graph of a function over the t-axis with non-negative second derivative, the third a straight-line segment, while the fifth is part of the graph of a function over the

t-axis with non-positive second derivative meeting the t-axis at a right angle. Note that the curve which only runs down the r-axis satisfies all of these conditions and so is always a Gromov-Lawson curve.

Part 2: The next step is to show that there is an isotopy from the metric g_Γ back to the original metric g. It follows from calculations done in [22], that the family of psc-metrics induced by performing the obvious homotopy, shown in Fig. 6 below from the curve Γ back to the vertical curve Γ_0 along the r-axis, is an isotopy. This homotopy consists of first shrinking to an infinitesimal size, the neck of the torpedo curve, and then performing a linear homotopy between the resulting curve and the vertical one. At each stage in the homotopy, the curve remains a Gromov-Lawson one.

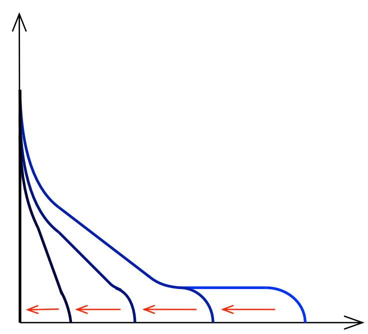

FIGURE 6. The homotopy of the curve Γ which induces an isotopy from g_Γ to g

Part 3: There is still a good deal of work to be done in making the metric standard. One obvious problem is that the metric g (and the resulting metric g_Γ) may not be a product metric. It is possible, however, to construct an isotopy g_Γ through psc-metrics to one which, on N_ρ, is a Riemannian submersion with base metric $g|_{S^p}$ and fibre metric g_{torp}^{q+1}. Using the formulae of O'Neill, it is shown that the positivity of the curvature on the disk factor allows us to isotopy through psc-submersion metrics to obtain the desired metric g_{std}. This process makes use of the shape of Γ which makes the resulting metric on fibres, for sufficiently small radius, arbitrarily close to a torpedo metric. As the space of psc-metrics is open, we may find a linear isotopy which moves the fibre metric to one which ends as precisely, a torpedo metric. This isotopy allows us to transition along the "neck" of the "almost torpedo" metric, rescaling as appropriate to obtain the desired metric; see Fig. 7. To ease with notation, we will retain the name g_Γ for this metric.

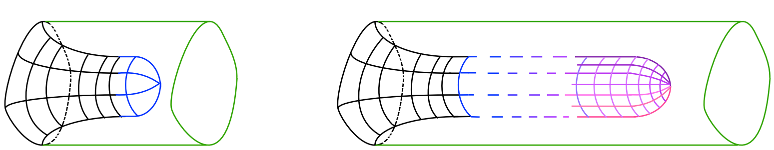

FIGURE 7. Adjusting the nearly torpedo-ended metric (left) to obtain a metric which ends as a torpedo (right)

Combining parts 1–3 gives rise to the isotopy γ described in Lemma 2.4.1, which we describe as a Gromov-Lawson isotopy on the original metric g and represent schematically as a gradual pushing out of standard torpedoes in Fig. 8 below.

We note that, given psc-metric g and a surgery datum ϕ, there is not a unique Gromov-Lawson isotopy, but rather a family of isotopies parameterised by a contractible space of choices. Details of the particular choices involved can be found in [22], although in our case we will be able to make certain simplifying assumptions. For a start, consider the case when the original metric $g|_N$ pulls back to a product metric $g_1 + g_2$ on $S^p \times D^{q+1}$. Part 3 above is now greatly simplified and only involves an adjustment of the fibre metric and a linear homotopy of the base metric. With this assumption in place we obtain the following lemma.

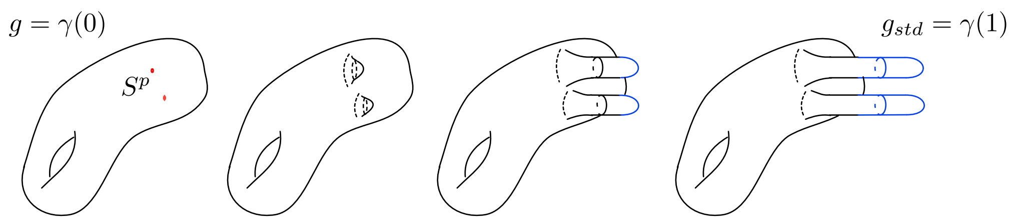

FIGURE 8. Some stages of the isotopy $\gamma(t)$ as t varies from 0 to 1

Lemma 2.4.2. *Suppose the psc-metric g takes the form of a product metric $g_1 + g_2$ on $N_\rho \cong S^p \times D^{q+1}$, for some $\rho \in (0, \bar{\rho}]$. Let $\gamma : I \to \mathcal{R}^+(X)$ be any Gromov-Lawson isotopy originating at g. Then γ can be specified by a choice of Gromov-Lawson curve Γ and a scaling parameter $\bar{\epsilon} > 0$.*

Proof. The specification of the curve Γ proceeds essentially as before. Recall this curve ends as a radius δ-"torpedo"-shaped curve for some $\delta > 0$. The resulting fibre metric can be made arbitrarily close to a δ-torpedo metric by choosing δ sufficiently small. This ensures, on each fibre, a linear homotopy through psc-metrics on the sphere S^q between the induced geodesic sphere metric and the standard one. In order to use this isotopy to transition on fibres to a standard torpedo at the end, we need a rescaling parameter $\bar{\epsilon}$ to potentially slow down change in the radial direction. Here we make use of the fact that, for any isotopy $g_t, t \in I$ of psc-metrics on a compact manifold X, there is a parameter $\bar{\epsilon} > 0$, so that the metric $g_{\epsilon t} + dt^2$ has positive scalar curvature, for all $\epsilon \in (0, \bar{\epsilon})$; see Lemma 1.3 of [22]. All that remains is to perform a linear homotopy on the base metric to make it a standard round one. This may necessitate an adjustment of Γ to make δ smaller. Importantly however, this is independent of the $\bar{\epsilon}$ choice above. Indeed, shrinking δ would actually allow for a larger value of $\bar{\epsilon}$. $\square$

2.5. **Review of Theorem 1.2.1.** Suppose now that we fix a surgery datum ϕ. We will be interested in the subspace of $\mathcal{R}^+(X)$:

$$\mathcal{R}^+_{\text{std}}(X)_{\bar{\rho}} = \{\ g \in \mathcal{R}^+(X) \mid g|_{N_\rho} = g_{\text{std}} \ \text{ for some } \rho < \bar{\rho}\ \},$$

although, to simplify notation, we write $\mathcal{R}^+_{\text{std}}(X) = \mathcal{R}^+_{\text{std}}(X)_{\bar{\rho}}$. There is a corresponding space $\mathcal{R}^+_{\text{std}}(X')$, defined with respect to the complementary surgery sphere $S^q \hookrightarrow X'$. The following lemma is immediate, as suggested by the correspondence depicted in Fig. 9.

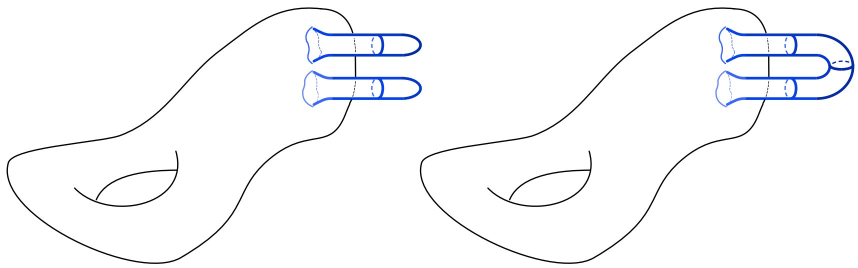

FIGURE 9. Corresponding elements of $\mathcal{R}^+_{\text{std}}(X)$ and $\mathcal{R}^+_{\text{std}}(X')$

Lemma 2.5.1. *The spaces $\mathcal{R}^+_{\text{std}}(X)$ and $\mathcal{R}^+_{\text{std}}(X')$ are homeomorphic.*

To prove Theorem 1.2.1, it suffices to show that $\mathcal{R}^+_{\text{std}}(X)$ and $\mathcal{R}^+(X)$ are homotopy equivalent. To do this we need to define an intermediary space. For a given $g \in \mathcal{R}^+(X)$, we say that a metric $h \in \mathcal{R}^+(X)$ is *reachable by a Gromov-Lawson isotopy on g* if there is a Gromov-Lawson isotopy $\gamma : I \to \mathcal{R}^+(X)$ so that $\gamma(0) = g$ and $\gamma(t) = h$ for some $t \in I$. With this in mind we define, for each $g \in \mathcal{R}^+(X)$, the subspace $\mathcal{GL}(X, g)$ of $\mathcal{R}^+(X)$, of psc-metrics on X which are reachable by some Gromov-Lawson isotopy on g. Consider the space

$$\mathcal{R}^+_{\text{Astd}}(X) := \bigcup_{g \in \mathcal{R}^+_{\text{std}}(X)} \mathcal{GL}(X, g).$$

This space is known as the space of *almost standard* psc-metrics on X and satisfies

$$\mathcal{R}^+_{\text{std}}(X) \subset \mathcal{R}^+_{\text{Astd}}(X) \subset \mathcal{R}^+(X).$$

Again, note that this space is defined with respect to a fixed surgery datum ϕ, a fixed reference metric and radial distance $\bar{\rho}$.

The need for such an intermediary space arises when utilising the Gromov-Lawson "standardising" construction as realised in Lemma 2.4.1. This construction goes through for compact families of psc-metrics, and so one would like to show that the relative homotopy groups $\pi_k(\mathcal{R}^+(X), \mathcal{R}^+_{\text{std}}(X))$ are all trivial by applying this process to families of psc-metrics which represent elements of these groups. Such a family would consist of a map from a disk into $\mathcal{R}^+(X)$ restricting on the boundary sphere to psc-metrics which are already standard. The problem however, once we start the Gromov-Lawson process on a family of metrics, is that metrics which are already standard are not fixed. Initially at least, the Gromov-Lawson construction moves such metrics outside of the standard space $\mathcal{R}^+_{\text{std}}(X)$. Moreoever, although the isotopy eventually finishes with a metric which is back in this standard space, the resulting metric may be different from the original one. That said, the damage done to a standard metric during this process is not severe and is contained in the larger space $\mathcal{R}^+_{\text{Astd}}(X)$. In particular, in [24], the author demonstrates that the space $\mathcal{R}^+_{\text{Astd}}(X)$ is homotopy equivalent to $\mathcal{R}^+_{\text{std}}(X)$. By then utilising the family version of the Gromov-Lawson construction, the author goes on to show that the groups $\pi_k(\mathcal{R}^+(X), \mathcal{R}^+_{\text{Astd}}(X))$ are all trivial. Via a well known theorem of Whitehead, this gives rise to the following result, which is crucial in the proof of Theorem 1.2.1.

Lemma 2.5.2. (see [24])

(1) *The homotopy type of the space* $\mathcal{R}^+_{\mathrm{std}}(X)$ *does not depend on the above choices of reference metric or constant* $\bar{\rho}$*.*
(2) *The spaces* $\mathcal{R}^+_{\mathrm{std}}(X)$ *and* $\mathcal{R}^+(X)$ *are homotopy equivalent.*

Reapplying this lemma with respect to the manifold X' and the complementary surgery sphere $S^q \hookrightarrow X'$, we obtain a homotopy equivalence between $\mathcal{R}^+_{\mathrm{std}}(X')$ and $\mathcal{R}^+(X')$. By combining this with Lemma 2.5.1, Theorem 1.2.1 follows.

2.6. **An observation about Gromov-Lawson "reachable" metrics.** Let us return again to the space $\mathcal{GL}(X,g)$ of psc-metrics which are reachable by Gromov-Lawson isotopy on g. We will assume that the metric g takes a product structure on the neighbourhood N_ρ. Thus, by lemma 2.4.2, we may think of each such metric as arising from g via a Gromov-Lawson curve Γ and some choice of rescaling parameter ϵ. With this in mind, we denote by $\tilde{\mathcal{GL}}(X,g)$, the space of all triples (h,Γ,ϵ), where $h \in \mathcal{GL}(X,g)$ is a psc-metric reachable from g by the Gromov-Lawson isotopy specified by Γ and ϵ. Let π denote the projection map

$$\pi : \tilde{\mathcal{GL}}(X,g) \longrightarrow \mathcal{GL}(X,g),$$

which sends each triple (h,Γ,ϵ) to the metric h, forgetting the extra data. We now obtain the following lemma.

Lemma 2.6.1. *The spaces* $\tilde{\mathcal{GL}}(X,g)$ *and* $\mathcal{GL}(X,g)$ *are contractible spaces.*

Proof. Recall that each triple (h,Γ,ϵ) specifies an isotopy $(h_t,\Gamma_t,\epsilon_t), t \in I$, where $h_0 = h, \Gamma_0 = \Gamma$ and $\epsilon_0 = \epsilon$ while $h_1 = g$, Γ_1 is the vertical curve and $\epsilon_1 = 1$. Thus, the family of maps $c_t : \tilde{\mathcal{GL}}(X,g) \to \tilde{\mathcal{GL}}(X,g)$ which sends (h,Γ,ϵ) to $(h_t,\Gamma_t,\epsilon_t)$ determines a deformation retract of $\tilde{\mathcal{GL}}(X,g)$ to the point $(g,\Gamma_1,1)$.

Let $s : \mathcal{GL}(X,g) \to \tilde{\mathcal{GL}}(X,g)$ be a map which satisfies the condition that $\pi \circ s$ is the identity map. The fact that π and s form part of a homotopy equivalence then follows from examination of the families of composition maps $\pi \circ c_t \circ s$ and $s \circ \pi \circ c_t$. $\square$

The space $\tilde{\mathcal{GL}}(X,g)$ will play an important role a little later on.

2.7. **A variation on the torpedo metric.** Before proceeding any further, there is a variation on the torpedo metric which we must introduce. Consider the torpedo metric g^{q+2}_{torp} and "stretched torpedo" metric g^{q+2}_{storp} on the disk D^{q+2} shown in Fig. 10. The metric g^{q+2}_{storp}, pictured in Fig. 10, is approximately given as follows. First, we split the disk D^{q+2} as

$$D^{q+2} = D^{q+2}_- \cup_{D^{q+1}} (D^{q+1} \times [r_-, r_+]) \cup_{D^{q+1}} D^{q+2}_+.$$

Then the metric g^{q+2}_{storp} is given as

$$g^{q+2}_{\mathrm{storp}}|_{D^{q+2}_\pm} = g^{q+2}_{\mathrm{torp}}|_{D^{q+2}_\pm}, \quad g^{q+2}_{\mathrm{storp}}|_{D^{q+1}\times[r_-,r_+]} = g^{q+1}_{\mathrm{torp}} + dr^2,$$

where g^q_{torp} is the torpedo metric on the equator, and r is the parameter along the interval in the product $D^q \times [r_-, r_+]$. Of course, some minor adjustments need to be made near certain parts of the boundaries of these individual pieces in order that they glue together smoothly. This is done in detail in [23] and so we will not concern ourselves with it here, suffice to say that these adjustments can be done so that the scalar curvature of the resulting metric is made arbitrarily close to the scalar curvature of the above piecewise approximation. We denote by $\hat{g}_{\mathrm{storp}}^{q+2}$ the metric g_{storp}^{q+2} restricted to the following manifold with corners

$$(D^{q+1} \times [r_-, r_+]) \cup_{D^{q+1}} D_+^{q+2}.$$

This is depicted in the rightmost image of Fig. 10.

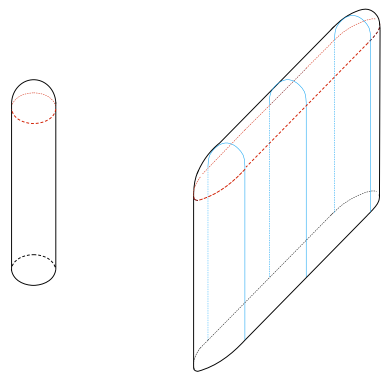

FIGURE 10. The torpedo metrics g_{torp}^{q+1}, g_{storp}^{q+1} and $\hat{g}_{\mathrm{storp}}^{q+1}$

2.8. **Extending the Gromov-Lawson Construction over the trace of a surgery.** Let us now consider the trace T_ϕ of the surgery on $\phi := \phi_\rho : N_\rho \cong S^p \times D_\rho^{q+1} \longrightarrow X$, where as usual $p+q+1=n$ and $q \geq 2$. Recall that this is obtained by attaching to $X \times I$, the disk product $D^{p+1} \times D^{q+1}$, via the embedding ϕ. Thus $\partial T_\phi = X \sqcup X'$ where X' is, as above, the manifold obtained by surgery on X. This was depicted earlier in Fig. 3. In [22] we describe in detail a procedure for extending a psc-metric g over T_ϕ to obtain an element $\bar{g} \in \mathcal{R}^+(T_\phi)_{g,g'}$ where g' is the metric obtained by a Gromov-Lawson surgery on g with respect to ϕ. For details, the reader is referred to Theorem 2.2 in [22]. Roughly, the metric $\bar{g}$ is constructed as follows.

1. Equip $X \times I$ with a concordance arising from the Gromov-Lawson isotopy γ described above. This is depicted in the upper left image in Fig. 11.
2. Attach a piece which is almost $(D^{p+1} \times D^{q+1}, g_{\mathrm{torp}}^{p+1} + g_{\mathrm{torp}}^{q+1})$ but which contains an extra smoothing region to avoid corners. This is depicted in the upper right of Fig. 11.
3. Of course, on this extra smoothing region, the metric is not a product. However, in the proof of Theorem 2.2 of [22], we show how to adjust the metric on this region so as to obtain one which is a product near the boundary.

In hindsight, this method can be made a little neater. Before attaching $D^{p+1} \times D^{q+1}$, we make a further preparation. On the standard part of the initial concordance, where the metric takes the form $dt^2 + ds_p^2 + g_{\mathrm{torp}}^{q+1}$, we adjust the metric so that, near the boundary it takes the form $ds_p^2 + \hat{g}_{\mathrm{storp}}^{q+1}$. The fact that such an adjustment is possible (and that the resulting psc-metric is isotopic to the original) is shown in detail later on in Lemma 7.0.13. We depict this in the lower left image of Fig. 11. We call this "putting on boots". Performing surgery whilst preserving the product structure on this metric is now trivial. The resulting psc-metric, $\bar{g}$, is shown in the lower right of Fig. 11. We call this metric a *Gromov-Lawson trace*.

Remark 2.8.1. In Theorem 2.2 of [22], we actually consider the more general case of a *Gromov-Lawson cobordism*, which consists of a union of Gromov-Lawson traces determined by an appropriate Morse function.

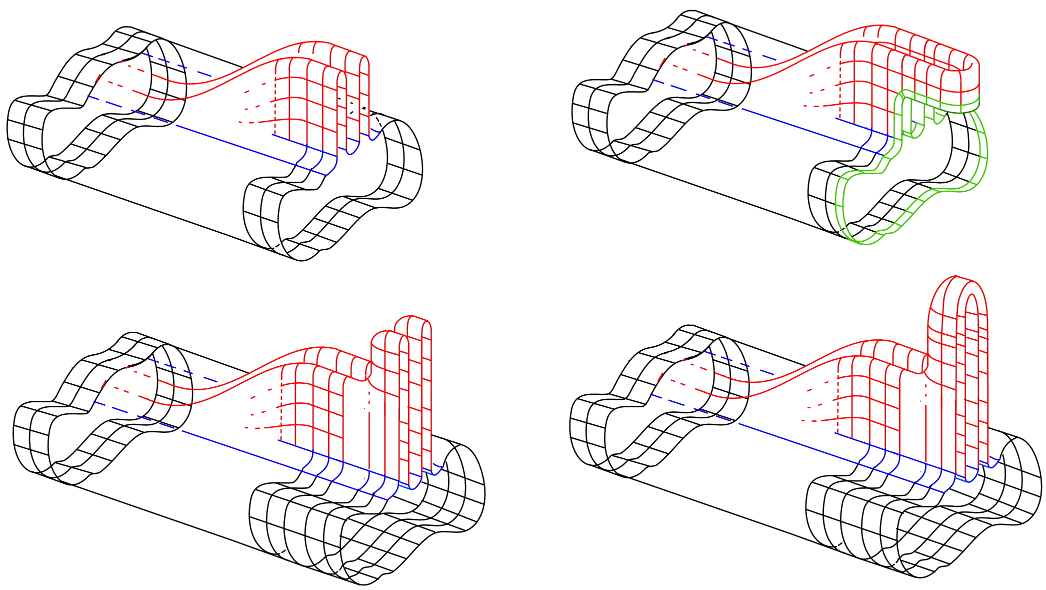

FIGURE 11. The concordance of g and g_{std} (upper-left), the original Gromov-Lawson trace construction (upper right), putting on boots (lower left) and the neater Gromov-Lawson trace (lower-right)

3. Surgery on the boundary

Recall, that in the proof of Theorem 1.2.1, there are are two surgeries we need to consider. The first one is a surgery on an embedded p-sphere S^p in X. The second one is a complementary surgery on the sphere S^q in X', the sphere factor of the attached handle. This undoes the first surgery and so returns (up to diffeomorphism) the original manifold X. Provided both surgeries are in co-dimension at least three i.e. $p, q \geq 2$, the conclusion of the theorem holds. Moreover, if the ambient manifold X is replaced by a manifold with non-empty boundary, W, and if the embedded surgery sphere S^p is in the interior of W, then, again, the theorem goes through without difficulty. The case where the embedded surgery sphere S^p lies in the boundary of W is slightly more complicated, in particular with regard to an appropriate second complementary surgery which undoes the first one. There are two types of surgery we must now consider on W.

Suppose that $i : S^p \to \partial W = X$ is an embedding, as above and depicted in the left image of Fig. 12, which extends to an embedding $\phi : S^p \times D^{q+1} \hookrightarrow \partial W = X$. Recall here that $p + q + 1 = n$, the dimension of X and that W has dimension $n + 1$. We will now examine two types of surgery: one where the image of the sphere S^p bounds a $(p+1)$-dimensional disk in W, the other involving no such assumption.

3.1. **Surgery Type 1.** Here we assume that the embedding $\phi : S^p \times D^{q+1} \hookrightarrow \partial W = X$ extends to an embedding

$$\bar{\phi} : S^p \times D_+^{q+2} \to W,$$

where D_+^{q+2} is the closed unit upper-half disk in $\mathbb{R}^{q+2}$ and the product

$$(S^p \times (D_+^{q+2} \setminus D^{q+1}))$$

is contained in the interior of W as shown in the middle image of Fig. 12. By doing surgery on the embedding $\bar{\phi}$ we obtain the manifold W' defined as

$$W' = (W \setminus \bar{\phi}(S^p \times \mathrm{int}(D_+^{q+2})) \cup_{\bar{\phi}} (D^{p+1} \times S_+^{q+1}).$$

Here S_+^{q+1} is the closed upper hemisphere of S^{q+1} in $\mathbb{R}^{q+2}$ and appropriate smoothing adjustments are made at the corners. The boundary $\partial W'$ of W', is X', the manifold obtained by surgery on X with respect to the embedded surgery sphere S^p; see right image of Fig. 12.

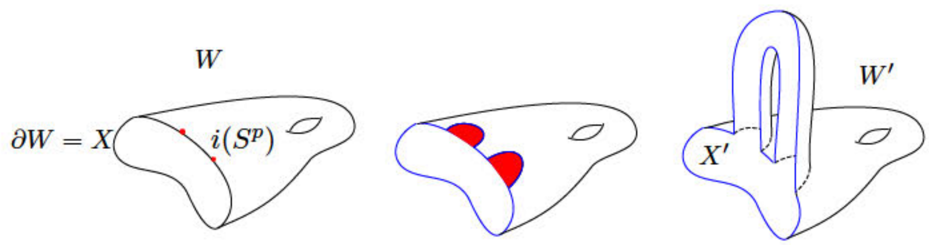

FIGURE 12. Surgery of type 1 on a manifold with boundary

The reader may be curious as to why we extend the embedding ϕ into the interior of W. Obviously, the manifold W' above obtained by a surgery of type 1 is diffeomorphic to that obtained by simply attaching along the boundary of W, the trace of a surgery on the embedding $\phi : S^p \times D^{q+1} \to \partial W$, as in Fig. 3, or indeed attaching the product $D^{p+1} \times D^{q+1}$ directly to ∂W with appropriate smoothing . When it comes to making metric adjustments however, it is often geometrically convenient to consider the surgery in the way we do.

3.2. **Surgery Type 2.** Here we consider the case where the embedded surgery sphere $i : S^p \hookrightarrow \partial W$ bounds a disk in W. More precisely, suppose that the embedding $i : S^p \hookrightarrow \partial W$ extends to an embedding $\bar{i} : S_+^{p+1} \hookrightarrow W$, and, in turn, the embedding $\phi : S^p \times D^{q+1} \hookrightarrow \partial W$ extends to an embedding

$$\bar{\phi} : S_+^{p+1} \times D^{q+1} \hookrightarrow W,$$

which satisfies

(i) $\bar{\phi}|_{S_+^{p+1} \times \{0\}} = \bar{i}$ and $\bar{\phi}|_{S^p \times D^{q+1}} = \phi$,
(ii) $\bar{\phi}(\mathrm{int}(S_+^{p+1}) \times D^{q+1})$ are contained in the interior of W,
(iii) $\bar{\phi}(S_+^{p+1} \times \{x\})$ intersects transversely with ∂W for all $x \in D^{q+1}$.

We now do surgery on W with respect to $\bar{\phi}$ to obtain

$$W' = (X \setminus \bar{\phi}(\mathrm{int}(S_+^{p+1}) \times D^{q+1})) \cup_{\bar{\phi}} (D_+^{p+2} \times S^q).$$

As in the previous case, the boundary $\partial W'$ is precisely the manifold X' obtained by the original surgery on X; see Fig. 13.

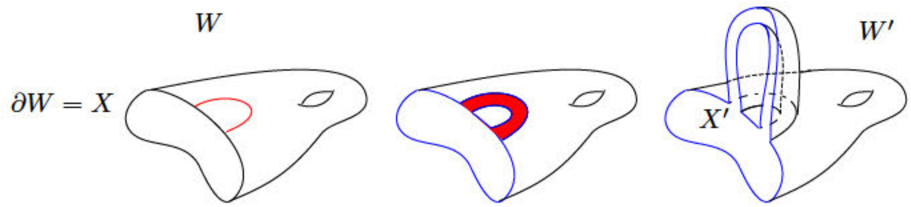

FIGURE 13. Surgery of type 2 on a manifold with boundary

We conclude our discussion by pointing out that a surgery of type 1 may be reversed by a complementary surgery of type 2 and vice versa. More precisely, we assume that W' is obtained from W by a surgery of type 1 on an embedding $\phi : S^p \times D^{q+1} \hookrightarrow \partial W$; see left and middle images of Fig. 14. Then we may undo this surgery and restore W by performing a surgery of type 2 on the resulting embedding $D^{p+1} \times S_+^{q+1}$; see rightmost image of Fig. 14.

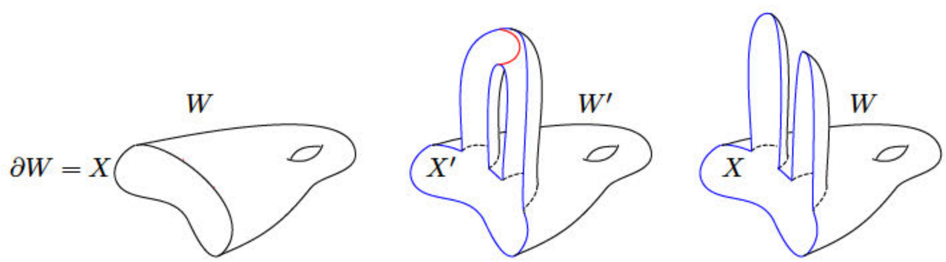

FIGURE 14. Reversing a surgery of type 1 with a surgery of type 2

In the case where W' is obtained from W by a surgery of type 2 on an embedded S^{p+1}, we may restore W by a surgery of type 1 on the embedded sphere S^q which lies naturally in the newly attached $D_+^{p+2} \times S^q$. This is illustrated in Fig. 15 although to simplify the picture we describe the situation only locally.

4. Geometric preparation for a surgery on the boundary

We will now describe the geometric setting for Theorem A. Recall, W is a smooth compact manifold with $\dim W = n+1$, and non-empty boundary $\partial W = X$. We recall that the manifold W comes with a collar $c : (-1, 0] \times X \hookrightarrow W$. We denote by $\mathcal{R}^+(W)$, the space of all psc-metrics on W which take a product structure near the boundary on the image $c\left((-\frac{1}{2}, 0] \times X\right)$. More precisely, $g \in \mathcal{R}^+(W)$ if $c^*g = dt^2 + h$ on $(-\frac{1}{2}, 0] \times X$ for some $h \in \mathcal{R}^+(X)$. Furthermore, given a metric $h \in \mathcal{R}^+(X)$, we define the subspace $\mathcal{R}^+(W)_h$ consisting of psc-metrics $g \in \mathcal{R}^+(W)$ which restrict on the boundary to h, i.e. so that $c^*g|_{\partial W} = h$.

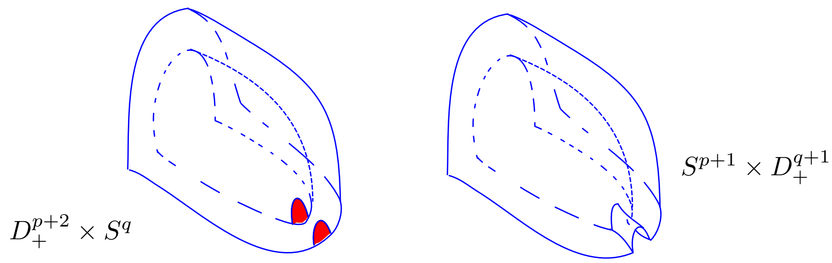

FIGURE 15. Reversing a surgery of type 2 with a surgery of type 1

4.1. **A quasifibration of spaces of psc-metrics.** At this point it is useful to recall the following observation by Chernysh in [6]. We will make use this several times in the paper, especially in the proof of Theorem C. Let res denote the restriction map

$$\begin{aligned} \mathrm{res} : \mathcal{R}^+(W) &\longrightarrow \mathcal{R}^+(\partial W) \\ g &\longmapsto g|_{\partial W}. \end{aligned} \tag{4.1}$$

Thus, for any $h \in \mathcal{R}^+(X)$, the space $\mathcal{R}^+(W)_h$ is precisely $\mathrm{res}^{-1}(h)$. We next denote by $\mathcal{R}^+_0(\partial W)$, the image space $\mathrm{res}(\mathcal{R}^+(W)) \subset \mathcal{R}^+(\partial W)$. It is a theorem of Chernysh, [6] Theorem 1.1, that the map

$$\mathrm{res} : \mathcal{R}^+(W) \longrightarrow \mathcal{R}^+_0(\partial W),$$

obtained by restriction of the codomain, is a quasifibration. This means that for any psc-metrics $h \in \mathcal{R}^+(\partial W)$ and $\bar{h} \in \mathcal{R}^+(W)_h$, there are isomorphisms

$$\pi_k(\mathcal{R}^+(W), \mathcal{R}^+(W)_h, \bar{h}) \cong \pi_k(\mathcal{R}^+_0(\partial W, h)),$$

for all $k \geq 0$. In particular, we obtain the following long exact sequence in homotopy

$$\cdots \pi_k(\mathcal{R}^+(W)_h, \bar{h}) \longrightarrow \pi_k(\mathcal{R}^+(W), \bar{h}) \xrightarrow{\mathrm{res}_*} \pi_k(\mathcal{R}^+_0(\partial W), h) \longrightarrow \pi_{k-1}(\mathcal{R}^+(W)_h, \bar{h}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(\mathcal{R}^+_0(\partial W), h) \longrightarrow \pi_0(\mathcal{R}^+(W)_h, \bar{h}) \longrightarrow \pi_0(\mathcal{R}^+(W), \bar{h}) \xrightarrow{\mathrm{res}_*} \pi_0(\mathcal{R}^+_0(\partial W), h).$$

We note here that the set $\pi_0(\mathcal{R}^+_0(\partial W), h)$ is usually not trivial. It is worth also mentioning that a consequence of Chernysh's observation, which we will make use of shortly, is the following sub-case of Theorem 1.2 in [6].

Lemma 4.1.1. [6] *Suppose $h_0, h_1 \in \mathcal{R}^+(X)$ are isotopic psc-metrics. Then the spaces $\mathcal{R}^+(W)_{h_0}$ and $\mathcal{R}^+(W)_{h_1}$ are weakly homotopy equivalent.*

4.2. **A geometric surgery of type 2.** We begin by considering the set up for a surgery of type 2. Suppose $\bar{\imath} : S^{p+1}_+ \hookrightarrow W$ is an embedded hemi-sphere extending the embedding $i : S^p \hookrightarrow \partial W$. Here the inclusion $\bar{\imath}$ extends to an embedding $\bar{\phi} : S^{p+1}_+ \times D^{q+1} \hookrightarrow W$ (which also extends the surgery datum $\phi : S^p \times D^{q+1} \hookrightarrow \partial W$) satisfying the conditions of a surgery of type 2 above and with $q \geq 2$. We will make an additional assumption about the behaviour of this embedding on an annular neighbourhood of the boundary $S^p = \partial S^{p+1}_+$ inside of S^{p+1}_+. For any $\epsilon \in (0, \frac{\pi}{2})$, we denote by $\mathsf{ann}^{p+1}_\epsilon$, the annulus consisting of all points in S^{p+1}_+ whose radial distance from the boundary S^p is ϵ. Denoting by r, the radial distance from the equator on the round unit hemisphere, we identify $\mathsf{ann}^{p+1}_\epsilon$ with $[0, \epsilon) \times S^p$. In particular, for each point $x \in \mathsf{ann}^{p+1}_{\epsilon}$, we denote by $(r(x), \theta(x)) \in [0, \epsilon) \times S^p$, the corresponding point in spherical coordinates. Identifying $[0, \frac{\pi}{2})$ with $[-1, 0)$ via the obvious linear rescaling leads to the following sequence of maps

$$\mathsf{ann}^{p+1}_{\frac{\pi}{2}} \times D^{q+1} \to [0, \tfrac{\pi}{2}) \times S^p \times D^{q+1} \to [-1, 0) \times S^p \times D^{q+1},$$

$$(x, y) \mapsto (r(x), \theta(x), y) \longmapsto (\tfrac{-2(r(x))}{\pi}, \theta(x), y).$$

For brevity, we denote this composition of maps by ψ. For each $\epsilon > 0$ we consider the space

$$\bar{\phi}(S^{p+1}_+ \times D^{q+1}) \cap c\left((-\tfrac{2\epsilon}{\pi}, 0] \times X\right) \subset W.$$

Let w be any point in this space. We will now compare the pre-image of w under c with its pre-image under $\bar{\phi}$ followed by the map ψ above. Note that $c^{-1}(w) \in (-1, 0] \times X$ while $\psi \circ \bar{\phi}^{-1}(w) \in [-1, 0) \times S^p \times D^{q+1}$. In both cases, the first factor lies inside of $(-1, 0]$. For our purpose, we will require that, for some $\epsilon > 0$, these first factors agree for all $w \in \bar{\phi}(S^{p+1}_+ \times D^{q+1}) \cap c\left((-\frac{2\epsilon}{\pi}, 0] \times X\right)$. In other words, we insist that $\bar{\phi}$ satisfies the following condition.

Condition on the embedding $\bar{\phi} : S^{p+1}_+ \times D^{q+1} \hookrightarrow W$: For some $\epsilon \in (0, \frac{\pi}{2})$, the maps $p_1 \circ c^{-1}$ and $p_1 \circ \psi \circ \bar{\phi}^{-1}$ agree on the subspace $\bar{\phi}(S^{p+1}_+ \times D^{q+1}) \cap c\left((-\frac{2\epsilon}{\pi}, 0] \times X\right)$, where p_1 is projection on the first factor.

The resulting embedded surgery disk $S^{p+1}_+ \hookrightarrow W$ is represented in the left hand image of Fig. 16. We use the composition

$$[-1, 0) \times S^p \times D^{q+1} \xrightarrow{\psi^{-1}} \mathsf{ann}^{p+1}_{\frac{\pi}{2}} \times D^{q+1} \xrightarrow{\bar{\phi}} \bar{\phi}(S^{p+1}_+ \times D^{q+1}) \cap c((-\tfrac{2\epsilon}{\pi}, 0] \times X)$$

to pull back the metric $dt^2 + h$. As a consequence of the construction above, we obtain

$$(\bar{\phi} \circ \psi^{-1})^*(dt^2 + h)|_{\bar{\phi}(S^{p+1}_+ \times D^{q+1}) \cap c((-\frac{2\epsilon}{\pi}, 0] \times X)} = dt^2 + (\bar{\phi}^* h)|_{(S^p \times D^{q+1})}, \tag{4.2}$$

where, recall, $S^p = \partial S^{p+1}_+$.

Now, consider the space of metrics

$$\mathcal{R}^+_{\text{std}}(W, S^{p+1}_+)_{h_1} := \{g \in \mathcal{R}^+(W)_{h_1} : \bar{\phi}^* g = g^{p+1}_{\text{torp}} + g^{q+1}_{\text{torp}}\}. \tag{4.3}$$

As we already described, the tubular neighbourhood $S^{p+1}_+ \times D^{q+1}$ is determined by some fixed reference metric. A typical element of this space is depicted on the right-hand image in Fig. 16. One method of constructing elements of $\mathcal{R}^+_{\text{std}}(W, S^{p+1}_+)_{h_1}$ is to apply the Gromov-Lawson construction near the hemisphere S^{p+1}_+ in a similar way as it was described in Lemma 2.4.1, to a psc-metric $g \in \mathcal{R}^+(W)_h$. In particular, we obtain a Gromov-Lawson isotopy γ so that $\gamma(0) = g$ and $\gamma(1) = g_1$, where g_1 is standard near S^{p+1}_+. Then we can use the decomposition (4.2) so that the resulting psc-metric $\gamma(t)$, at every stage in the isotopy, carries an appropriate product structure near the boundary. Thus, near the boundary, we obtain a product-wise Gromov-Lawson isotopy of the metric g, which here takes the form $dt^2 + h$. Moreover, the resulting standardized psc-metric g_1 takes the form $dt^2 + h_1$ near the boundary, where h_1 is the standard metric obtained by the Gromov-Lawson isotopy restricted to the boundary metric h with respect to surgery on the sphere $\partial S^{p+1}_+ = S^p \hookrightarrow X$. This is illustrated in the right-hand image of Fig. 16. We now obtain the following result.

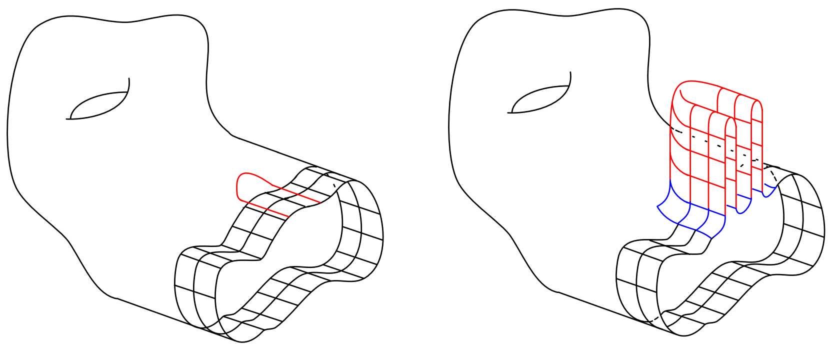

FIGURE 16. The well-behaved embedding of S^{p+1}_+ (left) and the result of a Gromov-Lawson isotopy (right)

Lemma 4.2.1. *There is a weak homotopy equivalence* $\mathcal{R}^+(W)_h \cong \mathcal{R}^+_{\mathrm{std}}(W, S^{p+1}_+)_{h_1}$.

We can immediately simplify matters by noticing that, via Lemma 4.1.1, the spaces $\mathcal{R}^+(W)_h$ and $\mathcal{R}^+(W)_{h_1}$ are weakly homotopy equivalent. Thus, it suffices to demonstrate a weak homotopy equivalence between $\mathcal{R}^+(W)_{h_1}$ and $\mathcal{R}^+_{\mathrm{std}}(W, S^{p+1}_+)_{h_1}$. Essentially, Lemma 4.2.1 follows from applying the method of Theorem 1.2.1. There are two important differences however. The first is that the embedded surgery sphere is replaced by a disk whose boundary sphere neatly embeds in ∂W. The second is that the spaces involved in this version of the theorem have fixed metrics near the boundary. The fact that the embedding $\bar{\phi}$ is well behaved near the boundary means that, near the boundary, a Gromov-Lawson isotopy on any metric $g \in \mathcal{R}^+(W)_{h_1}$ is precisely a slice-wise product of Gromov-Lawson isotopies on h_1. Thus, near the boundary, we are essentially applying a product-wise version of the proof of Lemma 2.5.2. Before we begin the proof, we need to define some relevant spaces.

Recall $\mathcal{GL}(X, h_1)$ denotes the space of psc-metrics on X which are "reachable" by a Gromov-Lawson isotopy emanating from h_1. As h_1 takes the form of a product near the embedded surgery sphere on X, we have the space $\tilde{\mathcal{GL}}(X, h_1)$ of triples (m, Γ, ϵ) which specify, for each metric $m \in \mathcal{GL}(X, h_1)$, an isotopy back to h_1. We denote by $\mathcal{R}^+(W)_{\mathcal{GL}(\partial W, h_1)}$, the subspace of $\mathcal{R}^+(W)$ consisting of metrics which restrict on the boundary ∂W as a metric in $\mathcal{GL}(\partial W, h_1)$. Furthermore, we denote by $\tilde{\mathcal{R}}^+(W)_{\mathcal{GL}(\partial W, h_1)}$, the corresponding space of triples (g, Γ, ϵ) where $(g|_{\partial W}, \Gamma, \epsilon) \in \tilde{\mathcal{GL}}(\partial W, h_1)$.

For any psc-metric $g \in \mathcal{R}^+(W)_{h_1}$, we denote by $\mathcal{GL}(W, g, h_1)$, the subspace of $\mathcal{R}^+(W)_{\mathcal{GL}(\partial W, h_1)}$ consisting of psc-metrics which are reachable by a Gromov-Lawson isotopy emanating from g. Thus, for each $m \in \mathcal{GL}(W, g, h_1)$, there is a Gromov-Lawson isotopy $\gamma : I \to \mathcal{R}^+(W)$ with $\gamma(0) = g$ and so that for some $t \in I$, $\gamma(t) = m$. As before, we will be interested in the simpler case where the metric g takes a product structure near the embedded hemisphere S^{p+1}_+. In fact, we really only care about the case when the metric $g \in \mathcal{R}^+_{\mathrm{std}}(W, S^{p+1}_+)_{h_1}$ and so is standard near S^{p+1}_+. This simplifies the choices made in determining a Gromov-Lawson isotopy. Thus, each such metric is specified by a Gromov-Lawson curve Γ and a scaling parameter ϵ. This in turn determines an isotopy from the metric back to g. For such a g, we denote by $\tilde{\mathcal{GL}}(W, g, h_1)$, the subspace of such triples (m, Γ, ϵ).

With this in mind, it is convenient to adopt the following notations

$$\mathcal{R}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)} := \bigcup_{g\in\mathcal{R}^+_{\mathrm{std}}(W,S^{p+1}_+)_{h_1}} \mathcal{GL}(W,g,h_1),$$

$$\tilde{\mathcal{R}}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)} := \bigcup_{g\in\mathcal{R}^+_{\mathrm{std}}(W,S^{p+1}_+)_{h_1}} \tilde{\mathcal{GL}}(W,g,h_1).$$

Finally, let p_1 and p_2 denote the projections

$$p_1 : \tilde{\mathcal{R}}^+(W)_{\mathcal{GL}(\partial W,h_1)} \longrightarrow \mathcal{R}^+(W)_{\mathcal{GL}(\partial W,h_1)}$$

$$p_2 : \tilde{\mathcal{R}}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)} \longrightarrow \mathcal{R}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)},$$

which send the relevant triple (m,Γ,ϵ) to the metric m.

Proof of Lemma 4.2.1. From the above definitions, it is clear that there exist maps

$$s_1 : \quad \mathcal{R}^+(W)_{\mathcal{GL}(\partial W,h_1)} \longrightarrow \tilde{\mathcal{R}}^+(W)_{\mathcal{GL}(\partial W,h_1)},$$

$$s_2 : \quad \mathcal{R}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)} \longrightarrow \tilde{\mathcal{R}}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)},$$

such that the compositions $p_1\circ s_1$ and $p_2\circ s_2$ are both identity maps. We now gather the various spaces defined above in the following commutative diagram.

$$\begin{tikzcd}
\mathcal{R}^+(W)_{h_1} \arrow[r, hook, "i_1"] & \mathcal{R}^+(W)_{\mathcal{GL}(\partial W,h_1)} \arrow[r, hook, "s_1"] & \tilde{\mathcal{R}}^+(W)_{\mathcal{GL}(\partial W,h_1)} \\
& \mathcal{R}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)} \arrow[u, hook, "i_2"] \arrow[r, hook, "s_2"] & \tilde{\mathcal{R}}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)} \arrow[u, hook, "i_3"'] \\
& \mathcal{R}^+_{\mathrm{std}}(W,S^{p+1}_+)_{h_1} \arrow[u, hook, "i_4"] &
\end{tikzcd}$$

Here, the maps i_1, i_2, i_3 and i_4 are all inclusions. To complete the proof, we must show that the maps $s_1\circ i_1$, i_3 and $s_2\circ i_4$ are homotopy equivalences.

The fact that i_3 is a homotopy equivalence, follows from application Gromov-Lawson construction on families, exactly as in the proof of Theorem 1.2.1. This shows that the relative homotopy groups

$$\pi_k(\tilde{\mathcal{R}}^+(W)_{\mathcal{GL}(\partial W,h_1)}, \tilde{\mathcal{R}}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W,h_1)}),$$

are all trivial. Once again, homotopy equivalence follows from Whitehead's theorem and the fact, due to Palais in [17], that these spaces are dominated by CW-complexes.

In the case of $s_1\circ i_1$, we will construct a homotopy inverse. Consider the triple $(g',\Gamma,\epsilon)\in \tilde{\mathcal{R}}^+(W)_{\mathcal{GL}(\partial W,h_1)}$. Denoting by h', the metric $g'|_{\partial W}$, we recall that $(g'|_{\partial W},\Gamma,\epsilon)$ is an element of $\tilde{\mathcal{GL}}(\partial W,h_1)$ and so there is a corresponding psc-istopy $h'_t, t\in I$ so that $h'_0 = h'$ and $h'_1 = h_1$. This isotopy gives rise to a concordance $\bar{h}$ between h_1 and h' via the map 2.2. Thus, we may specify a well-defined concordance arising from the triple (g,Γ,ϵ), which after an appropriate re-parameterisation, gives us a concordance $\bar{h}$ on $X\times[-1,2]$ which is h_1+dt^2

on $X \times [-1,0]$ and $h' + dt^2$ on $X \times [1,2]$. This gives rise to a map

$$\begin{aligned} r_1 : \tilde{\mathcal{R}}^+(W)_{\mathcal{GL}(\partial W, h_1)} &\longrightarrow \mathcal{R}^+(W)_{h_1} \\ (g, \Gamma, \epsilon) &\longmapsto g \cup \bar{h}, \end{aligned}$$

where $g \cup \bar{h}$ is the psc-metric obtained by attaching the concordance to g in the obvious way.

The map $r_1 \circ (s_1 \circ i_1)$ is not quite the identity map as, although the isotopy h'_t is trivial, we do attach another cylinder metric to our starting metric g. This is easily rectified by simply shrinking the cylinders. In the other direction, note that the concordance obtained by the map r_1 is slice wise. Thus, there is an obvious way to deform it back, through concordances, to the standard cylinder. This deformation gives us that the map $(s_1 \circ i_1) \circ r_1$ is homotopic to the identity map.

The fact that i_4 is a homotopy equivalence follows immediately from Lemma 3.3 of [24], which is used in the proof of Theorem 1.2.1. It remains to show that s_2 is a homotopy equivalence. The composition $p_2 \circ s_2$ is of course identity. Finally, the composition $s_2 \circ p_2$ is shown to be homtopic to the identity as follows. Suppose, for some $(g, \Gamma, \epsilon) \in \tilde{\mathcal{R}}^+_{\mathrm{Astd}}(W)_{\mathcal{GL}(\partial W, h_1)}$, $s_2 \circ p_2(g, \Gamma, \epsilon) = (g, \Gamma', \epsilon')$. Let γ and γ' be the respective psc-isotopies associated with these triples. Then $\gamma^{-1} \circ \gamma'$, where γ^{-1} denotes the isotopy γ in the reverse direction, moves (g, Γ', ϵ') back to (g, Γ, ϵ). This generalises to a well-defined family of maps deforming $s_2 \circ p_2$ back to the identity. □

4.3. **A geometric surgery of type 1.** We now consider the set-up for a surgery of type 1. As usual, we begin with an embedding $\phi : S^p \times D^{q+1} \hookrightarrow \partial W = X$. Recall here that $p + q + 1 = n$, the dimension of X and that W has dimension $n + 1$. We denote by W', the manifold obtained by attaching to W, the trace T_ϕ of a surgery on X with respect to ϕ, in the obvious way. Recall, the manifold W' has boundary $\partial W' = X'$, the manifold obtained from X by surgery on ϕ. Furthermore, we fix a collar $c' : (-1, 0] \hookrightarrow W'$ around the boundary X'. Thus, we may define the spaces $\mathcal{R}^+(W')$ and $\mathcal{R}^+(W')_{h'}$ as above, for some $h' \in \mathcal{R}^+(W')$.

Before going any further, let us recall what are trying to prove. We wish to show that for any $h \in \mathcal{R}^+(X)$ there are psc-metrics $h' \in \mathcal{R}^+(X')$ and $\bar{h} \in \mathcal{R}^+(W)_{h,h'}$ so that the map

$$\mu_{(T_\phi, \bar{h})} : \mathcal{R}^+(W)_h \to \mathcal{R}^+(W')_{h'}$$

which sends $g \in \mathcal{R}^+(W)_h$ to the element $g \cup \bar{h} \in \mathcal{R}^+(W')_{h'}$, is a weak homotopy equivalence. The existence of the psc-metrics h' and $\bar{h}$, is guaranteed by the Gromov-Lawson construction. In particular, we choose h' to be a psc-metric obtained from h via this construction and $\bar{h}$ to be a Gromov-Lawson trace between h and h'. The main work is in demonstrating that the map $\mu_{T_\phi, \bar{h}}$ is a weak homotopy equivalence.

As a first step, we consider a psc-metric $h_1 \in \mathcal{R}^+_{std}(X')$ obtained by standardizing h via a Gromov-Lawson isotopy. As described above, we utilise the well-known fact that, after an appropriate rescaling over a sufficiently long cylinder, this isotopy may be transformed into a concordance $m \in \mathcal{R}^+(M \times [-b, 0])_{h, h_1}$. Here $b > 0$ is a sufficiently large constant. Moreover, as we implied in the proof of Theoem 4.2.1 above, the slice wise nature of this concordance means that the map

$$\mu_m := \mu_{X \times [-b,0], m} : \mathcal{R}^+(W)_h \to \mathcal{R}^+(W \cup X \times [-b, 0])_{h_1},$$

is easily seen to be a homotopy equivalence. It is important to note that all aspects of the rescaling process may be chosen to be independent of the metric h except the choice of constant b. With this in mind, we will introduce the following technical tool, which will be of use to us shortly. Namely, we specify a family of embeddings $\sigma_b : (-b, 0] \times X \hookrightarrow W$ parameterised by $b > 0$, such that

(i) $\sigma_1 = c$, the collar map $c : (-1, 0] \times X \hookrightarrow W$,

(ii) $\sigma_{b'} = \sigma_b \circ \lambda_{b,b'}$, where $\lambda_{b',b} : (-b', 0] \times X \to (-b, 0] \times X$ is an obvious linear rescaling.

In particular, we have $\sigma_b(\{0\} \times X) = \partial W$. For each $b > 0$, we define the space

$$\mathcal{R}^+(W)_{b,h} = \{\ g \in \mathcal{R}^+(W) : \sigma_{b'}^* g = dt^2 + h \ \text{ on } (-b'/2, 0] \times X \text{ for some } b' > b\ \}.$$

The space $\mathcal{R}^+(W')_{b,h'}$ is defined analogously. It is easy to observe that the spaces $\mathcal{R}^+(W)_{b_1,h}$ and $\mathcal{R}^+(W)_{b_2,h}$ are homotopy equivalent for different $b_1, b_2 > 0$ via a collar stretching map λ_{b_1,b_2}. In particular, we observe

Lemma 4.3.1. *For any $b > 0$, there is a weak homotopy equivalence $\mathcal{R}^+(W)_h \cong \mathcal{R}^+(W)_{b,h}$.*

Thus, by Lemma 4.1.1, for any psc-metric $h_0 \in \mathcal{R}^+(X)$, which is isotopic to h, we may conclude the following

Lemma 4.3.2. *There is a weak homotopy equivalence $\mathcal{R}^+(W)_{b,h} \cong \mathcal{R}^+(W)_{b,h_0}$.*

For each $b > 0$, we define an embedding

$$\bar{\phi}_b : (-b, 0] \times S^p \times D^{q+1} \hookrightarrow (-b, 0] \times X,$$

given by the formula

$$\bar{\phi}_b : (x, y, t) \mapsto \sigma_b(\pi_X(\sigma_b^{-1} \circ \phi(x, y)), t),$$

where $\pi_X(-b, 0] \times X \to X$ is projection on the X factor.

Finally, we wish to generalise the definition of the space $\mathcal{R}^+_{\text{std}}(W, S_+^{p+1})_{h_1}$, defined in (4.3) above. Before doing so, we recall the metric $\hat{g}^n_{\text{torp}}$ on the disk with corners, constructed in section 2.7 and depicted in Fig. 10. For some $\beta \in (0, \frac{b}{2})$, we identify the product

$$[-\beta, 0] \times S^p \times D^{q+2} \subset [-b, 0] \times S^p \times D^{q+2}$$

with the product

$$S^p \times (D^{q+1} \times [r_-, r_+]) \subset S^p \times \left((D^{q+1} \times [r_-, r_+]) \cup D_+^{q+2}\right)$$

by switching order of the factors and by stretching linearly $[-\beta, 0]$ to $[r_-, r_+]$. This identification gives us the metric $ds_p^2 + \hat{g}^{q+2}_{\text{storp}(\beta)}$ on the manifold (with corners)

$$\begin{aligned} K :\ &= \ ([-\beta, 0] \times S^p \times D^{q+1}) \cup S^p \times D_+^{q+2} \\ &\cong \ S^p \times \left((D^{q+1} \times [r_-, r_+]) \cup_{D^{q+1}} D_+^{q+2}\right). \end{aligned} \tag{4.4}$$

This metric is depicted in the left image of Fig. 17 below. The quantity β denotes the length of the neck of the $q+1$-dimensional factor. On the right, we depict the related metric $g^{p+1}_{\text{torp}} + g^{q+2}_{\text{torp}(\beta)}$ obtained by removing the $S^p \times D_+^{q+2}$ factor and attaching a $D^{p+1} \times D^{q+2}$, equipped with the appropriate product metric.

The space $\mathcal{R}^+_{\text{std}}(W, S_+^{p+1})_{h_1}$ is now defined as follows

$$\begin{aligned} \mathcal{R}^+_{\text{std}}(W, S_+^{p+1})_{a,\beta,h_1} := \{&g \in \mathcal{R}^+(W) : \sigma_{a'}^* g = dt^2 + h_1 \text{ on } (-a'/2, 0] \times X \text{ for some } a' > a \\ &\text{and } \bar{\phi}^* g = g^{p+1}_{\text{torp}} \times g^{q+1}_{\text{torp}(\beta)}\}, \end{aligned}$$

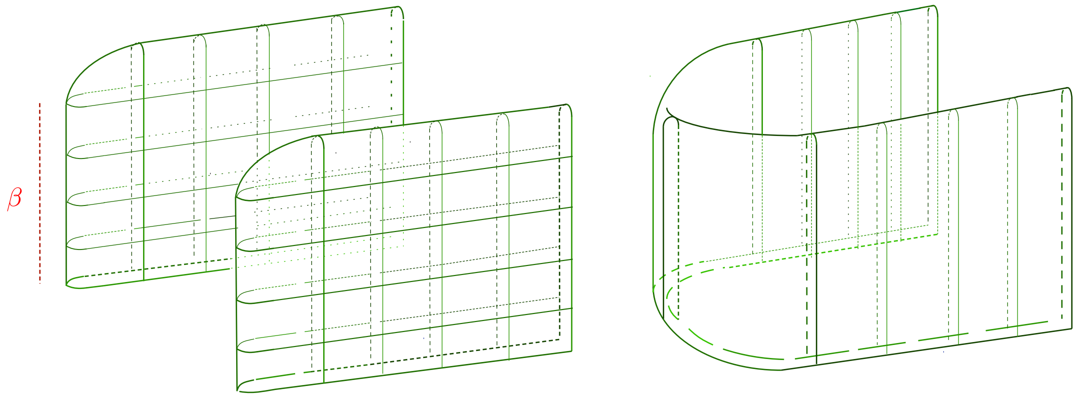

FIGURE 17. The metric $ds_p^2 + \hat{g}_{\mathrm{torp}(\beta)}^{q+1}$ (left) and $g_{\mathrm{torp}}^{p+1} + g_{\mathrm{torp}(\beta)}^{q+1}$ (right)

where, importantly, $0 < a < \beta$. To aid the reader, an arbitrary element of this space is depicted in Fig. 21 below. Recall here that $\bar{\phi} : S_+^{p+1} \times D^{q+1} \hookrightarrow W$ is the embedding, corresponding to a surgery of type 2, which extends the usual surgery embedding $\phi : S^p \times D^{q+1} \hookrightarrow \partial W$ and $h_1 \in \mathcal{R}^+(X)$. As with Lemma 4.3.1, the next lemma follows from a straightforward collar stretching isotopy.

Lemma 4.3.3. *For any a, β satisfying $0 < a < \beta$, there is a weak homotopy equivalence $\mathcal{R}_{\mathrm{std}}^+(W, S_+^{p+1})_{h_1} \cong \mathcal{R}_{\mathrm{std}}^+(W, S_+^{p+1})_{a,\beta,h_1}$.*

Henceforth, we will concentrate our attention on the collar $\sigma_b : (-b, 0] \times X \hookrightarrow W$ for some fixed large $b > 0$. To make our notation transparent, we use g for the metric $\sigma_a^* g$ on $(-a, 0] \times X$.

5. The main surgery and relevant spaces

We will now try to put the last section in context. Firstly, we return to the fixed boundary metric $h \in \mathcal{R}^+(X)$ and choose a Gromov-Lawson isotopy $\gamma : I \to \mathcal{R}^+(X)$, such that $\gamma(0) = h$ and $\gamma(1) = h_1 \in \mathcal{R}_{\mathrm{std}}^+(X)$, i.e. h_1 restricts to the standard metric $g_{\mathrm{std}} = ds_p^2 + g_{\mathrm{torp}}^{q+1}$ on $S^p \times D^{q+1} \subset X$. Let us consider now the space $\mathcal{R}^+(W)_{b,h_1}$ of psc-metrics which take the form of a product $h_1 + dt^2$ on sub-collar $X \times [-\frac{b}{2}, 0]$. For now, we will focus only on this sub-collar region. In particular, we consider the region $[-\frac{b}{2}, 0] \times S^p \times D^{q+1}$ near the embedded surgery sphere where the metric takes the form of the product $dt^2 + ds_p^2 + g_{\mathrm{torp}}^{q+1}$. We denote this region K' and, ignoring the metric momentarily, we note that $K \subset K'$ where K is the region defined above in (4.4). The region K' is represented on the left hand image of Fig.18 below. Provided b is large enough, it is possible to deform any psc-metric in $\mathcal{R}^+(W)_{b,h_1}$, making changes only in the standard region K', so that the resulting metric takes the form $ds_p^2 + \hat{g}_{\mathrm{torp}}^{q+2}$ on K; see right hand image in Fig. 18. This is Lemma 7.0.13, the proof of which is postponed until the appendix. In particular, the resulting psc-metric is an element of $\mathcal{R}^+(W)_{2\beta,h_1}$. As this construction only involves altering the standard region, we may apply the same construction to all psc-metrics thereby giving rise to a map

$$B_{b,\beta} : \mathcal{R}^+(W)_{b,h_1} \to \mathcal{R}^+(W)_{2\beta,h_1}.$$

We denote by $\mathcal{R}^+_{\rm std}(W)_{b,\beta,h_1}$, the subspace of $\mathcal{R}^+(W)_{2\beta,h_1}$ which is the image of this map. Recall here that b must be chosen sufficiently large and β satisfies $\beta < b/2$.

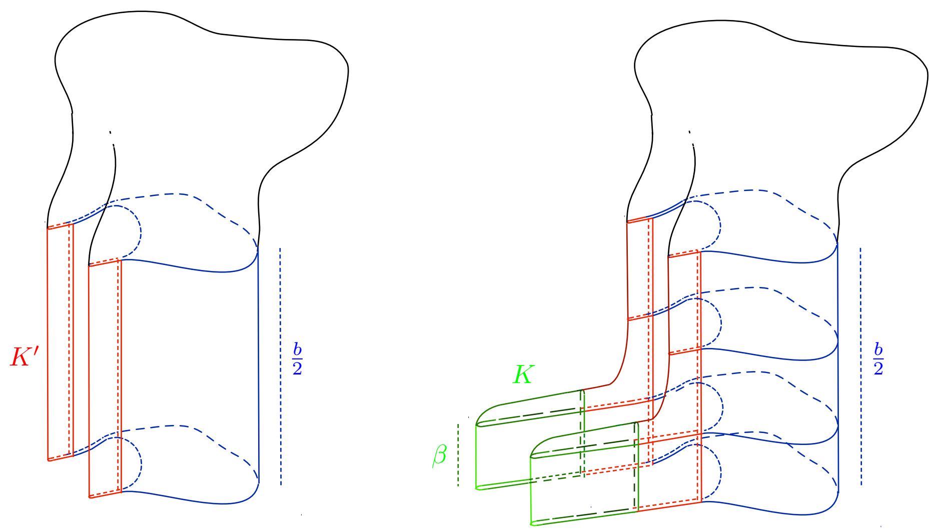

FIGURE 18. A typical element of $\mathcal{R}^+(W)_{b,h_1}$ (left) and the element resulting from application of $B_{b,\beta}$ (right)

5.1. **Performing surgery.** Elements of $\mathcal{R}^+_{\rm std}(W)_{b,\beta,h_1}$ are now ready for a surgery of type 1. By removing the $S^p \times D^{q+2}_+$ factor from the metric $ds_p^2 + \hat{g}^{q+2}_{\rm torp}$ and attaching a $D^{p+1} \times D^{q+1}$ equipped with the product $g^{p+1}_{\rm torp} + g^{q+1}_{\rm torp}$, we obtain an element of $\mathcal{R}^+(W')_{h_1'}$; see Fig. 19 below. Here $h_1' \in \mathcal{R}^+(X')$ is the element obtained by a standard surgery on the metric h_1, replacing the standard part $ds_p^2 + g^{q+1}_{\rm torp}$ with $g^{p+1}_{\rm torp} + \delta^2 ds_q^2$ for some appropriate $\delta > 0$.
This process determines a map

$$S_{b,\beta} : \mathcal{R}^+_{\rm std}(W)_{b,\beta,h_1} \longrightarrow \mathcal{R}^+(W')_{\beta,h_1'}.$$

We denote by $\mathcal{R}^+_{\rm std}(W')_{b,\beta,h_1'}$, the space which is the image of this map.

5.2. **The main technical lemma.** The following lemma plays the central role in the proof of our main result.

Lemma 5.2.1. *Let W, W', a, b, β be as above with $0 < \frac{a}{2} < \beta < \frac{b}{2}$.*

(i.) The spaces $\mathcal{R}^+_{\rm std}(W)_{b,\beta,h_1}$ and $\mathcal{R}^+_{\rm std}(W')_{b,\beta,h_1'}$ are homeomorphic.

(ii.) The inclusion $i : \mathcal{R}^+_{\rm std}(W')_{b,\beta,h_1'} \hookrightarrow \mathcal{R}^+_{\rm std}(W', S^{p+1}_+)_{a,\beta,h_1'}$ is a homotopy equivalence.

Proof. The first statement follows immediately from the fact that all metrics in $\mathcal{R}^+_{\rm std}(W)_{a,\beta,h_1}$ are identical on the standard region and the map $S_{a,\beta}$ only involves an adjustment on this standard part. The second statement requires a little more work. To aid the reader, and to make sure it is clear that this map makes sense, we begin by depicting an example of each type of metric in Fig. 20 below. Notice how elements of $\mathcal{R}^+_{\rm std}(W')_{b,\beta,h_1'}$ take a standard form on a larger region than elements of $\mathcal{R}^+_{\rm std}(W', S^{p+1}_+)_{a,\beta,h_1'}$. We will construct a homotopy

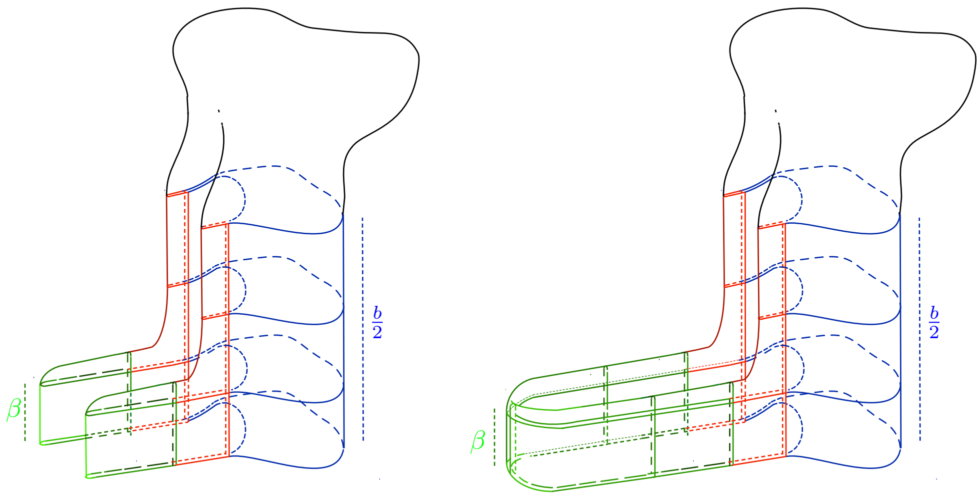

FIGURE 19. Doing surgery on an element of $g \in \mathcal{R}^+_{\text{std}}(W)_{b,\beta,h_1}$ (left) to obtain an element of $S_{b,\beta}(g) \in \mathcal{R}^+(W')_{\beta,h'_1}$ (right)

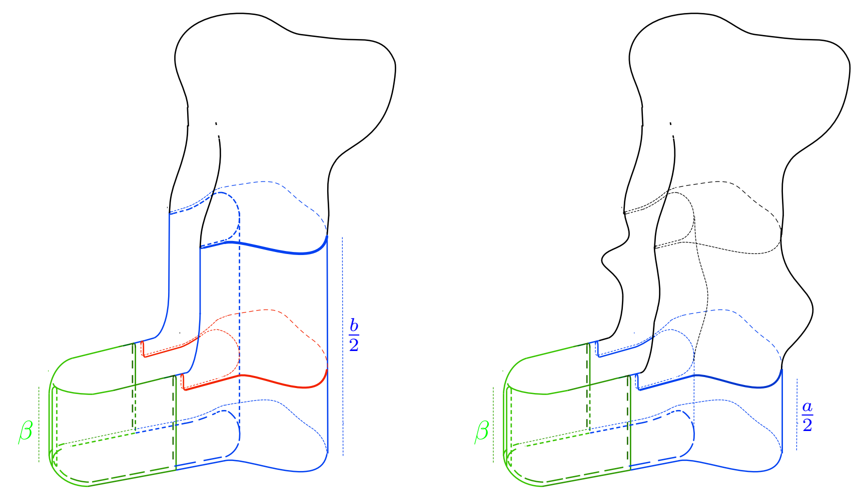

FIGURE 20. An element of $\mathcal{R}^+_{\text{std}}(W)_{a,\beta,h_1}$ and an arbitrary element of $\mathcal{R}^+_{\text{std}}(W', S^{p+1}_+)_{a,\beta,h'_1}$ (right)

inverse $j : \mathcal{R}^+_{\text{std}}(W', S^{p+1}_+)_{a,\beta,h'_1} \to \mathcal{R}^+_{\text{std}}(W')_{b,\beta,h'_1}$. Suppose $g \in \mathcal{R}^+_{\text{std}}(W', S^{p+1}_+)_{a,\beta,h'_1}$ is an arbitrary element. Working only where this metric is standard (and thus where it agrees with every other metric in $\mathcal{R}^+_{\text{std}}(W', S^{p+1}_+)_{a,\beta,h'_1}$), we may specify a region R_2 diffeomorphic to $X \times I$ which decomposes W' into 3 pieces R_1, R_2 and R_3 where $R_1 \sqcup R_2 = W' \setminus \text{int} R_1$; see Fig. 21. The first piece R_1 is diffeomorphic to the trace of the surgery on X with respect to ϕ, while the third piece is diffeomorpic to W. We will assume each piece contains its boundary and so $\{R_1, R_2, R_3\}$ is not strictly a partition of W'. We will now take a closer look at the

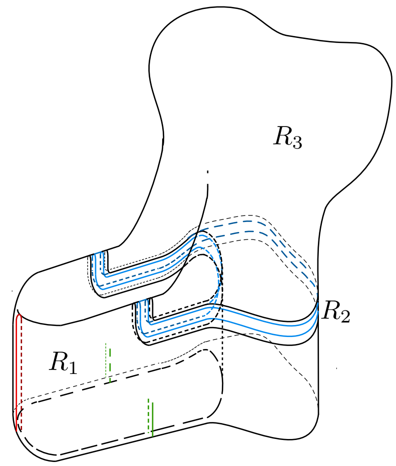

FIGURE 21. An element of $\mathcal{R}^+_{\mathrm{std}}(W)_{a,\beta,h_1}$ and an arbitrary element of $\mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)_{a,\beta,h_1'}$ (right)

region R_2. As shown in Fig. 22, this region can be decomposed into 2 pieces: R_2' and R_2''. One of these, R_2', highlighted in Fig. 22, is diffeomorphic to $I \times S^p \times D^{q+1}$. The other, R_2'' is diffeomorphic to $I \times (X \setminus (S^p \times D^{q+1}))$. More precisely, R_2'' is diffeomorphic to $I \times (X \setminus N_{\rho'})$ for some ρ' satisfying $0 < \rho' < \rho$, where N_ρ is the tubular neighbourhood of the embedded surgery sphere defined in (2.1). Thus on R_2'', any metric in $\mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)_{a,\beta,h_1'}$ takes the form $dt^2 + h_1|_{X \setminus N_{\rho'}}$. In particular, the fact that $\rho' < \rho$ means that near $R_2' \cap R_2'' \cong I \times S^p \times S^q$, any such metric is $dt^2 + ds_p^2 + \delta^2 ds_q^2$ for some appropriate $\delta > 0$.

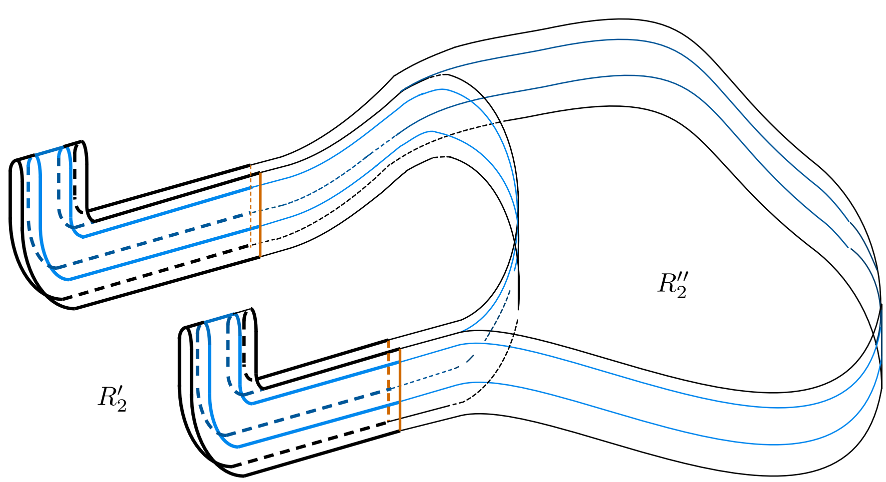

FIGURE 22. The decomposition of the region R_2 into subregions R_2' and R_2''

On the region R_2', all metrics in $\mathcal{R}^+_{\rm std}(W', S^{p+1}_+)_{a,\beta,h_1'}$ take the form of a restriction of $g^{p+1}_{\rm torp} + g^{q+1}_{\rm torp}$ to subregion of $\mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ of the type shown in Fig 23. In this setting, the cap of each torpedo is centered at the origin while the necks are infinite. The coordinates s and t in the diagram represent radial distance in $\mathbb{R}^{p+1}$ and $\mathbb{R}^{p+1}$ respectively. The boundary of R_2' consists of $\partial_1(R_2') = R_2' \cap R_2'' \cong I \times S^p \times S^q$ and a piece $\partial_2(R_2') \cong S^p \times D^{q+1} \times \{0\} \sqcup S^p \times D^{q+1} \times \{0\}$. We have further decomposed R_2' into 3 subregions. Near $\partial_1 R_2'$, there is a region we denote R_2'(outer) on which the metric takes the form $ds^2 + dt^2 + ds_p^2 + \delta^2 ds_q^2$. Next we have the region R_2'(middle) in which the metric takes the same form but where the boundary curves in the diagram bend over an angle of $\frac{\pi}{2}$. Finally we have remaining region, R_2'(inner) where the metric takes the form $ds^2 + ds_p^2 + g^{q+1}_{\rm torp}$.

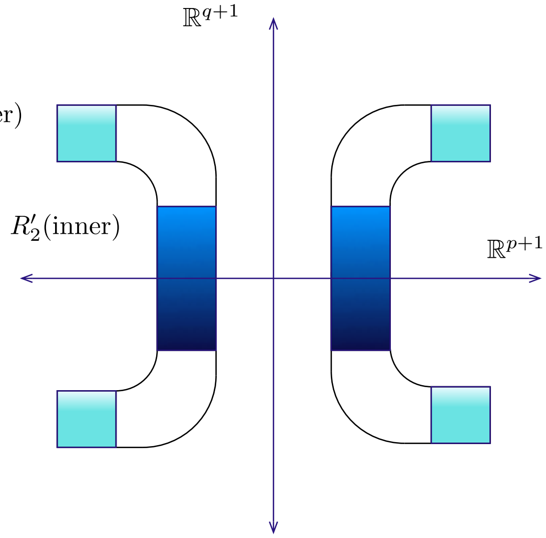

FIGURE 23. Representing the metric on R_2' as a subspace of $\mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ with the metric $g^{p+1}_{\rm torp} + g^{q+1}_{\rm torp}$

We will now describe an isotopy of the arbitrary metric $g \in \mathcal{R}^+_{\rm std}(W', S^{p+1}_+)_{a,\beta,h_1'}$ to an element of $\mathcal{R}^+_{\rm std}(W)_{a,\beta,h_1}$. This isotopy is denoted $s : I \times \mathcal{R}^+_{\rm std}(W', S^{p+1}_+)_{a,\beta,h_1'}$ and will satisfy the following conditions:

(1) $s(0) = g$.
(2) For all $t \in I$, the metric $s(t)$ restricts on the region R_2'' to a cylinder metric of the form $dr^2 + h_1|_{X \setminus N_{\rho'}}$ where $r \in [0, l]$ for some $l = l(t) > 0$.
(3) On the regions $R_1 \cup R_3$, $s(t) = g|_{R_1 \cup R_3}$ for all $t \in I$.

Thus, we make adjustments only on R_2. In order to satisfy condition (3.) above, we in fact work away from the boundary of R_2, as suggested by the subregion inscribed inside R_2 in Fig. 22. It remains to specify conditions for the isotopy s on the region R_2'. Obviously, near $R_2' \cap R_2''$, the metric $s(t)|_{R_2'}$ must take the form $dr^2 + ds_p^2 + \delta^2 ds_q^2$ as in the condition (2.) above. Thus on R_2'(outer) and R_2'(middle) we do nothing more than carefully stretch out the cylinder in line with the above condition (2). The non-trivial part happens inside R_2'(inner), Recall, here the metric takes the form $ds^2 + ds_p^2 + g^{q+1}_{\rm torp}$. We leave the ds_p^2 factor untouched. On the cylinder factor $ds^2 + g^{q+1}_{\rm torp}$ we perform two iterations of the boot metric isotopy described in detail in section 7. The resulting metric is depicted in Fig. 24 below. The existence of the isotopy s is guaranteed, when $q \geq 2$, by Lemma 7.0.13.

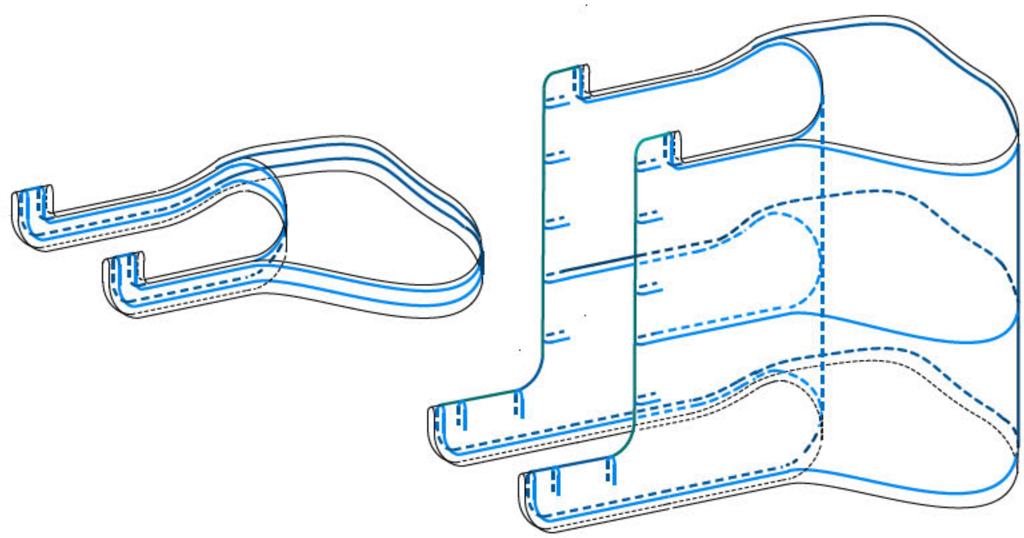

FIGURE 24. The restriction of the metrics $s(0) = g$ (left) and $s(1)$ (right) to the region R_2

As the isotopy s only makes alterations on a region where all metrics in $\mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)_{a,\beta,h'_1}$ agree, we may specify the map $j : \mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)_{a,\beta,h'_1} \to \mathcal{R}^+_{\mathrm{std}}(W')_{b,\beta,h'_1}$ by the formula $j(g) = s_g(1)$, where s_g is the isotopy above with respect to the metric g. The fact that j and i are homotopy inverses easily follows by retracting $s_g(1)$ along the isotopy. □

6. THE PROOF OF THEOREMS A, B AND C

At this stage, it is worth taking a look at how everything fits together. In order to prove Theorem A, we wish to show that for any $h \in \mathcal{R}^+(X)$, there is a psc-metric $h' \in \mathcal{R}^+(X')$ and a psc-metric $\bar{h} \in \mathcal{R}^+(T_\phi)_{h,h'}$, so that the map

$$\mu_{(T_\phi, \bar{h})} : \mathcal{R}^+(W)_h \to \mathcal{R}^+(W' = W \cup T_\phi)_{h'},$$

is a homotopy equivalence. As mentioned earlier, the metrics h' and $\bar{h}$ are, respectively, the psc-metric on X' obtained by the Gromov-Lawson construction and the corresponding Gromov-Lawson trace metric. Recall that, in Lemmas 4.3.1 and 4.3.2, we show that the map

$$b_1 : \mathcal{R}^+(W)_h \to \mathcal{R}^+(W)_{b,h_1},$$

is a weak homotopy equivalence. Indeed, this map sends a metric g on W to a metric $b_1(g)$ on $W \cup X \times [-2b, 0]$ by attaching a concordance $g' \in \mathcal{R}^+(X \times [-2b, 0])_{h,h_1}$ which was specified by a rescaled Gromov-Lawson isotopy between h and h_1. Recall here that $h_1 \in \mathcal{R}^+_{\mathrm{std}}(X)$ is the metric which is standard near the embedded surgery sphere. This is depicted in Fig. 25.

Next we consider the map

$$b_2 : \mathcal{R}^+(W)_{b,h_1} \to \mathcal{R}^+_{\mathrm{std}}(W)_{b',\beta,h_1},$$

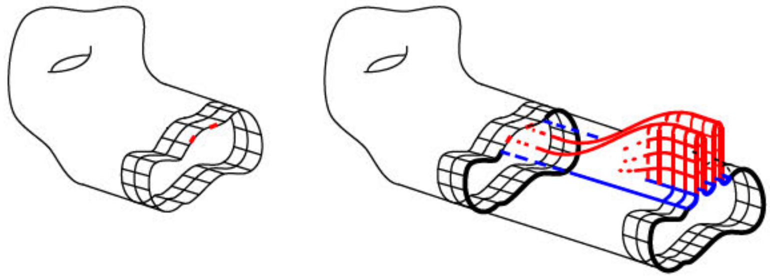

FIGURE 25. The metrics g and $b_1(g)$

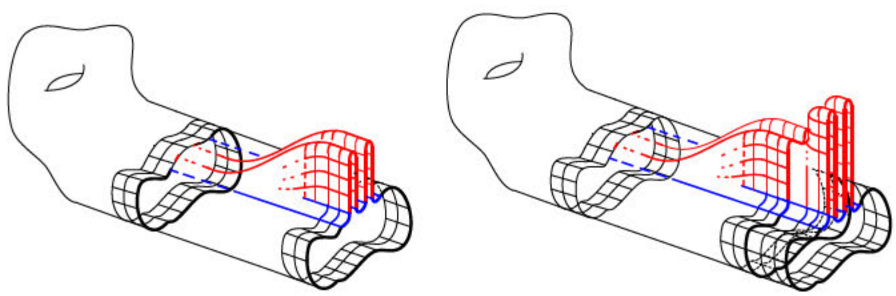

FIGURE 26. The metrics $b_1(g)$ and $b_2(g)$

by a attaching to $W \times [-2b, 0]$, a cylinder $X \times [-2b', 2(b'-b)]$ equipped with a metric which takes the form $ds_p^2 + \hat{g}_{\text{torp}}^{q+1}$ near the embedded surgery sphere at the end of the cylinder. This is depicted in Fig. 26, which specifically shows the effect of composing b_1 and b_2. An important technical piece of this work is the construction of this standard attaching metric which we describe suggestively as a "boot-shaped" collar. Up to a rescaling, such metrics lie in the space $\mathcal{R}^+(X \times I)_{h_1,h_1}$. It will also be important to see that this metric is isotopic in this space to the standard cylinder $dt^2 + h_1$ as suggested in Fig. 27 below.

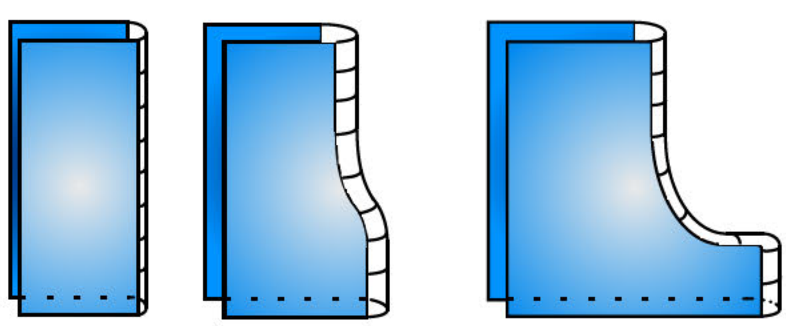

FIGURE 27. Pushing out the "toe" of a "boot" metric

Remark 6.0.2. Technically, the metrics on either end of the boot metric are not the same, as the necks of their torpedoes inevitably differ in length. However, to avoid yet further notation, we will ignore this problem which can be fixed by extending the space $\mathcal{R}^+(W)_{a,h_1}$ to one which allows for varying torpedo neck lengths on h_1 and is easily seen to be homotopy equivalent via a deformation retract.

We now denote by h_1', the metric obtained by the Gromov-Lawson surgery on h_1. The psc-metric h_1' is our choice for psc-metric h' and so hence forth we write $h' := h_1'$. After recalling that lemma 5.2.1 gives that $\mathcal{R}^+_{\text{std}}(W)_{b',\beta,h_1}$ and $\mathcal{R}^+_{\text{std}}(W')_{b',\beta,h'}$ are homeomorphic, we combine the maps b_1 and b_2 as part of the following commutative diagram.

$$\begin{tikzcd}
\mathcal{R}^+(W)_h \arrow[r, "\mu_{(T_\phi,\bar{h})}"] \arrow[d, "b_1"] & \mathcal{R}^+(W')_{h'} \\
\mathcal{R}^+(W)_{b,h_1} \arrow[d, "b_2"] & \mathcal{R}^+_{\text{std}}(W', S^{p+1}_+)_{a,\beta,h'} \arrow[u, hook, "rescale"'] \\
\mathcal{R}^+_{\text{std}}(W)_{b',\beta,h_1} \arrow[r, phantom, "\cong"] & \mathcal{R}^+_{\text{std}}(W')_{b',\beta,h'} \arrow[u, hook]
\end{tikzcd}$$

Recall that the constants a, b, b' and β are chosen so that $0 < \frac{a}{2} < \beta < \frac{b}{2} < \frac{b'}{2}$. Thus, the upper vertical map on the right-hand side is not quite an inclusion as, for example, the constant a may be less than 1. However a simple rescaling of the collar gives us an injective map. The lower vertical map on the right hand side is exactly the inclusion map.

6.1. **The Proof of Theorem A.** We now combine the results of previous sections to prove our main theorem.

Proof of Theorem A. It follows from Lemma 4.2.1 and Lemma 4.3.3 that the upper vertical map on the right-side is a homotopy equivalence. Furthermore, the lower vertical inclusion on the right is a homotopy equivalence by Lemma 5.2.1. We now turn our attention to the left hand side. The map b_1 is a weak homotopy equivalence by Lemmas 4.3.1 and 4.3.2.

To complete the proof of Theorem A, we need to show that the map $b_2 : \mathcal{R}^+(W)_{ab,h_1} \to \mathcal{R}^+_{\text{std}}(W)_{b,\beta,h_1}$ is a weak homotopy equivalence. This is based on a geometric construction which we will shortly deal with. For now, the important fact is that, on a cylinder $D^{q+1} \times I$, there is an isotopy in the space $\mathcal{R}^+(D^{q+1} \times I)_{g^{q+1}_{\text{torp}}, g^{q+1}_{\text{torp}}}$ which takes the metric $g^{q+1}_{\text{torp}} + dt^2$ to a metric which takes the form $\hat{g}^{q+1}_{\text{torp}}$ near one end of the cylinder. This was depicted in Fig. 27 above. A detailed proof of this fact, which is stated as Lemma 7.0.13, is given in the appendix. In particular, it means that any psc-metric in $\mathcal{R}^+(W)_{b,h_1}$ maybe moved via an isotopy to a psc-metric in $\mathcal{R}^+_{\text{std}}(W)_{b,\beta,h_1}$. Moreover, as this isotopy only makes adjustments on the standard part of the psc-metric, we can specify one adjustment to work for all psc-metrics in the space.

We will now consider the map $a_2 : \mathcal{R}^+_{\text{std}}(W)_{b,\beta,h_1} \to \mathcal{R}^+(W)_{b,h_1}$ which sends a psc-metric $g \in \mathcal{R}^+_{\text{std}}(W)_{b,\beta,h_1}$ to the psc-metric $a_2(g) \in \mathcal{R}^+(W)_{b,h_1}$ obtained by simply attaching another "upside down" boot metric to the boundary, as show below in Fig. 28. The previously mentioned geometric fact concerning the isotopy from cylinder to boot metrics means that the compositions $a_2 \circ b_2$ and $b_2 \circ a_2$ are easily seen to be homotopic to the identity map on their respective domains. □

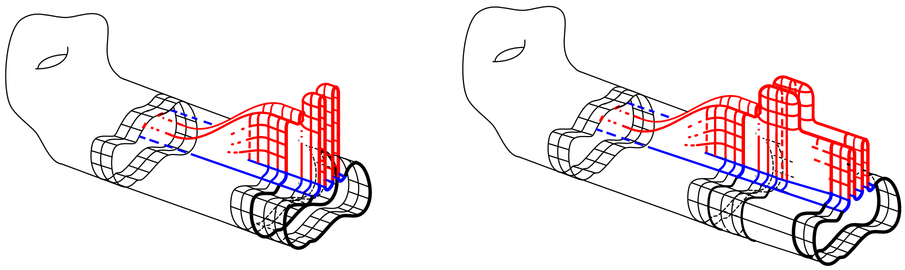

Figure 28. The metrics g and $a_2(g)$

6.2. **Proof of Theorem B.** We now turn our attention to the second main result.

Proof of Theorem B. The weak homotopy equivalence in (i) follows from the same reasoning employed by Gromov and Lawson in their proof of Theorem B in [11]. Here, the authors showed that any two spin bordant simply connected n-manifolds M and N are mutually obtainable by surgeries in codimension at least three. They demonstrate this by performing surgeries in the interior of a spin manifold Y^{n+1}, where $\partial Y = M \sqcup N$, to remove non-trivial elements in the integral homology groups $H_k(Y)$, where $k = 1, 2, n-1, n$. The spin and dimension conditions mean that all such elements are realised by embedded spheres with trivial normal bundle. In our case, as X is null-bordant, we may perform surgeries of type 1 on W to obtain a manifold W' with boundary X' diffeomorphic to S^n. As W' forms one half of a closed simply connected spin manifold (its double), there is a sequence of surgeries on the interior of W', which transform it into the disk D^{n+1}. As with the Gromov-Lawson proof in [11], all of these surgeries satisfy the appropriate co-dimension restrictions required by Theorem A and so the theorem follows.

Part (ii) is proved by the following argument. As h is Gromov-Lawson cobordant to the round metric ds_n^2, part (i) gives us that $\mathcal{R}^+(W)_h$ is weakly homotopy equivalent to $\mathcal{R}^+(D^{n+1})_{ds_n^2}$. In turn the space $\mathcal{R}^+(D^{n+1})_{ds_n^2}$ is homoemorphic to a subspace $\mathcal{R}^+_{\text{std}}(S^n) \subset \mathcal{R}^+(S^n)$, of psc-metrics which take the form of a standard torpedo on the southern hemisphere. The conclusion then follows from the fact, demonstrated in the proof of Theorem 1.2.1, that the inclusion $\mathcal{R}^+_{\text{std}}(S^n) \hookrightarrow \mathcal{R}^+(S^n)$ is a homotopy equivalence. $\square$

6.3. **Proof of Theorem C.** Before proving Theorem C, we recall the quasifibration which we defined in the introduction:

$$\begin{aligned} \text{res} : \mathcal{R}^+(W) &\longrightarrow \mathcal{R}^+_0(\partial W) \subset \mathcal{R}^+(\partial W) \\ g &\longmapsto g|_{\partial W.} \end{aligned}$$

Here the space $\mathcal{R}^+_0(\partial W = X)$ is of course the image of the above restriction map. Similarly, we define the space $\mathcal{R}^+_0(X')$ where $X' = \partial W'$, as above. We also denote by $\mathcal{R}^+_{\text{std}(0)}(X)$ and $\mathcal{R}^+_{\text{std}(0)}(X')$ the respective spaces $\mathcal{R}^+_0(X) \cap \mathcal{R}^+_{\text{std}}(X)$ and $\mathcal{R}^+_0(X') \cap \mathcal{R}^+_{\text{std}}(X')$. The following lemma is easy consequence of Theorem 1.2.1.

Lemma 6.3.1. *Let W and W' satisfy the hypotheses of Theorem A. Then the inclusions $\mathcal{R}^+_{\mathrm{std}(0)}(X) \hookrightarrow \mathcal{R}^+_0(X)$ and $\mathcal{R}^+_{\mathrm{std}(0)}(X') \hookrightarrow \mathcal{R}^+_0(X')$ are homotopy equivalences. In particular, the spaces $\mathcal{R}^+_0(X)$ and $\mathcal{R}^+_0(X')$ are homotopy equivalent.*

Proof. Suppose $g \in \mathcal{R}^+(X)$ extends to an element of $\bar{g} \in \mathcal{R}^+(W)_g$. Then, using the fact that isotopy implies concordance, any metric $h \in \mathcal{R}^+$, which is psc-isotopic to g, extends to an element $\bar{h} \in \mathcal{R}^+(W)_h$. This partitions the path components of $\mathcal{R}^+(X)$ into those consisting of psc-metrics which extend to elements of $\mathcal{R}^+(W)$ and those which do not. The former is of course the subspace $\mathcal{R}^+_0(W)$. Furthermore, if a psc-metric g is an element of $\mathcal{R}^+_0(X)$, then any metric g' obtained by Gromov-Lawson surgery is an element of $\mathcal{R}^+_0(X')$. In case this is unclear, recall that if $\bar{g} \in \mathcal{R}^+(W)_g$, the procedure described in Theorem A gives rise to a psc-metric $\bar{g}' \in \mathcal{R}^+(W')_{g'}$. Hence, the Gromov-Lawson construction gives rise to a one to one correspondence between path components of $\mathcal{R}^+(X)$ and $\mathcal{R}^+(X')$. In particular, subspaces $\mathcal{R}^+_0(X)$ and $\mathcal{R}^+_0(X')$ arise by simply removing corresponding (homotopy equivalent) path components of $\mathcal{R}^+(X)$ and $\mathcal{R}^+(X')$. The argument in the proof of Theorem 1.2.1 which uses a family version of the Gromov-Lawson construction to demonstrate that inclusion $\mathcal{R}_{\mathrm{std}}(X) \hookrightarrow \mathcal{R}^+(X)$ is a homotopy equivalence goes through just as well to show that $\mathcal{R}_{\mathrm{std}(0)}(X) \hookrightarrow \mathcal{R}^+_0(X)$ and $\mathcal{R}_{\mathrm{std}(0)}(X') \hookrightarrow \mathcal{R}^+_0(X')$ are homotopy equivalences. As $\mathcal{R}^+_{\mathrm{std}(0)}(X)$ and $\mathcal{R}^+_{\mathrm{std}(0)}(X')$ are clearly homeomorphic, the conclusion follows. $\square$

Suppose now that we fix psc-metrics $h \in \mathcal{R}^+_0(X)$ and $h_1 \in \mathcal{R}^+_{\mathrm{std}(0)}(X)$ where h_1 is obtained from h by a Gromov-Lawson isotopy. Denoting by $\mathcal{R}^+_{\mathrm{std}}(W)$, the pre-image $\mathrm{res}^{-1}(\mathcal{R}^+_{\mathrm{std}(0)}(X))$, we obtain the following commutative diagram.

$$\begin{tikzcd}
\mathcal{R}^+(W)_h \arrow[r, hook] \arrow[d, "w.h.e."] & \mathcal{R}^+(W) \arrow[r, "\mathrm{res}"] & \mathcal{R}^+_0(X) \\
\mathcal{R}^+(W)_{h_1} \arrow[r, hook] & \mathcal{R}^+_{\mathrm{std}}(W) \arrow[r, "\mathrm{res}"] \arrow[u, hook] & \mathcal{R}^+_{\mathrm{std}(0)}(X) \arrow[u, hook, "h.e."']
\end{tikzcd}$$

The vertical map on the left is the map b_1 from Theorem A followed by a rescaling. From the proof of Theorem A we know that this map is a weak homotopy equivalence. The rightmost vertical map is inclusion and is also a homotopy equivalence by Lemma 6.3.1 above. We will shortly show that the middle vertical map, which is also an inclusion, is a weak homotopy equivalence as well. In order to do so, we need the following fact.

Lemma 6.3.2. *The map* $\mathrm{res} : \mathcal{R}^+_{\mathrm{std}}(W) \to \mathcal{R}^+_{\mathrm{std}(0)}(X)$ *is a quasifibration.*

Proof. This map is in fact the pull-back under the inclusion $\mathcal{R}^+_{\mathrm{std}(0)}(X) \hookrightarrow \mathcal{R}^+_0(X)$, of the known quasifibration $\mathrm{res} : \mathcal{R}^+(W) \to \mathcal{R}^+_0(X)$. To avoid an abundance of notation we maintain the name res for both maps. Unfortunately, pull-backs of quasifibrations are not necessarily quasifibrations and so we do not get this fact directly. However, the original proof by Chernysh, in [6], goes through exactly as before when we replace $\mathcal{R}^+_0(X)$ with the homotopy equivalent space $\mathcal{R}^+_{\mathrm{std}(0)}(X)$ and so the conclusion follows. $\square$

This leads directly to the following lemma.

Lemma 6.3.3. *The inclusion map* $\mathcal{R}^+_{\mathrm{std}}(W) \to \mathcal{R}^+(W)$ *is a weak homotopy equivalence.*

Proof. As both rows of the above diagram are quasifibrations, we obtain the following diagram at the level of homotopy groups.

$$\begin{tikzcd}
\arrow[r] & \pi_k(\mathcal{R}^+(W)_h,\bar{h}) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \pi_k(\mathcal{R}^+(W),\bar{h}) \arrow[r, "\mathrm{res}_*"] & \pi_k(\mathcal{R}_0^+(X),h) \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & \pi_{k-1}(\mathcal{R}^+(W)_h,\bar{h}) \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & {} \\
\arrow[r] & \pi_k(\mathcal{R}^+(W)_{h_1},\bar{h_1}) \arrow[r] & \pi_k(\mathcal{R}^+_{\mathrm{std}}(W),\bar{h_1}) \arrow[r, "\mathrm{res}_*"] \arrow[u] & \pi_k(\mathcal{R}^+_{\mathrm{std}(0)}(X),h_1) \arrow[r] & \pi_{k-1}(\mathcal{R}^+(W)_{h_1},\bar{h_1}) \arrow[r] & {}
\end{tikzcd}$$

The rows of which are of course exact. We know in advance that the vertical homomorphisms are isomorphisms in all cases except $\pi_k(\mathcal{R}^+_{\mathrm{std}(0)}(X)) \to \pi_k(\mathcal{R}^+(W),\bar{h})$. That this is necessarily an isomorphism is implied by the 5-lemma. $\square$

Turning our attention to W', we let $h' = h_1' \in \mathcal{R}^+_{\mathrm{std}(0)}(X')$ denote the psc-metric obtained by Gromov-Lawson surgery on h. As before, we obtain the following commutative diagram.

$$\begin{tikzcd}
\mathcal{R}^+(W')_{h'} \arrow[r, hook] \arrow[d, leftrightarrow, "="] & \mathcal{R}^+(W') \arrow[r, "\mathrm{res}"] & \mathcal{R}_0^+(X') \\
\mathcal{R}^+(W')_{h'} \arrow[r, hook] & \mathcal{R}^+_{\mathrm{std}}(W') \arrow[r, "\mathrm{res}"] \arrow[u, hook] & \mathcal{R}^+_{\mathrm{std}(0)}(X') \arrow[u, hook, "h.e."']
\end{tikzcd}$$

Once again, $\mathcal{R}^+_{\mathrm{std}}(W') = \mathrm{res}^{-1}(\mathcal{R}^+_{\mathrm{std}(0)}(X'))$ and so, applying Lemmas 6.3.2 and 6.3.3, we see that the map $\mathrm{res} : \mathcal{R}^+_{\mathrm{std}}(W') \to \mathcal{R}^+_{\mathrm{std}(0)}(X')$ is a quasifibration and consequently, the map $\mathcal{R}^+_{\mathrm{std}}(W') \hookrightarrow \mathcal{R}^+(W')$ is a weak homotopy equivalence.

Next, we recall the spaces $\mathcal{R}^+_{\mathrm{std}}(W)_{b,\beta,h_1}$, $\mathcal{R}^+_{\mathrm{std}}(W')_{b,\beta,h_1'}$, defined immediately before Lemma 5.2.1, in which we show that these spaces are homeomorphic. We will now generalise these spaces by allowing the boundary metrics h_1 and h_1' to vary over standard psc-metrics on X and X'. We begin by defining the spaces

$$\mathcal{R}^+_{\mathrm{std}}(W)(b) := \bigcup_{g\in\mathcal{R}^+_{\mathrm{std}(0)}(X)} \mathcal{R}^+(W)_{b,g},$$

$$\mathcal{R}^+_{\mathrm{std}}(W')(b) := \bigcup_{g'\in\mathcal{R}^+_{\mathrm{std}(0)}(X')} \mathcal{R}^+(W)_{b,g'}.$$

The proof of Lemma 4.3.1, as it only makes scaling adjustments on the collar, generalises without difficulty to the following.

Lemma 6.3.4. *The spaces $\mathcal{R}^+_{\mathrm{std}}(W)(b)$ and $\mathcal{R}^+_{\mathrm{std}}(W)$ and are homeomorphic.*

We now generalise the spaces containing the "boot" versions of these metrics.

$$\mathcal{R}^+_{\mathrm{std}}(W)(b,\beta) := \bigcup_{g\in\mathcal{R}^+_{\mathrm{std}(0)}(X)} \mathcal{R}^+(W)_{b,\beta,g},$$

$$\mathcal{R}^+_{\mathrm{std}}(W')(b,\beta) := \bigcup_{g'\in\mathcal{R}^+_{\mathrm{std}(0)}(X')} \mathcal{R}^+(W')_{b,\beta,g'}$$

where, recall, $0 < \beta < \frac{b}{2}$. Lemma 7.0.13, applied as in the proof of Theorem A and combined with Lemma 6.3.4 above, gives us the following.

Lemma 6.3.5. *The spaces $\mathcal{R}^+_{\mathrm{std}}(W)(b,\beta)$ and $\;^+_{\mathrm{std}}(W)$ are homotopy equivalent.*

Part (i) of Lemma 5.2.1 easily generalises to give us the following fact.

Lemma 6.3.6. *The spaces* $\mathcal{R}^+_{\mathrm{std}}(W)(b,\beta)$ *and* $\mathcal{R}^+_{\mathrm{std}}(W')(b,\beta)$ *are homeomorphic.*

Finally, we generalise the space $\mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)_{a,\beta,h'}$ to obtain

$$\mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)(a,\beta) := \bigcup_{g' \in \mathcal{R}^+_{\mathrm{std}(0)}(X')} \mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)_{a,\beta,g'}$$

As the proof of part (ii) of Lemma 5.2.1 only makes adjustments in the region where metrics in these spaces are mutually standard, we obtain the following generalisation of this lemma.

Lemma 6.3.7. *The inclusion* $\mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)(a,\beta) \hookrightarrow \mathcal{R}^+_{\mathrm{std}}(W')(b,\beta)$*, where* $0 < \frac{a}{2} < \beta < \frac{b}{2}$*, is a homotopy equivalence.*

Proof of Theorem C. We will now put all of the preceding facts together. By Lemma 6.3.3, we see that $\mathcal{R}^+_{\mathrm{std}}(W)$ and $\mathcal{R}^+(W)$ and also, $\mathcal{R}^+_{\mathrm{std}}(W')$ and $\mathcal{R}^+(W)$, are weakly homotopy equivalent. Thus, to prove the theorem it suffices to show that $\mathcal{R}^+_{\mathrm{std}}(W)$ is weakly homotopy equivalent to $\mathcal{R}^+_{\mathrm{std}}(W')$. Next, we see that $\mathcal{R}^+_{\mathrm{std}}(W)$ is weakly homotopy equivalent to $\mathcal{R}^+_{\mathrm{std}}(W')(b,\beta)$, by combining Lemmas 6.3.5 and 6.3.6. In turn, Lemma 6.3.7 gives that $\mathcal{R}^+_{\mathrm{std}}(W')(b,\beta)$ is weakly homotopy equivalent to $\mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)(a,\beta)$. The fact that $\mathcal{R}^+_{\mathrm{std}}(W', S^{p+1}_+)(a,\beta)$ is weakly homotopy equivalent to $\mathcal{R}^+_{\mathrm{std}}(W')$ then follows from application of the argument of Theorem 1.2.1 as in Lemma 4.2.1. □

Finally, Corollary D. follows easily by argument used to prove part (i) of Theorem B.

7. Appendix: Boot metrics and bent cylinders

We begin with the psc-metric $g_{\rm torp}^{n} + dt^2$ on the cylinder $D^n \times I$. This metric is an element of the space $\mathcal{R}^{+}(D^n \times I)_{g_{\rm torp}^{n}, g_{\rm torp}^{n}}$. We wish to construct an isotopy of this metric which "pushes out", near one end, a standard "toe", as shown in Fig. 27 above, to obtain the metric depicted in Fig. 29 below. The resulting metric is known as a "boot metric" and denoted $g_{\rm boot}$. In our case, the torpedo metric in the starting product has small radius and so the original metric has large positive scalar curvature. However, we would like to adjust the metric without adjusting the radius. As indicated by the arrowed arc in Fig. 29, there is a region where potential exists to introduce negative curvature. We must therefore show

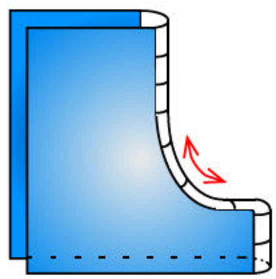

FIGURE 29. The boot metric $g_{\rm boot}$

that positivity can always be maintained, provided this bending arc is sufficiently long. In proving this, we consider the more general case of bending a Riemannian cylinder of positive scalar curvature around an arc. It will be clear that the controlling positivity of the scalar curvature in the problem region above is a sub case of this situation.

Bending the cylinder. We begin by equipping the cylinder $S^n \times (0, \frac{\pi}{2})$ with the standard Riemannian metric $g_{\rm cyl} = \delta^2 ds_n^2 + dl^2$, where $l \in (0, \frac{\pi}{2})$. Provided $n \geq 2$ this metric has positive scalar curvature given by the formula

$$R = \frac{n(n-1)}{\delta^2}.$$

We now consider a particular bending of this cylinder over an angle of $\frac{\pi}{2}$ to obtain the metric shown in the left image of Fig. 30 below. We will show that, provided $n \geq 2$ and δ is chosen to be sufficiently small and or the bend is done particularly slowly, the resulting "bent cylinder", $g_{\rm bcyl}$ also has positive scalar curvature. Finally, we will extend this procedure to the case when the sphere factor S^n is equipped with a more general rotationally symmetric metric. Here we will need to assume that $n \geq 3$. An example of this is shown in the right image of Fig. 30. This example illustrates the case of bending a cylinder of "double torpedo" metrics.

Realising rotationally symmetric metrics on the sphere via embeddings. The standard round metric on S^n, which we denote ds_n^2, is induced by the embedding

$$\begin{aligned} F_{n\text{-round}} : (0, \pi) \times S^{n-1} &\longrightarrow \mathbb{R} \times \mathbb{R}^n, \\ (t, \theta) &\longmapsto (\cos t, \sin t.\theta), \end{aligned}$$

and computed in these coordinates as

$$ds_n^2 = dt^2 + \sin^2(t) ds_{n-1}^2. \tag{7.1}$$

Strictly speaking this metric is defined on the cylinder $(0,\pi)\times S^{n-1}$. However, the behaviour of the function sin near the end points of the interval $(0,\pi)$ gives the cylinder $(0,\pi)\times S^{n-1}$ the geometry of a round n-dimensional sphere S^n which is missing a pair of antipodal points. Such a metric extends uniquely onto the sphere. More generally, consider the map F which takes the form

$$\begin{aligned} F : (0,b)\times S^{n-1} &\longrightarrow \mathbb{R}\times\mathbb{R}^n, \\ (t,\theta) &\longmapsto (\alpha(t), \beta(t).\theta). \end{aligned} \tag{7.2}$$

Here α and β are smooth functions $(0,b)\to(0,\infty)$. We will now establish conditions on α and β whereby this map is an embedding inducing a psc-metric on the sphere S^n. Let $\mathcal{B}_{(0,b)}$ denote the space of all smooth functions $\beta : (0,b)\to(0,\infty)$ which satisfy the following conditions:

$$\begin{aligned} \beta(0) &= 0, & \beta(b) &= 0, & & \\ \dot\beta(0) &= 1, & \dot\beta(b) &= -1, & & \\ \beta^{(even)}(0) &= 0, & \beta^{(even)}(b) &= 0, & & \\ \ddot\beta &\leq 0, & \dddot\beta(0) &< 0, & \dddot\beta(b) &> 0, \\ \ddot\beta(t) &< 0, & &\text{when } t \text{ is near but not at } 0 \text{ and } b. \end{aligned} \tag{7.3}$$

We now define $\mathcal{E}_{(0,b)}$ to be the space of smooth maps of the form given in (7.2) where $\beta\in\mathcal{B}(0,b)$ and α is defined as

$$\alpha(t) = \int_0^t \sqrt{1-\dot\beta(u)^2}\,du. \tag{7.4}$$

Lemma 7.0.8. *The set $\mathcal{E}_{(0,b)}$ is a subspace of the space of embeddings*

$$\mathrm{Emb}((0,b)\times S^{n-1}, \mathbb{R}^{n+1})$$

Proof. Let $F\in\mathcal{E}$. Injectivity of F is guaranteed by the fact that $\beta > 0$ and that α is strictly monotonic. The maximality of the rank of the derivative map F_* follows from an easy calculation. □

Finally, we define $\mathcal{B}$ and $\mathcal{E}$ as

$$\mathcal{B} = \bigcup_{b>0} \mathcal{B}_{(0,b)} \text{ and } \mathcal{E} = \bigcup_{b>0}\mathcal{E}_{(0,b)}. \tag{7.5}$$

A straightforward calculation gives that the induced metric, obtained by pulling back the standard Euclidean metric on $\mathbb{R}^{n+1}$ via $F\in\mathcal{E}$, is given by the formula

$$\begin{aligned} F^*(dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2) &= (\dot\alpha^2(t) + \dot\beta^2(t))dt^2 + \beta^2(t)ds_{n-1}^2 \\ &= dt^2 + \beta^2(t) ds_{n-1}^2, \end{aligned} \tag{7.6}$$

following the definition of α. It is worth recalling that the scalar curvature R of this metric is given by the formula

$$R = -2(n-1)\tfrac{\ddot\beta}{\beta} + (n-1)(n-2)\tfrac{1-\dot\beta^2}{\beta^2}. \tag{7.7}$$

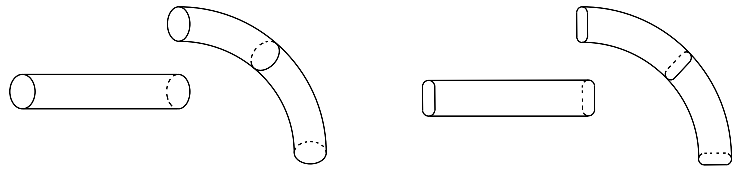

FIGURE 30. Bending a standard cylinder (left) and a more general cylinder of a rotationally symmetric metric (right)

Proposition 7.0.9. *For each function $\beta \in \mathcal{B}$, the metric $dt^2 + \beta^2(t)ds_{n-1}^2$ extends uniquely to a smooth psc-metric on the sphere S^n.*

Proof. Smoothness follows immediately from [22, Proposition 1.5]. The fact that each metric has positive scalar curvature is shown in [22, Proposition 1.6]. □

It follows from Proposition 7.0.9 that the space of embeddings $\mathcal{E}$ can be identified with a subspace of the space of embeddings $\mathrm{Emb}(S^n, \mathbb{R}^{n+1})$. Henceforth we will regard $\mathcal{E}$ as a space of embeddings of S^n into $\mathbb{R}^{n+1}$. Moreover we can now define a subspace $\mathcal{W}$ of the space of psc-metrics on S^n, $\mathcal{R}^+(S^n)$, as follows.

$$\mathcal{W} := \{g = dt^2 + \beta^2 ds_{n-1}^2 : \beta \in \mathcal{B}\} = \{g = F^*(g_{Euc}^{n+1}) : F \in \mathcal{E}\}.$$

Here, g_{Euc}^{n+1} denotes the standard Euclidean metric on $\mathbb{R}^{n+1}$. This information is summarised in the commutative diagram below, where the horizontal maps are inclusions while the vertical ones send an embedding to the metric obtained by pulling back the standard Euclidean metric.

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathrm{Emb}(S^n, \mathbb{R}^{n+1}) \arrow[d] \\ \mathcal{W} \arrow[r] & \mathcal{R}^+(S^n) \end{tikzcd}$$

Lemma 7.0.10. *The spaces $\mathcal{W}$ and $\mathcal{E}$ are path-connected subspaces of the spaces $\mathcal{R}^+(S^n)$ and $\mathrm{Emb}(S^n, \mathbb{R}^{n+1})$ respectively.*

Proof. The fact that $\mathcal{W}$ is path connected is proved in [23, Proposition 1.7]. Now we suppose F_0 and F_1 are elements of $\mathcal{E}$. Let $g_0 = dt^2 + \beta_0^2(t)ds_{n-1}^2$ and $g_1 = dt^2 + \beta_1^2(t)ds_{n-1}^2$ denote the respective psc-metrics in $\mathcal{W}$ obtained by pulling back the standard Euclidean metric on $\mathbb{R}^{n+1}$. From [23, Proposition 1.7], there is a path $\beta_s, s \in I$ through functions in $\mathcal{B}$, giving rise to a path $g_s = dt^2 + \beta_s^2(s)ds_{n-1}^2$ in W which connects the metrics g_0 and g_1. We define

$$\alpha_s(t) = \int_0^t \sqrt{1 - \dot{\beta_s}^2(u)}du.$$

Then the formula $F_s(t, \theta) = (\alpha_s(t), \beta_s(t).\theta)$ gives the desired path in $\mathcal{E}$. □

We briefly turn our attention from the sphere to the disk to recall an important example of a warped product metric: the *torpedo metric.* Consider a function $\eta_1 : (0, b) \to (0, \infty)$ which satisfies the following conditions:

(i) $\eta_1(t) = \sin t$ when t is near 0.
(ii) $\eta_1(t) = 1$ when $t \geq \frac{\pi}{2}$.
(iii) $\ddot{\eta}_1(t) \leq 0$.

The constant b is assumed to be larger than $\frac{\pi}{2}$. More generally, we define a function $\eta_\delta : (0,b) \to (0,\infty)$ by

$$\eta_\delta(t) = \delta\eta_1(\tfrac{t}{\delta}).$$

Then the metric $g_{\text{torp}} = dt^2 + \eta_\delta(t)ds_{n-1}^2$ is the standard torpedo metric of radius δ on the disk D^n. It is evident from formula 7.7 that, provided $n \geq 3$, a sufficiently small choice of δ guarantees an arbitrary large positive lower bound for the scalar curvature of this metric.

We can obtain a metric on S^n by taking the double of g_{torp}^n. Such a metric is given by the formula $dt^2 + \bar{\eta_\delta}(t)^2 ds_{n-1}^2$, where $\bar{\eta_\delta} : (0,b) \to (0,\infty)$ agrees with η_δ on $(0,\frac{b}{2})$ and is given by the formula $\bar{f_\delta}(t) = f_\delta(b-t)$ on $(\frac{b}{2}, b)$. Here we assume that $\delta\frac{\pi}{2} < \frac{b}{2}$. Such a metric is called a *double torpedo metric of radius* δ and denoted $g_{\text{dtorp}}^n(\delta)$. It is easy to check that $\bar{\eta_\delta}$ is an element of $\mathcal{B}$ and moreover that $g_{\text{dtorp}}^n(\delta)$ is an element of $\mathcal{W}$ obtainable as an embedding in the usual way.

Remark 7.0.11. In general, we will suppress the δ term when writing $\bar{g}_{\text{torp}}^n(\delta)$, $g_{\text{dtorp}}^n(\delta)$ etc and simply write $\bar{g}_{\text{torp}}^n$, g_{dtorp}^n etc, knowing that we may choose δ to be arbitrarily small if necessary.

We will denote by $F_{n\text{-dtorp}}$ the embedding in $\mathcal{E}$ which corresponds to the metric g_{dtorp}^n and by $F_{n\text{-torp}}$, the restriction of this embedding to the disk D^n (taken as the hemisphere corresponding to the interval $(0,\frac{b}{2})$). Finally, an obvious corollary of Lemma 7.0.10 is that the metric $g_{\text{dtorp}}^n(\delta)$ is isotopic to standard round metric and that there is a corresponding isotopy in the space $\mathcal{E}$ between $F_{n\text{-dtorp}}$ and $F_{n\text{-round}}$.

Bending Cylinder Metrics. We will consider metrics on the cylinder $S^n \times I$ of the form $g + dt^2$ where $g \in \mathcal{W}$ as defined above. Let $F : (0,b) \times S^{n-1} \to \mathbb{R} \times \mathbb{R}^n$ be the embedding described above, defined $F(r,\theta) = (\alpha(r), \beta(r).\theta)$, which corresponds to the metric g. Here α and β satisfy the conditions described in the previous section. We are particular interested in the case of a cylinder of double torpedo metrics. This case is a sub-case of the following more general situation.

The cylinder metric $g_{\text{cyl}} = g + dt^2$ on $S^n \times (0,\frac{\pi}{2})$ is obtained via the embedding

$$\begin{aligned} \text{cyl} : (0,b) \times S^{n-1} \times (0,\tfrac{\pi}{2}) &\longrightarrow (0,\infty) \times \mathbb{R}^n \times (0,\infty), \\ (r,\theta,l) &\longmapsto (c + \alpha(r), \beta(r).\theta, l). \end{aligned}$$

where $S^{n-1} \subset \mathbb{R}^n$ and consists of the set of points $\theta = (\theta_1, \cdots, \theta_n)$ which satisfy

$$\theta_1^2 + \theta_2^2 + \cdots + \theta_n^2 = 1,$$

and $c >$ diameter(g). In particular, the round cylinder of radius δ is obtained by the embedding

$$\begin{aligned} \text{cyl} : (0,\delta\pi) \times S^{n-1} \times (0,\tfrac{\pi}{2}) &\longrightarrow (0,\infty) \times \mathbb{R}^n \times (0,\infty), \\ (r,\theta,l) &\longmapsto (c + \delta\cos\tfrac{r}{\delta}, \theta.\delta\sin\tfrac{r}{\delta}, l), \end{aligned}$$

where $c > 2\delta > 0$.

To obtain the bent cylinder we must compose the embedding, cyl, with a certain diffeomorphism of Euclidean space, bend, defined as follows

$$\begin{aligned} \text{bend} : (0,\infty) \times \mathbb{R}^n \times (0,\infty) &\longrightarrow (0,\infty) \times \mathbb{R}^n \times (0,\infty), \\ (x_0, x, l) &\longmapsto (x_0 \cos l, x, x_0 \sin l). \end{aligned}$$

Here $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. The metric g_{bcyl} is then obtained by pulling back the standard Euclidean metric on $(0,\infty) \times \mathbb{R}^n \times (0,\infty)$ via the composition bend $\circ$ cyl. These maps and their effect on the cylinder are represented, in the case of the round cylinder, in Fig. 31 below.

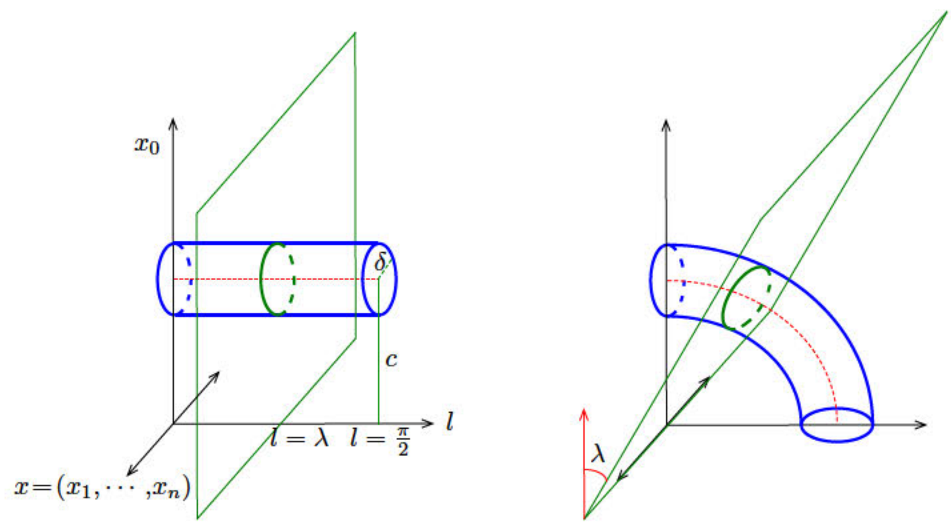

FIGURE 31. The embedding cyl (left) followed by the diffeomorphism bend (right)

We will now prove a lemma which will complete the proof of Lemma B.1.7.

Lemma 7.0.12. *Let $n \geq 3$. For some sufficiently large $c > 0$, the bent cylinder metric g_{bcyl} has positive scalar curvature.*

Proof. We will now compute the induced metric g_{bcyl}. Letting g_{Euc} denote the standard Euclidean metric on $(0,\infty) \times \mathbb{R}^n \times (0,\infty)$, we note that

$$g_{\text{bcyl}} = (\text{bend} \circ \text{cyl})^*(g_{\text{Euc}}) = \text{cyl}^*(\text{bend}^*(g_{\text{Euc}})).$$

We will begin by computing $\text{bend}^*(g_{\text{Euc}})$. Letting $g_{\text{Euc}} = dx_0^2 + (dx_1^2 + \cdots + dx_n^2) + dl^2$, we obtain

$$\begin{aligned} \text{bend}^*(g_{\text{Euc}}) &= \text{bend}^*(dx_0^2 + (dx_1^2 + \cdots + dx_n^2) + dl^2) \\ &= d(x_0 \cos l)^2 + (dx_1^2 + \cdots + dx_n^2) + d(x_0 \sin l)^2 \\ &= (\cos l.dx_0 - x_0 \sin l dl)^2 + (dx_1^2 + \cdots + dx_n^2) + (\sin l.dx_0 + x_0 \cos l.dl)^2 \\ &= \cos^2 l dx_0^2 + x_0^2 \sin^2 l.dl^2 - 2x_0. \cos l \sin l.dx_0 dl \\ &\quad + (dx_1^2 + \cdots + dx_n^2) \\ &\quad + \sin^2 l dx_0^2 + x_0^2 \cos^2 l.dl^2 + 2x_0 \cos l \sin l dx_0 dl \\ &= dx_0^2 + (dx_1^2 + \cdots + dx_n^2) + x_0^2 dl^2. \end{aligned}$$

Next we compute $\mathrm{cyl}^*(\mathrm{bend}^*(g_{\mathrm{Euc}}))$ as follows.

$$
\begin{aligned}
\mathrm{cyl}^*(\mathrm{bend}^*(g_{\mathrm{Euc}})) &= \mathrm{cyl}^*(dx_0^2 + (dx_1^2 + \cdots + dx_n^2) + x_0^2 dl^2)\\
&= \mathrm{cyl}^*(dx_0^2 + \sum_{i=1}^{n} dx_i^2 + x_0^2 dl^2)\\
&= \mathrm{cyl}^*(dx_0^2 + \sum_{i=1}^{n} \delta_{ij} dx_i dx_j + x_0^2 dl^2)\\
&= d(c+\alpha(r))^2 + \sum_{i=1}^{n} \delta_{ij} d(\theta_i \beta(r)) d(\theta_j \beta(r)) + (c+\alpha(r))^2 dl^2\\
&= \dot{\alpha}(r)^2 dr^2 + \sum_{i=1}^{n} \delta_{ij} (\theta_i \dot{\beta}(r) dr + \beta(r).d\theta_i)(\theta_j \dot{\beta}(r) dr + \beta(r).d\theta_j)\\
&\qquad + (c+\alpha(r))^2 dl^2.
\end{aligned} \tag{7.8}
$$

We will temporarily consider the term

$$
\sum_{i=1}^{n} \delta_{ij} (\theta_i \dot{\beta}(r) dr + \beta(r).d\theta_i)(\theta_j \dot{\beta}(r) dr + \beta(r).d\theta_j).
$$

This simplifies as follows.

$$
\begin{aligned}
&\sum_{i=1}^{n} \delta_{ij} (\theta_i \dot{\beta}(r) dr + \beta(r).d\theta_i)(\theta_j \dot{\beta}(r) dr + \beta(r).d\theta_j)\\
=&\dot{\beta}(r)^2 \sum_{i=1}^{n} \theta_i^2 dr^2 + 2\sum_{i=1}^{n} \beta(r)\dot{\beta}(r)\theta_i d\theta_i dr + \beta(r)^2 \sum_{i=1}^{n} d\theta_i^2\\
=&\dot{\beta}(r) dr^2 + \beta(r)^2 \sum_{i=1}^{n} d\theta_i^2\\
=&\dot{\beta}(r) dr^2 + \beta(r)^2 ds_{n-1}^2.
\end{aligned}
$$

The third line above follows from the fact that $\sum_{i=1}^{n} \theta_i^2 = 1$ and so $\sum_{i=1}^{n} \theta_i d\theta_i = 0$. The fourth line then follows from the fact that $\sum_{i=1}^{n} d\theta_i^2$ restricted to S^{n-1} is precisely the standard round metric of radius 1. Returning to (7.8), we now have that

$$
\begin{aligned}
\mathrm{cyl}^*(\mathrm{bend}^*(g_{\mathrm{Euc}})) &= \dot{\alpha}(r)^2 dr^2 + \dot{\beta}(r)^2 dr^2 + \beta(r)^2 ds_{n-1}^2 + (c+\alpha(r))^2 dl^2\\
&= dr^2 + \beta(r)^2 ds_{n-1}^2 + (c+\alpha(r))^2 dl^2,
\end{aligned}
$$

as, recall, $\dot{\alpha}^2 + \dot{\beta}^2 = 1$. So, in the case of the round cylinder $g + dt^2 = \delta^2 ds_n^2 + dt^2$, this metric is $dr^2 + \delta^2 \sin^2 \frac{r}{\delta} ds_{n-1}^2 + (c + \delta \cos \frac{r}{\delta})^2 dl^2$.

The scalar curvature R of a metric, on $(0,a) \times S^{n-1} \times (0,b)$ of the form

$$
dr^2 + \phi(r)^2 ds_{n-1}^2 + \psi(r)^2 dl^2,
$$

where $\phi, \psi : (0,b) \to (0,\infty)$ are smooth functions and $r \in (0,a)$ and $l \in (0,b)$ is given by the formula, see [23, proof of Lemma 2.1]:

$$
R = \tfrac{(n-1)(n-2)}{\phi^2}(1-\phi_r^2) - \tfrac{2(n-1)}{\phi}\left(\phi_{rr} + \tfrac{\phi_r.\psi_r}{\psi}\right) - 2\tfrac{\psi_{rr}}{\psi}.
$$

In our case $\phi(r) = \beta(r)$ while $\psi(r) = c + \alpha(r)$. Hence we obtain the following formula for the scalar curvature of g_{bcyl}.

$$\begin{aligned} R^{g_{\mathrm{bcyl}}}(r,\theta,l) &= \tfrac{(n-1)(n-2)}{\beta(r)^2}(1-\dot{\beta}(r)^2) - \tfrac{2(n-1)}{\beta(r)}\left(\ddot{\beta}(r) + \tfrac{\dot{\beta}(r).\dot{\alpha}(r)}{c+\alpha(r)}\right) - 2\tfrac{\ddot{\alpha}(r)}{c+\alpha(r)} \\ &= \tfrac{1}{\beta(r)}\left(\tfrac{(n-1)(n-2)}{\beta(r)}(1-\dot{\beta}(r)^2) - 2(n-1)\ddot{\beta}(r)\right) \\ &\quad -\tfrac{2(n-1)}{\beta(r)}\tfrac{\dot{\beta}(r).\dot{\alpha}(r)}{c+\alpha(r)} - 2\tfrac{\ddot{\alpha}(r)}{c+\alpha(r)}. \end{aligned}$$

The first line of this formula, featuring the terms inside the large parentheses, is strictly positive everywhere. This follows from the fact that $\ddot{\beta}(r) \leq 0$ for all r. Any negativity arising from the remaining terms can be minimised by choosing a sufficiently large $c > 0$ and so, provided $n \geq 3$, the bend can always be made to preserve positive scalar curvature. $\square$

Note that in the case of the standard round cylinder of radius δ. Here scalar curvature takes the following form.

$$\begin{aligned} R^{g_{\mathrm{bcyl}}} &= \tfrac{(n-1)(n-2)}{\delta^2\sin^2\frac{r}{\delta}}\left(1-\cos^2\tfrac{r}{\delta}\right) - \tfrac{2(n-1)}{\delta\sin\frac{r}{\delta}}\left(\tfrac{-1}{\delta}\sin\tfrac{r}{\delta} + \tfrac{\cos\frac{r}{\delta}.\sin\frac{r}{\delta}}{c+\delta\cos\frac{r}{\delta}}\right) - 2\tfrac{\cos\frac{r}{\delta}}{\delta(c+\delta\cos\frac{r}{\delta})} \\ &= \tfrac{1}{\delta}\left(\tfrac{(n-1)(n-2)}{\delta} + \tfrac{2(n-1)}{\delta} - \tfrac{2(n-1)\cos\frac{r}{\delta}}{c+\delta\cos\frac{r}{\delta}} - \tfrac{2\cos\frac{r}{\delta}}{c+\delta\cos\frac{r}{\delta}}\right) \\ &= \tfrac{1}{\delta}\left(\tfrac{(n)(n-1)}{\delta} - \tfrac{2n\cos\frac{r}{\delta}}{c+\delta\cos\frac{r}{\delta}}\right). \end{aligned}$$

Since

$$\left|\tfrac{2n\cos\frac{r}{\delta}}{c+\delta\cos\frac{r}{\delta}}\right| \leq \tfrac{2n}{c},$$

it follows immediately that by choosing sufficiently small δ and sufficiently large c, we may obtain a metric with everywhere positive scalar curvature. Notice that in this case it is sufficient to assume that $n \geq 2$.

Returning to the boot metric case, we conclude with the following lemma.

Lemma 7.0.13. *Provided $n \geq 3$, there is an isotopy in the space $\mathcal{R}^+(D^n \times I)_{g_{\mathrm{torp}}^n, g_{\mathrm{torp}}^n}$ which moves the metric $g_{\mathrm{torp}}^n + dt^2$ to a boot metric g_{boot}.*

References

1. A. L. Besse, *Einstein Manifolds,* Springer-Verlag (Berlin, 1987)
2. B. Botvinnik, Johannes Ebert, Oscar Randal-Williams, *Infinite Loop Spaces and Positive Scalar Curvature,* Preprint, November 2014
3. B. Botvinnik, B. Hanke, T. Schick, M. Walsh, *Homotopy groups of the moduli space of metrics of positive scalar curvature,* Geom. Topol. 14 (2010), no. 4, 2047-2076.
4. B. Botvinnik and P.B. Gilkey, *The eta invariant and metrics of positive scalar curvature,* Math. Ann **302**, no. 3, 1995, 507-517.
5. V. Chernysh, *On the homotopy type of the space $\mathcal{R}^+(M)$,* Preprint, arXiv:math.GT/0405235
6. V. Chernysh, *A quasifibration of spaces of positive scalar curvature metrics.* Proc. AMS **134**, no. 9, September 2006, 2771-2777
7. R. Carr, *Construction of manifolds of positive scalar curvature*, Trans. A.M.S. **307**, no. 1, May 1988, 63-74.
8. D. Crowley and T. Schick, *The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature*, Geom. Topol., 17(3):17731789, 2013.
9. P. Gajer, *Riemannian Metrics of Positive Scalar Curvature on Compact Manifolds with Boundary,* Ann. Global Anal. Geom. Vol. 5, No. 3 (1987), 179-191
10. S. Galatius and O. Randal-Williams *Stable moduli spaces of high-dimensional mani- folds,* Acta Math., 212 (2) :257377, 2014.
11. M. Gromov and H. B. Lawson, Jr., *The classification of simply-connected manifolds of positive scalar curvature,* Ann. of Math. **111** (1980), 423–434.
12. B. Hanke, T. Schick and W. Steimle, *The Space of Metrics of Positive Scalar Curvature*, To appear in Publ.Math. IHES. Preprint at arXiv:1212.0068
13. A. Hatcher, *Algebraic Topology,* Cambridge University Press (2002)
14. M. W. Hirsch, *Differential Topology,* Springer (1980)
15. N. Hitchin, *Harmonic Spinors,* Advances in Math. 14 (1974), 155.
16. F.C. Marques, *Deforming Three-Manifolds with Positive Scalar Curvature*, Ann. of Math. **176** (2012) Issue 2, 815–863.
17. R. S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16.
18. P. Peterson, *Riemannian Geometry 2nd Edition,* Springer (1998).
19. J. Rosenberg and S. Stolz, *Metrics of positive scalar curvature and connections with surgery,* Surveys on Surgery Theory, Vol. 2, Ann. of Math. Studies 149, Princeton Univ. Press, 2001.
20. R. Schoen and S.-T. Yau, *On the structure of manifolds with positive scalar curvature,* Manuscripta Math. **28** (1979), 159–183. MR535700 (80k:63064)
21. S. Stolz, *Simply connected manifolds with positive scalar curvature,* Ann. of Math., 2nd Series **136**, no. 3 (1979), 511–540.
22. M. Walsh, *Metrics of positive scalar curvature and generalised Morse functions, Part I,* Memoirs of the American Mathematical Society. Volume 209, No. 983, January 2011
23. M. Walsh, *Metrics of positive scalar curvature and generalised Morse functions, Part II,* Trans. Amer. Math. Soc., 366(1):150, 2014.
24. M. Walsh, *Cobordism Invariance of the homotopy type of the space of positive scalar curvature metrics* Proc. Amer. Math. Soc., 141(7):24752484, 2013.

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