Horizontal Bundles and Connections

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Abstract. A manifold is a mathematical space that locally resembles standard Euclidean space. In order to study the geometry of such a space, it is necessary to prescribe a connection on the manifold. A connection describes how tangent spaces to the manifold change with respect to infinitesimal changes in the manifold. In 1950, Charles Ehresmann defined a connection to be an abstract object called a horizontal bundle, with a special property called horizontal path lifting. By including this additional property in his definition, Ehresmann implicitly acknowledged that it was nontrivial: not all horizontal bundles satisfy it. We give the first complete characterization of the horizontal bundles which have this property, and hence are connections.

1. A Brief History of Geometry

Geometry began as the study of shapes and objects that people could touch and see around them, long before the beginning of recorded history. Around 300 BC, Euclid collected the logical foundations of geometry in his *Elements*. The axioms he presented in these thirteen volumes were accepted as law for nearly 2000 years, and are still taught as such to many of today's school children.

During the late 18*th* century some experts began to suspect that other geometries might exist, and by 1817 C.F. Gauss had become assured that such non-Euclidean geometries did exist. Over the next decade, Schweikart, Boylai, and Lobachevsky found a version of geometry that did not match Euclid's, and it eventually became clear to the experts that there were at least two geometries.

In 1854, Bernhard Riemann gave his inaugural lecture at Göttingen on the topic of Gauss's choosing: the foundations of geometry. Here he presented an infinity of new geometries, whilst also laying the framework for what has become modern differential geometry. The principal object of study is now an *n-dimensional manifold*: a space that is intuitively built out of many blobs of *n*-dimensional Euclidean space that are glued together in a specific way.

One challenge of studying geometry on such a space is that one must be able to compare directions at different points in some meaningful way. This is done using a *connection* on the manifold, intuitively named because its job is to connect distance points *via* parallel transport. Connections were studied from different points of view by E.B. Christoffel, T. Levi-Civita, H. Weyl, and E. Cartan, among others, throughout the late 19*th* and early 20*th* centuries.

In 1950, Charles Ehresmann [1] gave a simple, although abstract, definition of a connection that encompassed all of the previous. He defined a connection to be a horizontal subbundle of the second tangent bundle that has a property called *horizontal path lifting*. These connections have been studied extensively, but until now the following question had not been completely answered: *which horizontal bundles are connections?*

2. Horizontal Bundles *versus* Connections

A horizontal bundle on a manifold acts like a second-order differential equation: it describes how tangent vectors to M (directions) change from point to point. A horizontal bundle is a connection if this differential equation has a solution for every set of initial data. Thus to answer the question of which horizontal bundles are connections, we set out to reduce the problem from the abstract setting of a horizontal bundle on a manifold to the more fundamental setting of a differential equation on a portion of Euclidean space. We were then able to apply the well-known theory of differential equations. The remainder of this section summarizes the details of this reduction.

At every point p in a manifold M, the set of all tangent vectors to the manifold at that point determines the *tangent space* to M at p, denoted by T_pM. The collection of all of these tangent spaces is called the *tangent bundle* of M, and is denoted by TM. The tangent bundle is itself a manifold, hence it has a tangent bundle of its own, denoted by TTM.

Given a vector v in T_pM, the tangent space T_vTM can be split into two factors: those vectors which are tangent to T_pM, and those which are not. The space V_v of all vectors that are tangent to T_pM is called the *vertical space*, and the collection of all vertical spaces V is called the *vertical bundle* on TM.

Definition. A *horizontal space* at v in TM is a space H_v such that $H_v + V_v = T_vTM$. If these vary smoothly from point to point, then the collection of all is called a *horizontal bundle* on M, and is denoted by H . We write $H + V = TTM$.

Let M be a manifold, p in M a point, U a Euclidean neighborhood of p, and H a horizontal bundle on M. The pair (H,U) determines a function C called a *Christoffel form* that takes in a direction field X on M and a fixed direction v in T_pM as its arguments, and returns another direction field on M. We write $C(X,v) = V$ to denote this relationship.

Suppose $g : [0,1]$ to U is a path contained in U, emanating from p, with $g(1) = q$. A *horizontal lift* of g is a path G : $[0; 1]$ to TU that is defined over the entire interval [0,1], and satisfies $G(t) = C(g',v)(t)$ for all t in [0,1], where the g' indicates a derivative. One should think of G as sitting directly above the original path g, hence the name lift. The path g is said to have *horizontal lifts* if such a G exists for every initial value v in T_pM.

Definition. A horizontal bundle H on M is said to have *horizontal path lifting*, or HPL, if and only if every path g in M has horizontal lifts. A horizontal bundle with HPL is called a *connection* on M.

Standard differential equation theory tells us that a horizontal bundle will have HPL if and only if the maps C are bounded as v varies in T_pM. To measure this, it is necessary to compare the vertical and horizontal spaces along T_pM. In [2], we showed that the Christoffel forms will be bounded if and only if H has the following property.

Definition. A horizontal bundle H is *uniformly vertically bounded*, or UVB, if and only if the horizontal spaces H_v are bounded away from the vertical spaces V_v, uniformly along tangent spaces T_pM.

For this definition to be useful, one must have a way to measure the distance between the horizontal spaces H_v and vertical spaces V_v in T_vTM. This can be done using a collection of *Wong angles*, which are adapted from the angles used by Wong to study the geometry of Grassmann manifolds in [3]. Combining these techniques and properties, we were able to give a definitive answer to our question.

3. Summary of Results

Using the methods described above, we were able to prove that uniform vertical boundedness is both necessary and sufficient for a horizontal bundle on M to be a connection [2].

Theorem. *A horizontal bundle H on a manifold M has HPL if and only if it is UVB.*

In fact, we proved that this result actually holds for horizontal bundles on more abstract spaces called fiber bundles. A *fiber bundle* is a manifold M with another manifold F of information attached at every point in a very specific way. Tangent bundles are a special example of these more abstract spaces. Fiber bundles have applications in algebraic topology, high energy particle physics, and gauge theories.

The methods developed in proving this theorem have also led to further insight into the intrinsic nature of connections, and should lead to more interesting geometric results in the near future. In particular, the Wong angles and Christoffel forms have made it possible to define a class of connections on fiber bundles that deviates the least from the *linear connections* that fill the literature.

4. Acknowledgements

I wish to thank Professor Phil Parker for introducing this problem to me, and sharing his insight, advice, and ideas. I would also like to acknowledge that the historical content of the introduction largely comes from his post *Some History of Geometry*, which can be found on his personal web page.

5. References

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