MULTIVARIATE QUALITY CONTROL: STATISTICAL PERFORMANCE AND ECONOMIC FEASIBILITY

A Dissertation by

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I have examined the final copy of this dissertation for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy with a major in Industrial Engineering

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DEDICATION

To my beloved parents,
Asem Khalidi and Hania Jauni,
for their continuous encouragement and unconditional support,
which made the completion of this dissertation possible

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ABSTRACT

Shewhart control charts have been used to monitor uncorrelated quality characteristics. Advancement in manufacturing technology and increased complexity of products and systems raise the need to monitor correlated characteristics. The literature provides numerous examples of research pertaining to the misuse of traditional charts when the charted characteristics are correlated. This research is aimed at quantifying the statistical and economic consequences of utilizing the Hotelling's T^2 multivariate control chart as an alternative to the traditional Shewhart $\bar{x}$ chart. Consequently, there were two main objectives of this research. The first objective was to identify the levels of correlation between the charted variables where the statistical performance of the $\bar{x}$ chart deteriorates compared to that of an equivalent T^2 chart. Statistical analyses of simulated data generated under varying levels of process and chart variables indicated a correlation threshold value of $\pm$ 0.48, outside of which the T^2 chart is better. The second objective was to assess the economic feasibility of utilizing a T^2 chart as an alternative to the two $\bar{x}$ charts. Knappenberger and Grandage's (1969), and Montgomery and Klatt's (1972) economic design models for $\bar{x}$ and T^2 charts were utilized, respectively, in constructing an incremental cost model to examine the cost and worth of switching from the $\bar{x}$ charts to a T^2 chart under specified levels of process and chart parameters. Results indicated that the switch to multivariate T^2 chart would result in economic savings under all levels of the process and chart variables considered. It is hoped that this research will encourage practitioners to implement appropriate multivariate statistical techniques in monitoring their processes.

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LIST OF SYMBOLS

μ	population mean vector
Σ	population covariance matrix
$\mathbf{X}$	random vector of quality characteristics
T^2	statistic plot on control chart
$T^2_{(\alpha,p,n-p)}$	upper α percentage point of Hotelling's T^2 distribution
$\mathbf{S}$	estimate of population covariance matrix
μ_0	value of μ corresponding to the in-control state
μ_1	value of μ corresponding to the in-control state
$\bar{\mathbf{X}}$	sample mean vector of quality characteristics
δ	vector of difference between the in-control and out-of-control states
$E(C_1)$	expected cost per unit of sampling and testing
$E(C_2)$	expected cost per unit of investigating and correcting the process
$E(C_3)$	expected cost per unit associated with producing defectives
a_1	fixed cost per sample
a_2	per-unit cost of sampling
a_3	mean cost of investigating and correcting a process which is out-of-control
a_4	penalty cost of producing a defective units
k	number of units produced between successive samples
λ^{-1}	mean time between shifts to the out-of-control state
ρ_i	conditional probability that the test procedure indicates that the process is out-of-control given that the process is in state μ_i (i = 0, 1)

LIST OF SYMBOLS (continued)

β_i	probability that the process is in state μ_i (i = 0, 1) at the time the test is performed
ϕ_i	conditional probability of producing a defective unit given that the process is in state μ_i (i = 0, 1)
γ_i	probability that the process is in state μ_i (i = 0, 1) at any point in time
N	sample size
n	subgroup size
k	number of units produced between successive samples
G	probability of the process shifting from state μ_i (i = 0, 1) during the production of k units
l	lower specification vector
u	upper specification vector
q	row vector representing values of probabilities q_i (the probability of rejecting H_0 when $\mu = \mu_i$,
α^t	transpose of the row vector representing the steady-state probability that the process is in state i (that is, $\mu = \mu_i$) at the time of the test
R	production rate per hour
K	$\lambda k / R$
λ`	λ / R
A_i	$(a_i\lambda / R)/ a_4$, i = 1, 2, 3

CHAPTER 1

INTRODUCTION

Since the pioneering work of Shewhart in 1931, control charts have been successfully used to monitor process performance over time. They have been a foundation for maintaining and achieving new unprecedented levels of quality. However, these are generally classified as univariate charts that can only be used to monitor a single characteristic of a stationary process. Advancements in technology and increased customer expectations have raised the need to monitor correlated variables simultaneously. This requires the utilization of multivariate control charts, enabling engineers and manufacturers to monitor the stability of their systems. Under these conditions, achieving a state of statistical control requires a higher level of knowledge regarding the process variables, the level of correlation among them, and the accuracy by which they can be controlled. The original work in multivariate quality control can be attributed to Hotelling (1947). His work led to a number of multivariate techniques presented in the literature.

There are many situations where simultaneous monitoring or control of two or more correlated quality characteristics is necessary. Using independent univariate charts is not always the best method for monitoring correlated characteristics, because the correlations between variables result in degrading the statistical performance of these charts.

With the advancement in technology and increased complexity of processes, customers' demand of higher quality, and market competition, it is necessary to use multivariate statistical process control (SPC). Furthermore, with the greatly increased availability of high-speed computers and multivariate software, many users can now apply multivariate techniques.

Despite the renewed interest in multivariate SPC, these techniques have not been fully utilized in practice. Some questions remain unanswered: the levels of correlation that mandate the use of multivariate charts, and the statistical effect of mis-specifying the process model while applying traditional Shewhart charts. In addition, the economic consequences of implementing multivariate SPC as an alternative procedure to Shewhart charts have not been studied.

Chapter 2 presents a review of the literature of multivariate statistical process control and the underlying assumptions. Chapter 3 provides a discussion leading to the research gap, objectives, and procedures. Chapter 4 is devoted to the initial investigations to quantify the effect of correlation on the statistical performance of the Shewhart $\bar{x}$ chart and Hotelling T^2 chart leading to Chapter 5, which presents the characteristics of statistical performance of the Shewhart $\bar{x}$ chart and Hotelling T^2 chart and their implementation boundaries. The incremental cost model depicting the cost and worth of switching from Shewhart $\bar{x}$ charts to Hotelling T^2 chart is presented in Chapter 6. The summary and conclusions of this research including recommended future research are provided in Chapter 7.

CHAPTER 2

LITERATURE REVIEW

This chapter presents a review of publications in the area of multivariate control charts and their applications. This review is divided into four sections. The first section presents a review of traditional statistical process control and process capability measures in the univariate domain. The second section presents a definition of correlation and a review of the various methods of quantifying its presence. The third section reviews multivariate statistical process control methods, including a review of Hotelling T^2 control charts and their schemes, the first application of a Hotelling T^2 control chart and its interpretation, a recent review of more sensitive multivariate charts such as Multivariate Cumulative Sum (MCUSUM) and Multivariate Exponentially Weighted Moving Average (MEWMA) control charts, a review of process capability in the multivariate domain, and the statistical performance of Hotelling T^2 control charts. The fourth section presents a review of traditional economic models.

2.1 Traditional Statistical Process Control

Control charts were developed in 1931 by Shewhart to be utilized for process monitoring. They have been widely used to distinguish between assignable causes and chance causes of variation. The literature revealed several definitions of control charts. Shewhart (1931) gave the control chart the following definition: "The control chart may serve, first, to define the goal or standard for a process that management strives to attain; second, it may be used as an instrument for attaining that goal and third, it may serve as a means of judging whether the goal has been reached." Control charts may also be viewed as a statistical tool as defined by Duncan in 1956: ". . . a statistical device principally used for the study and control of repetitive processes." Moreover, Feigenbaum (1983) defined control charts as ". . . a graphical comparison of the actual product-characteristics with limits reflecting the ability to produce as shown by past experience on the product characteristics."

Therefore, the control chart is a graphical display used to monitor a process. It usually consists of a horizontal centerline corresponding to the in-control value of the parameter that is being monitored and the lower and upper control limits. Control limits are not determined arbitrarily, nor are they related to specification limits but rather by statistical criteria. If the sample points fall within the control limits, the process is deemed to be in-control, or free from any assignable causes. Points beyond the control limits indicate an out-of-control process, i.e., assignable causes are likely present. This signals the need for a corrective action to find and remove the assignable causes. The assignable causes, also called special causes, are the portion of the variability in a set of observations that can be traced to specific causes, such as, operators, materials, or equipment. On the other hand, the chance causes, also called common causes, are the portion of the variability in a set of observations that is due only to random forces and cannot be traced to specific sources, such as, operators, materials, or equipment.

The average run length (ARL) is used to evaluate the performance of control charts. The ARL can be calculated from

$$\mathrm{ARL}_0 = \frac{1}{\alpha} \tag{2.1}$$

where α is the probability that any point will exceed the control limits. For the Shewhart $\bar{x}$ chart with 3 σ limits, $\alpha = 0.0027$ is the probability that a single point will fall outside the limits when the process is in-control. Therefore, the ARL of the $\bar{x}$ chart when the process is in-control, called ARL_0, is

$$\mathrm{ARL}_0 = \frac{1}{\alpha} = \frac{1}{0.0027} = 370$$

Even if the process remains in-control, an out-of-control signal will be generated on the average every 370 samples. Moreover, the expected number of samples taken before the shift is detected, called ARL_1, is

$$ARL_1 = \frac{1}{1-\beta} \tag{2.2}$$

where β is the probability of points falling within the control limits after a shift in the process. Therefore, the probability a shift will be detected on the first subsequent sample is 1 - β (Montgomery, 2001).

2.1.1 Capability in Univariate Domain

Statistical process control procedures are widely used in industrial environments. A standard practice in SPC is to measure the process capability using Shewhart control charts. Capability indices, such as C_p, C_{pk}, and C_{pm}, typically are used as measures of the process capability.

A sample vector containing (n) univariate observations of a single product characteristic is represented by **x**. Assume that summary statistics, $\bar{x}$ and s, the process sample mean and sample deviation, respectively, are estimated from this sample and used to estimate a capability index of the process. While the original motivation may have been to estimate the expected proportion of production not conforming to engineering specification (Wang et al., 2000), a variety of univariate capability indices are currently available and used as decision making tools, such as vendor or process selection (Kotz and Lovelace, 1998). In fact, Kane (1986) has shown that these indices do not uniquely define the percentage nonconforming. Consider the two very popular univariate indices, C_p and C_{pk}. The process capability ratio, C_p, is the ratio of allowable process dispersion and observed process dispersion or

$$C_p = \frac{USL - LSL}{6\sigma} \quad (2.3)$$

where USL and LSL are the upper and lower specification limits, respectively. In using this index correctly, it is assumed that the underlying process characteristic measured is normally distributed. Moreover, if the process mean is centered within the tolerance region, then the index value provides an estimate of the proportion of nonconforming product. For example, for a process centered in the middle of the tolerance region, a C_p of 1.0 implies that the percentage of nonconforming product is 0.0027. If the process mean is far from the center of the engineering specification, it is possible that the process could be yielding as much as 100 percent nonconforming products. Similarly, the index, C_{pk}, takes process centering into account and is defined as

$$C_{pk} = \min\,(CPL, CPU) \quad (2.4)$$

$$C_{pk} = \min\left(\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right)$$

As is evident by the form of the ratio, C_{pk} is sensitive to the magnitude of the process variance and the location of the process mean relative to the specification limits (Montgomery, 2001). Kane (1986) stated that the presence of special causes of variation make prediction impossible and the meaning of a capability index unclear.

Despite these problems of interpretation, capability indices and their use in capability analysis are widely accepted in the implementation of univariate quality control monitoring scheme. In practice, the shortcoming of the indices are typically overcome by using graphical procedures to visualize the process data relative to the interval defined by engineering specifications and by percentage of nonconforming product, given an assumed underlying distribution of the process measurements. If the process is deemed to be capable, then the computed index value and the estimated percentage of nonconforming product are acceptable. The acceptance regions for both these statistics are usually specified as part of an organization's quality control system.

By examining the graphical displays of the estimated distribution functions in comparison to engineering specifications, the ambiguity of the univariate capability indices can be explained. Therefore, it is reasonable to compare the bell-shaped curve of the assumed normal distribution to the location of the upper and lower specification limits. The univariate indices provide a comparison of the length of the intervals (Walpole and Myers, 1993). However, in the multivariate domain, the comparison is somewhat more complex.

2.2 Correlation

The use of statistical process control has spread widely in industrial applications for improving processes, estimating process parameters, and determining capability. A primary assumption in the typical application of the standard Shewhart control charts is that observations are independent or uncorrelated. Moreover, processes may be classified as stationary or non-stationary. For stationary processes, Shewhart univariate charts are used to monitor single variables. On the other hand, non-stationary processes are autocorrelated (Del Castillo, 2002). Thus, an autocorrelated variable is a variable that is correlated "with itself" over time. Unfortunately, the independent assumptions are often violated in many types of manufacturing and production processes.

Correlation analysis is a statistical technique that can show whether and how strongly pairs of variables are related. Correlation refers to the departure of two or more variables from independence (Del Castillo, 2002). It is the degree to which two or more quantities are associated (Montgomery, 2001). For example, height and weight are related; taller people tend to be heavier than shorter people. However, people of the same height vary in weight; moreover, there are people where the shorter one is heavier than the taller one. Nevertheless, the average weight of people 5'5" tall is less than the average weight of people 5'6" tall, and their average weight is less than that of people 5'7" tall, and so on. Correlation can tell just how much of the variation in peoples' weights is related to their heights and whether this relationship is adversely or positively proportional. Correlation in industrial process data could be elucidated the same way.

Although correlation is fairly obvious in some industrial processes data, many may contain unsuspected correlations. Also correlations may be suspected without knowing which are the strongest. A correlation analysis can lead to a greater understanding of such data. Like all statistical techniques, correlation analysis is only appropriate for certain types of data, in which numbers are meaningful, usually quantities of some sort. It cannot be used for purely categorical data, such as gender. Various methods are used to quantify the presence of correlation.

When two or more random variables are defined on a probability space, it is usful to describe how they vary together; that is, it is useful to measure the relationship between the variables. A common measure of the relationship between two random variables is the covariance. The covariance between random variables X and Y, denoted as cov (X, Y) or σ_{xy} is

$$\sigma_{xy} = E[(X - \mu_X)(Y - \mu_Y)] \tag{2.5}$$

Covariance gives an idea of the strength of the correlation. For two variables X and Y, if the correlation is very strong means that if X is far from its mean, so should Y. Therefore, the covariance between X and Y describes the variation between the two variables. In the multivariate domain, the population covariance is represented in a matrix denoted as **Σ**. The covariance matrix, also called the variance-covariance matrix, is a symmetrical matrix that contains the variance and covariance among a set of random variables. The main diagonal elements of the matrix are the variances of the random variables, and the off-diagonal elements are the covariance between the p variables (Neter et al., 1996). The (p x p) sample variance-covariance matrix **S** is formed as

$$\mathbf{S} = \begin{bmatrix} S_1^2 & S_{12} & \cdots & S_{1P} \\ S_{12} & S_2^2 & \cdots & S_{2P} \\ \vdots & \vdots & \cdots & \vdots \\ S_{1P} & \cdots & \cdots & S_P^2 \end{bmatrix} \qquad (2.6)$$

In a two-dimensional plot, the degree of correlation between the values on the two axes is quantified by the so-called correlation coefficient. The most common correlation coefficient is the Pearson Product-Moment Correlation Coefficient, which is found by dividing the covariance of the two variables by the product of their standard deviation. This correlation coefficient (r) is a measure of the degree of linear relationship between two variables X and Y. In regression, the emphasis is on predicting one variable from the other; in correlation, the emphasis is on the degree to which a linear model may describe the relationship between two variables. In regression, the interest is directional, one variable is predicted and the other is the predictor. On the other hand, in correlation, the interest is non-directional; the relationship is the critical aspect. The square of (r) is called the Coefficient of Determination and denotes the portion of total variance explained by the regression model (Walpole and Myers, 1993). The sample correlation coefficient (r) is calculated by

$$r_{xy} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{(n-1) s_x s_y} \qquad (2.7)$$

where $\overline{x}$ and $\overline{y}$ are the sample means of x_i and y_i , s_x and s_y are the sample standard deviation of x_i and y_i, and the sum is from i = 1 to (n). As for the population, the correlation coefficient ρ_{xy} can be estimated from the sample r_{xy} and defined as

$$\rho_{xy} = \frac{COV(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \tag{2.8}$$

The correlation coefficient may take any value between - 1.0 and + 1.0. It is because of Cauchy-Schwarz inequality that the correlation cannot exceed 1 in absolute value (Neter et al., 1996). It is a useful inequality encountered in many different settings, such as linear algebra applied to vectors, in analysis applied to infinite series, integration of products, and in probability theory applied to variance and covariance. The inequality states that if x and y are elements of real or complex inner product space, then

$$|(x, y)|^2 \leq (x,x)\ (y,y) \tag{2.9}$$

The two sides are equal if and only if x and y are linearly dependent (or parallel). This contrasts with a property that the inner product of two vectors is zero if they are orthogonal (or perpendicular) to each other (Johnson and Wichern, 1998).

A correlation coefficient of ($r = 0.50$) indicates a stronger degree of linear relationship than one of ($r = 0.40$). Likewise, a correlation coefficient of ($r = -0.50$) shows a greater degree of relationship than one of ($r = -0.40$). Thus, a correlation coefficient of zero ($r = 0.0$) indicates the absence of a linear relationship and correlation coefficients of ($r = +1.0$) and ($r = -1.0$) indicate a perfect linear relationship.

A limitation to the measures of correlation presented is noted; their value could be 0 while, in fact, there is a relationship between the variables. The reason may be because this relationship is quadratic or of a higher order. Thus, it should be noted that correlation measures represent the strength of the linear relationship of the variables (Neteret al., 1996).

2.3 Multivariate Statistical Process Control

Process monitoring using control charts can be seen as a two-stage process, Phase I and Phase II (Woodall, 2000). The goal of Phase I is to evaluate the stability of the process and, after coping with any assignable causes, to estimate the in-control values of the process parameters. In Phase II, the main concern is to monitor the online data to quickly detect shifts in the process from the baseline established in Phase I. Different types of statistical methods are appropriate for the two phases, with each type requiring different measures of statistical performance. In Phase I, it is important to assess the probability of deciding that the process is unstable. However, in Phase II, the emphasis is on detecting process changes as quickly as possible. This is usually measured by parameters of the run-length distribution, where the run length is the number of samples taken before an out-of-control signal is given. The average run length is often used to compare the performance of computing control chart methods.

Hotelling (1947) developed the multivariate T^2 control chart as a direct analog of the Shewhart $\bar{x}$ control chart. This chart can be used to monitor the mean vector of multiple quality characteristics of a process in both Phase I and Phase II operations.

2.3.1 Hotelling T^2 Control Charts

The multivariate process control problem involves a repetitive process in which each characteristic is represented by random variables, $\mathbf{X_1, X_2, \ldots, X_p}$. The probability distribution of the process characteristics is assumed to be multivariate normal with a mean vector $\boldsymbol{\mu}$ and a covariance matrix $\boldsymbol{\Sigma}$. Multiple measurements of each process are assumed to be drawn from a population with standard values for $\boldsymbol{\mu_0}$ and $\boldsymbol{\Sigma_0}$. When changes in the process cause elements of $\boldsymbol{\mu}$

or $\mathbf{\Sigma}$ to shift from the standard values, it is necessary to detect and correct the change to ensure a stable process.

The T^2 control chart combines several quality characteristics for each item into a single quality measurement of the overall performance of the item. Hotelling formulated T^2 on the basis of a generalized Student Ratio (t) that was introduced in 1931 for testing multivariate hypotheses when the sample variance-covariance matrix **S** is unknown. Hotelling applied T^2 to the quality-control problem of testing bombsights. The advantage of the T^2 control chart is that the status of the process can be characterized by one value. However, if an out-of-control process does exist, one must go back to the original data to determine the nature of this problem.

In controlling industrial processes, it is not sufficient to monitor only the process mean. The process variability should be monitored and controlled as well. Montgomery and Wadsworth (1972) proposed a control chart for the multivariate dispersion that is based on a normal approximation of log |**S**|, where **S** is the sample variance-covariance matrix. This chart can be constructed by using data from the same preliminary samples used to develop the T^2 control chart. The variance-covariance matrix for each sample can then be computed from preliminary samples. To construct the log |**S**| chart, first the determinant of the variance-covariance matrix for each sample is computed, then the logarithm of the determinant of each of these matrices is taken, and the mean and standard deviation of this logarithm is determined. A control chart can then be constructed using the upper control limit (UCL) and the lower control limit (LCL) calculated as

$$\text{UCL} = \overline{\mathbf{Y}} + Z_{\alpha/2}\, \mathbf{S_y} \tag{2.10}$$

$$\text{LCL} = \overline{\mathbf{Y}} - Z_{\alpha/2}\, \mathbf{S_y} \tag{2.11}$$

where $Z_{\alpha/2}$ is the percentage point of the normal distribution, and $\bar{\mathbf{Y}}$ and $\mathbf{S_y}$ are the mean and the standard deviation of the logarithm of the determinant of each variance-covariance matrix. This chart, in conjunction with the T^2 control chart, could monitor, diagnose, and control procedures for multivariate control between and within sample variations.

Assume that there are (p) process characteristics that are jointly distributed according to the p-variate normal distribution, and a random sample of size (n) is available from the process. Then the multivariate analogue of (t) is

$$t^2 = \frac{(\bar{X} - \mu_0)^2}{\frac{s^2}{n}} \tag{2.12}$$

$$t^2 = n\left(\bar{\mathbf{X}} - \boldsymbol{\mu}_\mathbf{0}\right)'\left(\mathbf{s}^2\right)^{-1}\left(\bar{\mathbf{X}} - \boldsymbol{\mu}_\mathbf{0}\right)$$

When t^2 is generalized to (p) variables, it becomes

$$T^2 = n\left(\bar{\mathbf{X}} - \boldsymbol{\mu}_\mathbf{0}\right)'\left(\boldsymbol{\Sigma}_0^{-1}\right)\left(\bar{\mathbf{X}} - \boldsymbol{\mu}_0\right) \tag{2.13}$$

where

$\boldsymbol{\mu}_\mathbf{0}$ is a (p x 1) vector of population mean

$\bar{\mathbf{X}}$ is a (p x 1) vector of sample mean

$\boldsymbol{\Sigma}_\mathbf{0}$ is a (p x p) variance-covariance matrix

If the observed statistical distance T^2 is too large, that is, if $\bar{\mathbf{X}}$ is "too far" from $\boldsymbol{\mu}_\mathbf{0}$, then the hypotheses H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_\mathbf{0}$ is rejected. Since T^2 is distributed as $\frac{p(n-1)}{n-p} F_{\alpha(p,n-p)}$, then the T^2 statistic can be used for testing the hypotheses about the mean vector $\boldsymbol{\mu}_\mathbf{0}$ such as

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_\mathbf{0}$$

$$H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu_0}$$

T^2 can be computed and compared with $\frac{p(n-1)}{n-p} F_{\alpha(p,n-p)}$

When multiplying a T^2 statistic by a constant $\frac{n(n-p)}{p(n+1)(n-1)}$, it follows an F-distribution, where $F_{\alpha(p,n-p)}$ refers to the F-distribution with (p) and (n – p) degrees of freedom and a probability of Type 1 error of α . The null hypothesis would be rejected if

$$T^2 > \frac{p(n-1)}{n-p} F_{\alpha(p,n-p)} \tag{2.14}$$

Thus, the control limits of the T^2 control chart can be formed as

$$UCL = \frac{p(n-1)}{n-p} F_{\alpha(p,n-p)} \tag{2.15}$$

and

$$LCL = 0 \tag{2.16}$$

Since the test statistic is a generalized measurement of distance, the lower control limit is always zero. The reason for this is that any shift in the mean will always lead to an increase in the T^2 statistic, and thus the LCL may be ignored. If the computed statistic T^2 exceeds the upper control limit, the process mean is out-of-control, and assignable causes of variation are sought. In practice, $\boldsymbol{\mu_0}$ is generally unknown, so it is necessary to estimate it from a set of preliminary samples, which are taken when the process is assumed to be in-control.

If $\boldsymbol{\mu_0}$ and $\boldsymbol{\Sigma_0}$ are estimated from a relatively large number (more than 25) of preliminary samples, then it is customary to use $\chi^2_{\alpha,p}$ as the upper control limit on the T^2 control chart, where

$\chi^2_{\alpha,p}$ is the upper α percentage point of the Chi-square distribution with (p) degrees of freedom (Montgomery, 2001).

2.3.2 First Application

Hotelling (1947) conducted a study on dropping bombs from airplanes for the purpose of testing bombsights. Air testing is only one in a series of tests and inspections to which a bombsight is subjected. It is the final step and an exceptionally costly one. Because of the high cost and uncertainty of air testing with relative accuracy, only a very small number of bombsights were tested. Two sights were randomly selected from each lot of twenty sights. Four bombs on each sight from two flights were dropped for this experiment. Two measurements were targeted for the accuracy of each bomb dropping. The range error is an error in the direction of the airplane's heading at the time of releasing the bombs on the sights. The deflection error is an error in a direction perpendicular to the airplane's heading to the bombsight location.

There were three testing alternatives. The first alternative was to accept the bomb sight for which the univariate scheme applied for acceptance. Hence, the probability of Type I error (α) is maintained on each scheme. The true probability of Type I error for the joint control procedure is $\alpha' = 1 - (1 - \alpha)^p$. Therefore, the probability that both range and deflection are acceptable for α = 0.0027 is

$$\text{P(Acceptance)} = (1-0.0027)^2 = (0.9973)^2 = 0.9946 \tag{2.17}$$

Another alternative was rejection, which would require that both variables take such values as to call for rejection. For two independent variables, a probability of rejection intended to be 0.9 would actually be only 0.81in such a case. Thus, rejection occurs if both range and deflection are unacceptable. For β = 0.10,

$$\text{P(Rejection)} = (1-0.10)^2 = 0.81 \tag{2.18}$$

Hotelling suggested that the probabilities could be adjusted so as to become equal to 0.05, or such level as is chosen, by altering the acceptance level for each variable separately. However, this introduces additional difficulties. The variables may not be mutually independent, and calculations such as the aforementioned must be altered to take into account the multivariate distribution. Furthermore, it will often not be known whether they are independent or not; or if they are mutually dependent, the character of the dependence may be known only imperfectly. Thus, the correlation coefficient may have to be estimated from the preliminary sample size, so small as to leave its value somewhat uncertain. Any acceptance probabilities based on such a correlation coefficient will then, likewise, be uncertain. Another defect of such assumptions mentioned earlier is that an article close to the margin of acceptability with respect to one variable may well be marked for acceptance or rejection on the basis of the other variable involved. Unusual excellence in one respect may often occur for a slight departure in another way from what would otherwise be considered satisfactory.

As a result, Hotelling proposed a third alternative, which was a combined measure of accuracy T^2 that serves as a measure of the deviation of the particular bomb from the center of the target. This measure is more accurately interpreted in terms of the probability than is the actual distance. By adding the values of T^2 for all bombs dropped on a particular bombsight, a measure is useful in obtaining the accuracy of the bombsight, which achieves specified levels of (α) and (β) risks (Hotelling, 1947).

2.3.3 Chart Interpretation

The objective of performing multivariate SPC is to monitor process performance over time in order to detect any unusual events. It is essential to be able to track the cause of an out-of-control signal to maintain acceptable levels of quality and to allow for process improvements.

However, the complexity of multivariate control charts and cross-correlation among variables makes it difficult to analyze assignable causes leading to the out-of-control signals. Several techniques have been developed that assist in the interpretation of out-of-control signals. Following the same sensitivity of the Shewhart $\bar{x}$ control chart, the Hotelling T^2 is more efficient in detecting larger process shifts. Mason and Young (1999) introduced a modification procedure for the T^2 control charts in order to enhance sensitivity toward detecting a small process shift.

A T^2 control chart is used primarily to monitor the mean vector of quality characteristics of a process. There are two versions of the T^2 chart, one for subgrouped data and the other for individual observations. They can be used not only in achieving a state of statistical control (Phase I) but also in maintaining control over the process (Phase II).

In some cases, the multivariate data can be grouped into rational subgroups, relying on properties of the production process that creates homogeneity within subgroups. When rational subgroups are present, a shift in the mean vector is presumed to be more likely to take place between subgroups (variability in the process over time) than within a subgroup (instantaneous process variability at a given time). This can be used to advantage by forming the sample covariance matrix for each subgroup, then averaging them to get an estimate of the process covariance matrix. The mean vectors for each subgroup can be examined for a shift, thus detecting assignable causes for the shift in the mean vector (Sullivan and Woodall, 1996).

Mason et al. (2001) studied the effectiveness of using the T^2 control charts for batch (subgrouped) processes. His study recommended that when the batch data are collected from the same multivariate normal distribution, T^2 statistic is recommended for detecting out-of-control signals. When the batch data are collected from multivariate normal distributions with different mean vectors, the translation of the different batches to a common origin again allows the usage of T^2 statistic to identify out-of-control signals. Translation to a common origin involves the subtraction of individual batch mean vectors from the corresponding batch observations.

However, sometimes the rational subgroup size is one, that is, data are structured only as individual observations, and process characteristics do not necessarily produce homogeneous subgroups of large size. In the case of individual observations, Sullivan and Woodall (1996) recommended using the sample mean vector and covariance matrix if any value of the T^2 statistic exceeds an upper control limit resulting in an out-of-control signal generated. In some industrial situations, such as chemical and process industries, it is either impractical or difficult to obtain a subgroup size of more than one unit, since these industries frequently have multiple quality characteristics that must be monitored. Therefore, the T^2 control chart with n = 1 would be appropriate to use.

Mason et al. (1997) presented a multivariate profile chart by superimposing an $\bar{x}$ chart of univariate statistics on top of the T^2 chart. By performing discrimination analysis, this allows the distinguishing of in-control conditions from out-of-control conditions to determine where assignable causes of variation are occurring. This analysis works by partitioning the multivariate control chart based on the contribution of each variable.

There are also graphical solutions to interpretation difficulty. Lowry and Montgomery (1995) proposed poly plots and multivariate control webs to superimpose univariate statistics on multivariate statistics in order for the user to test trends in individual statistics and realize how they affect other variables.

Jackson (1956) suggested that the multivariate control region be displayed as an ellipse for two variables (p = 2). However, when Jackson's control ellipse is used, the time sequence of the plotted points is lost. The results obtained from Jackson's control ellipse are exactly the same as those obtained from using the T^2 control chart. If an observation is outside the ellipse, it will also be above the control limit specified on the T^2 control chart. On the other hand, if an observation is inside the control ellipse, it will be below the control limit specified on the T^2 control chart. However, if an observation is exactly on the parameter of the ellipse, it will be exactly on the control limit line of the T^2 control chart. The results obtained by both methods are identical. Nevertheless, the T^2 control chart retains the time scale and summarizes the process condition by one value, while use of the control ellipse indicates pictorially the nature of the out-of-control conditions.

Figure 2.1 presents the control region for two variables with different levels of correlations. Here, it can be seen that when (r = + 0.8), the control ellipse is tilted to the right from the horizontal axis; on the other hand, when (r = - 0.8), the ellipse becomes tilted to the left from the horizontal axis. However, when (r = 0), the ellipse becomes a circle.

Jackson (1959) considered the case of investigating two or more (p - 1) related variables to analyze a multivariate process. The basic concept of the technique is to break up the T^2 statistic into a sum of its principal components, the linear portions of the original variables. Principal component analysis (PCA) is a reliable technique to interpret out-of-control signals, whereby components can be examined to understand why the process is out-of-control. This could be accomplished by expressing the T^2 statistic as the normalized principal component of the multinormal variables. Hence, when an out-of-control signal is received, components with abnormally high values are detected. Plots of these variables can be used to determine exactly what occurred in the original sets of data that contributed to the signal in the multivariate set of T^2 statistics (Mason et al., 1997).

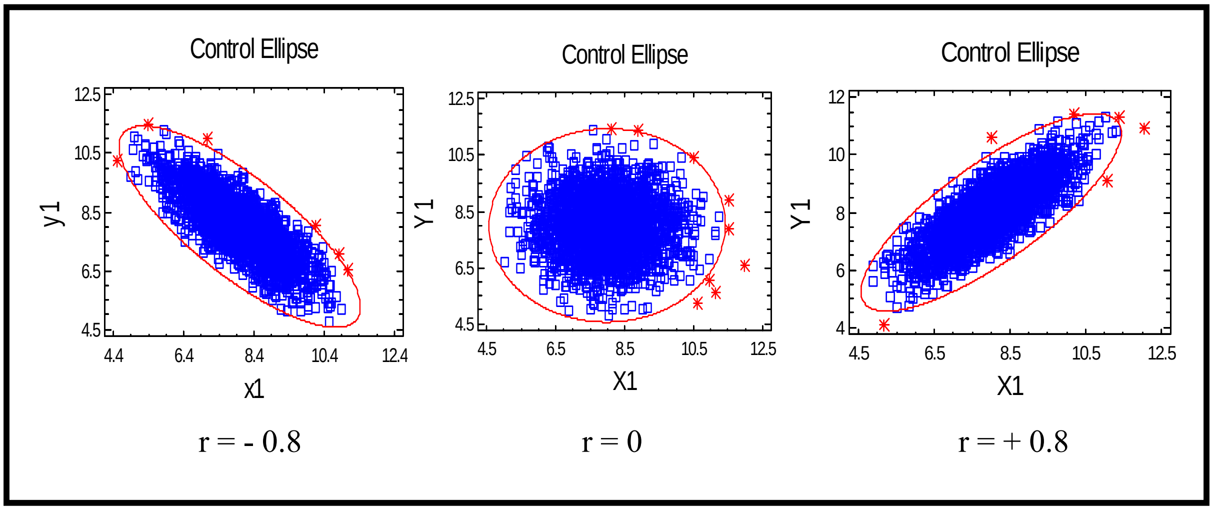


Figure 2.1 Ellipse control region
(Source: Montgomery, 2001)

Another method of interpreting out-of-control signals is to view the corresponding univariate charts of a multivariate process to determine which statistic is causing the assignable cause of variation. Some concerns are associated with the adaptability of this technique. First, when there are multiple variables being measured, this technique tends to be tedious due to the interpretation of multiple univariate charts. Second, in multivariate quality control, an out-of-control signal is usually not caused by one variable but rather is a function of several correlated variables that act interdependently. Therefore, in many circumstances, the respective univariate charts may show no signs of being out-of-control; however, multivariate charts would show this (Kourti and MacGregor, 1996; Mason et al., 1997). It is important to understand that there are other effective interpretation techniques that could be used with this technique to perform a better analysis of the out-of-control signals. The user should not be limited to this technique merely because it is a simple approach to the interpretation.

Runger (1996) proposed another approach to diagnose out-of-control signals. It includes decomposition of the T^2 statistic into components that reflect the contribution of each individual variable. The variable with the relatively higher contribution to the overall statistic should be the focus of attention.

2.3.4 More Sensitive Charts

Hotelling's multivariate control chart procedure is based on only the most recent observation; it is insensitive to small and moderate shifts in the mean vector. Hotelling's work paved the way for further developments in the multivariate field. Several multivariate CUSUM and multivariate EWMA procedures have appeared in the literature since then.

2.3.4.1 Multivariate CUSUM Control Charts

The Cumulative Sum (CUSUM) chart was first developed by Page (1954) to detect slight but sustained shifts in the process level (1.5 σ or less). The CUSUM chart is constructed for monitoring the mean of a process. It can be constructed for both individual observations $n = 1$ and the averages of rational subgroups $n > 1$ (Johnson, 1994). The multivariate CUSUM (MCUSUM) chart can be derived from the univariate versions to serve multivariate process monitoring purposes. There are two different approaches of applying CUSUM: one is the simultaneous analysis of multiple univariate CUSUM procedures; the other involves modifying the CUSUM scheme itself to form MCUSUM procedures. A MCUSUM can be derived from CUSUM based on two strategies. The first strategy involves reducing each multivariate observation to a weighted measurement and then forming a CUSUM of these measurements. The second strategy involves forming a MCUSUM directly from the observations by accumulating the **X** vectors before reducing it to weighted measurements. MCUSUM procedures are mostly dependent on the non-centrality parameter, which reports the shift size in terms of a quantity and defined as

$$\tau = (\mu' \Sigma^{-1} \mu)^{1/2} \quad (2.19)$$

Large values of τ correspond to larger shifts in the mean. The value $\tau = 0$ is the in-control state. The MCUSUM chart is designed for various shifts. Another MCUSUM chart is simply the square root of the T^2 statistic. The choice of this chart rather than a CUSUM chart of the T^2 is based on forming a CUSUM of distance rather than the CUSUM of the squared distance (Crosier, 1988).

Crosier proposed two MCUSUM charts. The one with the better ARL properties is based on the statistics

$$C_i = \{(\mathbf{S}_{i-1} + \mathbf{X}_i)^{`}\, \mathbf{\Sigma}^{-1}\, (\mathbf{S}_{i-1} + \mathbf{X}_i)\}^{1/2} \qquad (2.20)$$

where

$$\mathbf{S}_i = 0 \qquad \text{if } C_i \le k_1$$

$$\mathbf{S}_i = (\mathbf{S}_{i-1} + \mathbf{X}_i)\,(1 - k_1/C_i) \qquad \text{if } C_i > k_1$$

$i = 1, 2,\ldots,$ $\mathbf{S_0} = 0$, and $k_1 > 0$. This MCSUM chart signals when

$$\gamma_i = \{\mathbf{S}_i^{`}\, \mathbf{\Sigma}^{-1}\, \mathbf{S}_i\}^{1/2} > h, \qquad h > 0 \qquad (2.21)$$

For this procedure, (h) is chosen to achieve a specified in-control ARL. The MCUSUM procedure forms a CUSUM vector directly from the observations and gives an indication of the direction in which the mean has shifted. This scheme detects small shifts in the mean vector more quickly than does the Hotelling multivariate procedure. Moreover, it is directionally invariant.

Smith (1987) developed a MCUSUM procedure based on the likelihood ratio test, which is used to study shifts in the mean vector of a multivariate normal process. The procedure is adapted to study shifts in the covariance matrix of a multivariate normal process and to study shifts in the probabilities of a multinomial process. Because of its cumulative nature, this method is much better at detecting small shifts in the covariance matrix. Moreover, it continues to operate well for large shifts in variability. When a trend occurs in one direction of the target mean and a resulting shift occurs in the other direction, the MCUSUM chart will not detect the shift immediately. A combination of the MCUSUM chart and the T^2 limits will improve the chart sensitivity to large shifts (Lowry and Montgomery, 1995).

2.3.4.2 Multivariate EWMA Control Charts

The scheme of the exponentially weighted moving average chart developed by Roberts (1959), is similar to the moving average chart and could be extended to multivariate quality control problems (Montgomery, 2001). Shewhart's control charts have been the traditional tools for detecting larger shifts in the process mean (1.5 σ or more). For the univariate case, the EWMA is more effective than Shewhart control charts in detecting smaller shifts in the process mean. When (n) measurements from each item are required, these univariate control charts ignore the dependency among the (p) variables.

The multivariate exponentially weighted moving average control chart accumulates information from past observations making it sensitive to shifts in the variance as well as shifts in the mean. It allows the user to specify weights for each variable being measured. Although MEWMA is used commonly for controlling a multivariate process mean, Alt and Smith (1988) proposed three control charts for monitoring the covariance matrix, which is analogous to EWMA for the variance. Prabu and Runger (1997) have provided a thorough analysis of the average run-length performance of the MEWMA control chart. The MEWMA chart given by Lowry et al. (1992) is a natural extension to the univariate EWMA, defined by vectors of EWMAs and based on the statistics as

$$\mathbf{G}_i = \lambda\, \mathbf{x}_i + (1 - \lambda)\, \mathbf{G}_{i-1} \tag{2.22}$$

where $G_0 = 0$, $0 < \lambda_j \leq 1.0$, and $i = 1, 2, \ldots$, $\boldsymbol{\lambda} = \text{diagonal}\,(\lambda_1, \lambda_2, \ldots, \lambda_p)$, and $j = 1, 2, \ldots, p$.

The MEWMA chart gives an out-of-control signal as soon as

$$\mathbf{T}_i^2 = \mathbf{G}_i' \boldsymbol{\Sigma}_{G_i}^{-1} \mathbf{G}_i > h \tag{2.23}$$

where (h > 0) is chosen to achieve a specified in-control ARL, and Σ_{Gi} is the covariance matrix of $\mathbf{G}_i$. If there is no reason to weight past observations differently for the (p) quality characteristics being monitored, then $\lambda_1 = \lambda_2 = \ldots = \lambda_p = \lambda$. MCUSUM procedures weight past observations in the same way for each quality characteristic. However, this MEWMA chart depends only on the non-centrality parameter. The practitioner may use unequal weighting constants, but then the ARL depends on the direction of the shift, not just the value of the non-centrality parameter (Lowry and Montgomery, 1995).

2.3.5 Capability in Multivariate Domain

In the usual statistical-thinking paradigm, process capability improvement occurs by reducing common cause variation through some fundamental improvement in the process. These concepts translate easily from univariate to multivariate settings (Boyles, 1996). Assuming a multivariate normality of the process data, the elliptical contours in the two dimensions and ellipsoids in the higher dimensions, for probability levels, define the regions (areas or volumes), and these regions are analogs to the interval of the univariate case.

In a general multivariate case, define **X** as a (p x n) sample matrix, where (p) is the number of product quality characteristics measured on a part, and (n) is the number of parts measured. Each column in the matrix represents the (p) measurements recorded from a sampled part. These (n) observations are assumed to be independent and represent a sample drawn from a multivariate distribution with correlation among the (p) variates. The (p) vector $\overline{\mathbf{X}}$ contains the sample means of the observations, and the (p x p) matrix **S** contains the unbiased sample variances and covariances of the observations estimated in the usual way for the underlying process mean $\boldsymbol{\mu_0}$, and variance covariance matrix $\boldsymbol{\Sigma}$. Engineering specifications are assumed to exist for each of the (p) dimensions. The vector $\boldsymbol{\mu_0}$ contains the target values for the (p) product characteristics. In the multivariate domain, the objective is to use the $\mathbf{X}$, $\bar{\mathbf{X}}$, $\mathbf{S}$, or the underlying distribution in comparison to the engineering specifications to arrive at some acceptable definition of capability in the multivariate domain (Wang et al., 2000).

A multivariate capability vector was proposed by Shahriari et al. (1995), based on the original work of Hubele et al. (1991). The multivariate capability vector consists of three components. Two components use the assumption that the process data is from a multivariate normal distribution with elliptical contours defining the probability regions. The third component is based on the geometric understanding of the process relative to the engineering specifications. The first component of the vector is a ratio of areas or volumes equivalent to the ratio of lengths of the univariate C_p index. The numerator is the area (two-dimensional case) or the volume (three or more dimensions) defined by the engineering tolerance region. The denominator is the area or volume of a "modified process region," defined as the smallest region similar in shape to the engineering tolerance region, circumscribed about a specified probability contour. The number of dimensions of the process data is captured by taking the p^{th} root of the ratio. The first component $C_{pM,}$ is defined as

$$C_{pM} = \left[\frac{vol.of\ engineering\ tolerance\ region}{vol.of\ \mathrm{mod}ified\ process\ region} \right]^{1/p} \tag{2.24}$$

The engineering specifications define a rectangular tolerance region, and bivariate normal process measurements define an elliptical probability contour denoted as a "process region." This method forms a "modified process region" by drawing the smallest rectangle around the ellipse. The edges of the rectangle are defined as the lower process limits and the upper process limits (LPL_i and UPL_i, respectively, where i= 1, 2…, p) and are determined by solving the system of equations of first derivatives, with respect to each x_i, of the quadratic form

$$(\mathbf{X} - \mu_0)' \, \Sigma \, (\mathbf{X} - \mu_0) = \chi^2_{(p,\alpha)} \tag{2.25}$$

The distribution of the statistic follows a multivariate normal distribution. When the process data is a multivariate normal, the distribution of the statistic will follow a χ^2 distribution. The two solutions to this equation for each dimension provide the upper and lower limits

$$\mathrm{UPL_i} = \mu_i + \sqrt{\frac{x^2_{(p,\alpha)} \det(\mathbf{\Sigma}_i^{-1})}{\det(\mathbf{\Sigma}^{-1})}} \tag{2.26}$$

$$\mathrm{LPL_i} = \mu_i - \sqrt{\frac{x^2_{(p,\alpha)} \det(\mathbf{\Sigma}_i^{-1})}{\det(\mathbf{\Sigma}^{-1})}} \tag{2.27}$$

where i=1, 2, …, p, and $\chi^2_{(p,\alpha)}$ is the upper $100(\alpha)$ percentile of a χ^2 distribution with (p) degrees of freedom associated with the probability contour and det ($\mathbf{\Sigma_i^{-1}}$) is the determinant of $\mathbf{\Sigma_i^{-1}}$, a matrix obtained from $\mathbf{\Sigma^{-1}}$ by deleting the i^{th} row and column. Estimates from larger samples may be used instead of μ and Σ (Johnson and Wichern, 1992).

The idea is to construct a modified process region with the same general geometric shape as the engineering tolerance region. Thus,

$$C_{pM} = \left[\frac{\prod_{i=1}^{p} (\mathrm{USL_i} - \mathrm{LSL_i})}{\prod_{i=1}^{p} (\mathrm{UPL_i} - \mathrm{LPL_i})} \right]^{\frac{1}{p}} \tag{2.28}$$

To interpret the results, values higher than 1 indicate that the circumscribed modified process region is smaller than the engineering specified region "goodness." The limits UPL and

LPL are derived from the projection of probability ellipse onto the respective axes (Nickerson, 1994).

Also, when the engineering specifications are intervals and the product of the length of the intervals forms the volume, then C_{pM} could be calculated by multiplying univariate capacity indices

$$C_{pM} = \left[\prod_{i=1}^{p} \frac{(\text{allowable process spread})_i}{(\text{actual process spread})_i} \right]^{\frac{1}{p}} \tag{2.29}$$

The second component of the vector is based on the assumption that the center of the engineering specifications is the true underlying mean of the process. A Hotelling T^2 statistic is computed, and the second component is defined to be the significance level of the observed value. That is,

$$T^2 = n \left(\overline{\mathbf{X}} - \boldsymbol{\mu}_0\right)' \mathbf{S}^{-1} \left(\overline{\mathbf{X}} - \boldsymbol{\mu}_0\right) \tag{2.30}$$

with the second component defined as

$$PV = P\left(T^2 > \frac{p(n-1)}{n-p} F_{(p,\, n-p)}\right) \tag{2.31}$$

and PV is a probability value which never exceeds 1. A PV value closer to zero indicates that the center of the process is "far" from the engineering target value.

The third component summarizes a comparison of the location of the modified process region and the tolerance region (L1). It indicates whether any part of the modified process region falls outside the engineering specifications. It has a binary value of (0, 1). L1 has the value of 1 if the entire modified process region is contained within the tolerance region, otherwise L1 = 0.

The three components [$C_{pM,}$ PV, L1] represent a comparison of the volumes of regions, locations of centers, and location of regions. This multivariate index requires the assumption of multivariate normality (Wang et al., 2000).

2.3.6 Statistical Performance

When comparing multivariate control schemes, a performance aspect should be discussed. This aspect concerns the question of how quickly the scheme generates a signal when an actual change in the process has occurred. The quicker a scheme responds to a real change, the more advantageous. A control scheme that can quickly detect real changes while not being overly sensitive to false alarm is desired. In particular, it is possible to identify two different situations. With Type I error probability (α), or false positive, the control chart indicates an out-of-control signal but the process is in-control. With Type II error probability (β), or false negative, the control chart fails to indicate an out-of-control signal, while the process is out-of-control. The number of samples required to detect a real change in the process is measured by the run length. The expected value is then the average run length (ARL) (Montgomery, 2001). Therefore, a good performance of a control scheme is obtained if the ARL is low in out-of-control situations. As was pointed out in equation (2.1), $ARL_0 = \frac{1}{\alpha}$, where

$$\alpha = P\left[T^2 > UCL \mid \boldsymbol{\mu} = \boldsymbol{\mu_0}\right] \tag{2.32}$$

and from equation (2.2), $ARL_1 = \frac{1}{1-\beta}$, where

$$\beta = P\left[T^2 < UCL \mid \boldsymbol{\mu} \neq \boldsymbol{\mu_0}\right] \tag{2.33}$$

Figures 2.2 and 2.3 illustrate the probability of Type I and Type II errors respectively.

The probability of Type II error depends on the distribution of the statistic T^2 when $\boldsymbol{\mu} \neq \boldsymbol{\mu_0}$. Anderson (1958) shows that if $\boldsymbol{\mu} \neq \boldsymbol{\mu_0}$, then T^2 follows the generalized T^2 distribution with (p) and (n – p) degrees of freedom, denoted $T^2_{p,\ n-p}$. Moreover, it may be shown that the random variable

$$F' = \frac{n-p}{p(n-1)} T^{2'} \tag{2.34}$$

which has the non-central F-distribution, with (p) and (n – p) degrees of freedom and the non-centrality parameter $\tau^2 = N\left(\boldsymbol{\mu} - \boldsymbol{\mu_0}\right)'\left(\boldsymbol{\Sigma}^{-1}\right)\left(\boldsymbol{\mu} - \boldsymbol{\mu}_0\right)$. The probability density function of T^2 is

$$p(t) = \frac{e^{-\frac{1}{2}\tau^2}}{(N-1)\,\Gamma\left[\frac{1}{2}(N-p)\right]} \sum_{i=0}^{\infty} \frac{\left(\frac{\tau^2}{2}\right)^i \left[\frac{(t^2)}{(N-1)}\right]^{\frac{1}{2}p+i-1} \Gamma\left(\frac{1}{2}N+i\right)}{i!\,\Gamma\left(\frac{1}{2}p+i\right)\left[1+\frac{(t^2)}{(N-1)}\right]^{\frac{1}{2}N+i}} \tag{2.35}$$

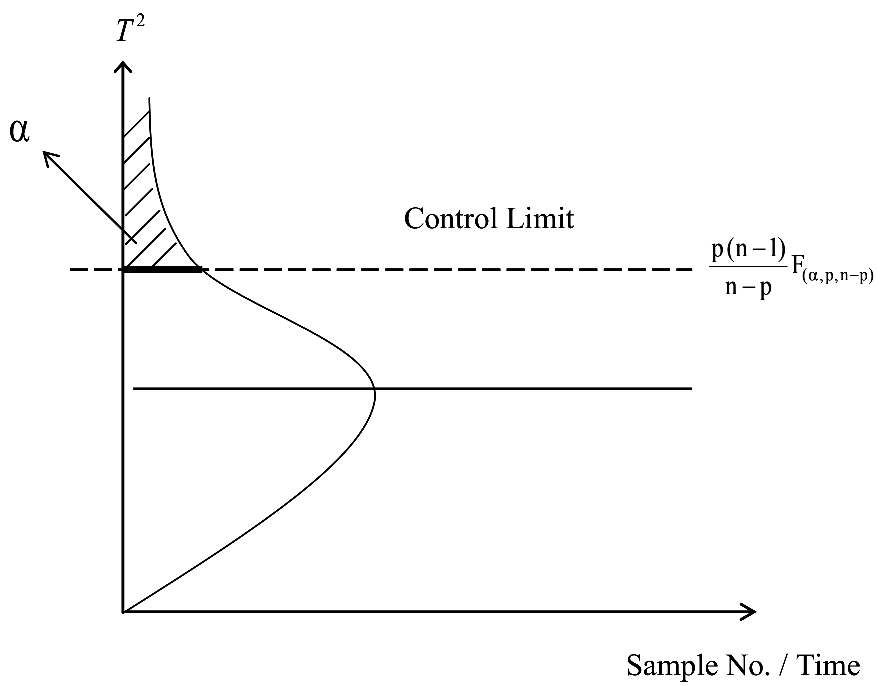


Figure 2.2 Probability of Type I error (α)

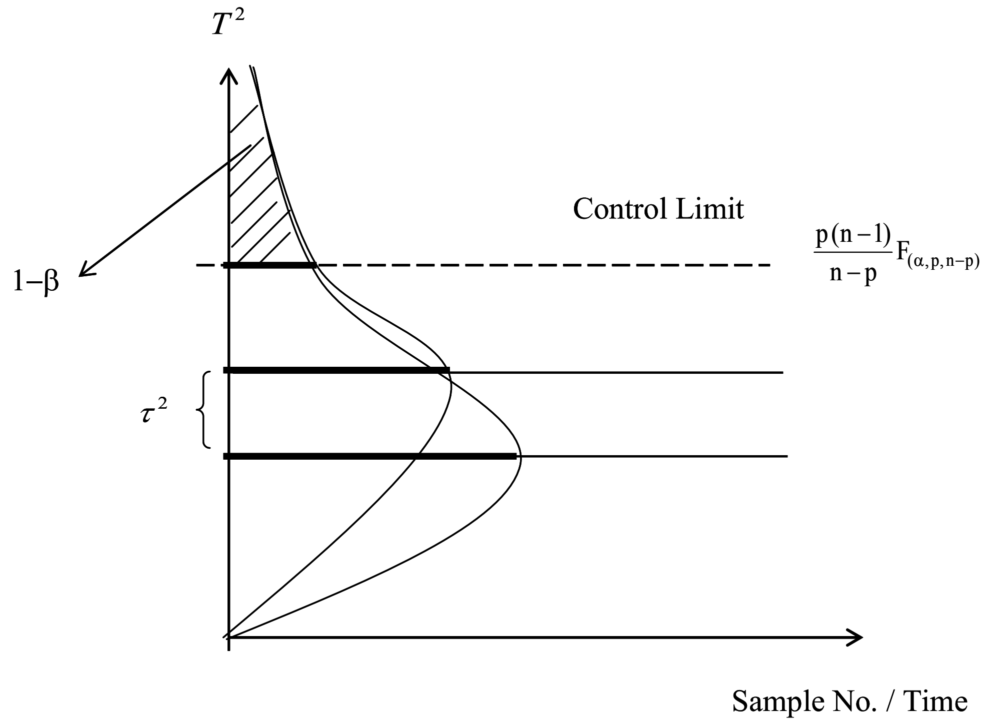


Figure 2.3 Probability of Type II error (β)

Kay (1998) provided the probability density function of the non-central F-distribution as

$$p(f)=\sum_{i=0}^{\infty}\left[\frac{e^{-\frac{\tau^2}{2}}\left(\frac{\tau^2}{2}\right)^{i}}{\frac{\Gamma\left(\frac{\nu_2}{2}\right).\Gamma\left(\frac{p}{2}+i\right)}{\Gamma\left[\frac{\nu_2}{2}+\left(\frac{p}{2}+i\right)\right]}.i!}\left(\frac{p}{\nu_2}\right)^{\frac{p}{2}+i}\left(\frac{\nu_2}{\nu_2+p.f}\right)^{\frac{(p+\nu_2)}{2}+i}.f^{\frac{p}{2}-1+i}\right] \tag{2.36}$$

when $f \geq 0$ and zero otherwise. In equation (2.36), (p) is the number of variables or quality characteristics being measured, and (f) is the inverse cumulative probability of the F-distribution

$F_{(1-\alpha), p, \nu 2}$, with (p) and (ν_2) degrees of freedom, where $\nu_2 = (n - p + 1)/2$. The degrees of freedom are positive. When $\tau^2 = 0$, the non-central F-distribution becomes the F-distribution.

2.3.7 Advantages of Multivariate Statistical Process Control

Multivariate SPC has several advantages over univariate SPC. As noted by Hotelling, (1947); Alt, (1984); and Lowry and Montgomery, (1995), multivariate SPC requires no additional data accumulated for univariate control charts. Hotelling (1947) indicated that multivariate SPC has the ability to combine measures in several dimensions into a single measure of performance. In addition, multivariate SPC offers an easier graphical tool to examine; the practitioner can only use one chart instead of multiple univariate charts to evaluate the product or system quality as a whole rather than the sum of many individual parts (Hotelling, 1947, and Montgomery, 2001). Moreover, Montgomery (2001) demonstrated that multivariate control charts will produce an acceptable Type I error or in-control run length while maintaining the original data means, variances, and correlations. Multivariate statistics consider the relationship between the variables since the variance-covariance matrix is part of the computations (Hotelling, 1947). As such, multivariate control charts can detect changes in the relationships among the variables being monitored, which would not be noticeable from separate univariate charts (Lowry and Montgomery, 1995).

Another advantage is that multivariate SPC provides the appropriate control region for the application. If the assumption of independence does not hold, then the assumed performance of traditional Shewhart approaches can be misleading. The multivariate approach, however, can guarantee error protection for a variety of different types of shifts in the process. Also, in the multivariate domain, an advantage of the multivariate statistic is that it moves away from the application of run rules (Sullivan and Woodall, 1996).

2.3.8 Disadvantages of Multivariate Statistical Process Control

While the literature provides strong evidence for the benefits of applying multivariate SPC, a number of limitations were sited. As pointed out by Mason et al. (1997), Ryan (2000), and Montgomery (2001), multivariate control charting procedures are computationally intensive. Furthermore, multivariate control charts work well when the number of process variables is not too large, i.e., ($p > 10$). As the number of variables grows, multivariate control charts lose efficiency with regard to shift detection. Moreover, multivariate control chart procedures do not directly provide the information an operator needs when the control chart signals an out-of-control condition. It doesn't provide information on which variable or set of variables is out-of-control (Hawkins, 1991). Jackson and Mudholkar (1979) proposed the transformation of correlated quality characteristic variables into a set of independent variables. Known as principle component analysis (PCA), this approach reduces the dimensionality of the problem. In addition, when applying Shewhart control charts, the use of averages of subgroups substantially improves control chart performance. However, this is not always the case when using MCUSUM (Montgomery, 2001).

2.4 Economic Models

Control charts have been used traditionally to establish and maintain statistical control of a process. However, the design of a control chart has economic consequences, which are all affected by the choice of the control chart parameters such as the selection of the sample size (n), the width coefficient of the control limits (k), and the time interval between samples (h). Three categories of costs are customarily considered in the economic design of control charts. These categories are the cost of sampling and testing, the cost associated with investigating out-of-control signals and correcting the assignable causes, and the costs of allowing nonconforming units to reach the customer.

2.4.1 Duncan's Model

Duncan (1956) was the first to propose an economic model of a Shewhart control chart. Duncan defined the process of net income as the difference between total income and total cost. Total income has two elements: net income per hour of operation in the in-control state V_0, and net income per hour of operation in the out-of-control state V_1. Moreover, the total cost consists of three parts: the cost of looking for an assignable cause when there is none, (a_3), the cost of looking for an assignable cause when there is one (W), and the cost of maintaining the chart ($(a_1+a_2*n)/h$). Parameters (a_1) and (a_2) represent the fixed and variable costs of measurements, respectively. Duncan considered the production cycle shown in Figure 2.4 to develop his economic model. He developed expressions for the proportion of time when the process is in-control (γ_0) and when the process is out-of-control (γ_1), and determined the average number of times the process goes out-of-control (ε) and the expected number of false alarms ($\gamma_0*\alpha/h$). He assumed that assignable causes occurred according to a Poisson process, with an intensity of λ occurrence per hour and a cause a shift of $\pm\,\delta$ in the process average. Moreover, he assumed that production continues while investigating and correcting the process. The average net income per hour is

$$E(A) \cong \gamma_0 V_0 + \gamma_1 V_1 - (\gamma_0\, \alpha a_3)/h - \varepsilon W - (a_1 + a_2 * n)/h \qquad (2.37)$$

This can be written as

$$E(A) \cong V_0 - E(L) \qquad (2.38)$$

The expression E(L) represents the expected loss per hour incurred by the process. E(L) is a function of the control chart parameters (n), (k), and (h). Maximizing the expected net income per hour V_0 is equivalent to minimizing E(L).

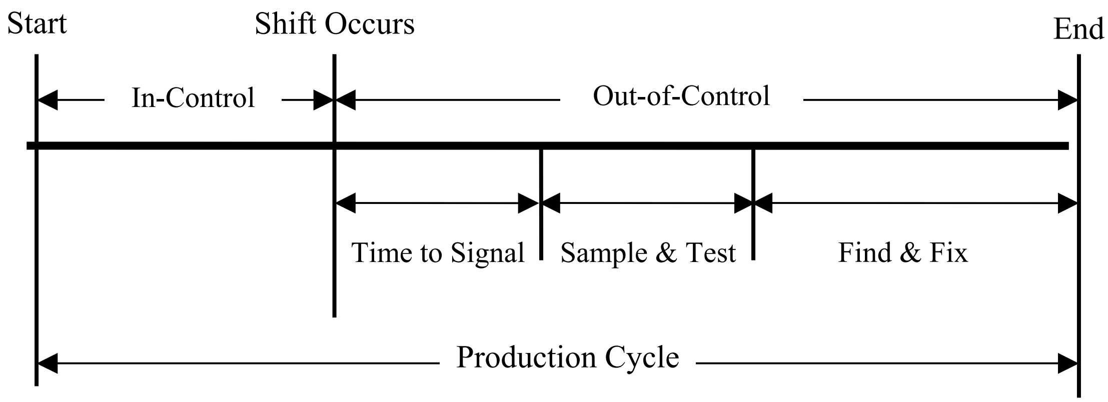


Figure 2.4 Production cycle in Duncan's model

Duncan incorporated formal optimization methodology into determining the control chart parameters. Several numerical approximations were used in the structure and optimization of this model. An optimization procedure was developed based on using a numerical approximation to the system of first partial derivatives of E(L) with respect to (n), (k), and (h). Duncan compared the optimum design with the heuristic design of n = 5, k = 3, and h = 1 for a set of 25 examples at different levels of input parameters. He concluded that using the heuristic design in some cases might result in vary large penalties.

Duncan's research was the stimulus for much of the research that followed in this area. Several interventions were conducted to investigate further optimization methods, model sensitivity, and its application to other Shewhart control charts.

Alexander et al. (1995) embellished Duncan's cause model with the Taguchi loss function that defines losses owing to the variability caused by both chance and assignable causes. Through sensitivity analysis, they indicated that the design parameters for the $\bar{x}$ chart are fairly robust when the cost of finding assignable cause and the frequency of occurrence of an assignable cause are not very high. They found that (n) increases and (h) decreases to steady-state values as the frequency of the process shift decreases. Moreover, they stated that the rate of convergence to the steady-state depends on the cost of searching for an assignable cause. Therefore, the higher the cost, the slower the convergence rate. Also, they indicated that (n) and (h) must be adjusted based on the size of the process shift that is investigated. Therefore, small process shifts require large values of (n) and (h), while for large process shifts, small (n) and (h) values are recommended.

2.4.2 Lorenzen and Vance's Model

In 1986, Lorenzen and Vance developed an economic model for the design of control charts. They used a different approach in developing their model. Instead of using the Type I error probability (α) and Type II error probability (β) risks, they based their approach on developing their model on the in-control and out-of-control average run lengths. They considered the production cycle shown in Figure 2.5. Moreover, they developed expressions for estimating the in-control and out-of-control expected times. They considered the following costs in their model: (1) cost incurred during the in-control period due to process sampling (a_1+a_2.n), where nonconforming units produced C_0, and False alarms Y; (2) cost incurred during the out-of-control period, including the cost of nonconforming units that produced C_0 and (C_1>C_0); and (3) cost of locating and repairing the assignable cause (W) and that of process sampling. The total cost per production cycle E(C) was

$$E(C) = \frac{C_o}{\lambda} + C_1 E(t_1) + YE(t_2) + W + \frac{a_1 + a_2.n}{h} E(T) \tag{2.39}$$

where t_1 is the time of operation in the out-of-control state, t_2 is the time spent searching for a false alarm, and (T) represents values of the total cycle time. The expected cost per hour is obtained as the ratio of the expected cost per cycle to the expected cycle time in hours.

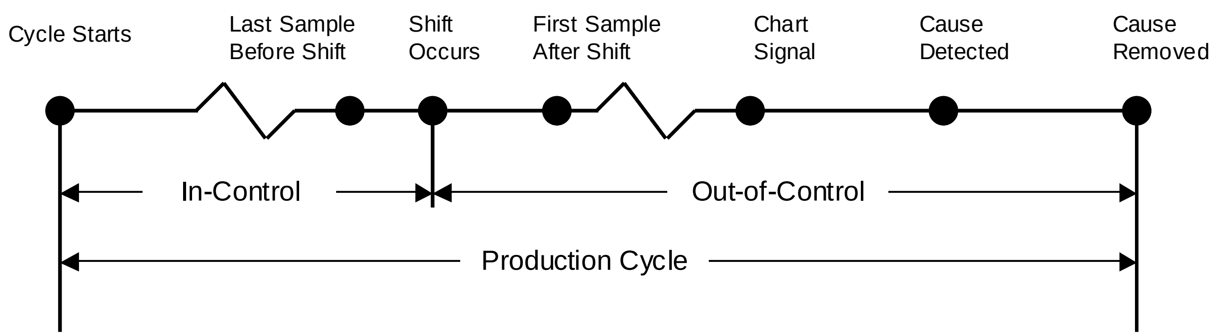


Figure 2.5 Production cycle in Lorenzen and Vance's model

This model has two important assumptions. The first is that the time in-control is a negative exponential random variable with a parameter (1/λ). The second assumption is that only one assignable cause of known magnitude can affect the process. The advantage of this model is that it allows for the use of other control charts simply by changing the probability distribution function that generates ARLs. A combination of three minimization techniques was combined into a general algorithm for minimizing the cost function. A sensitivity analysis of the optimal plan to changes in the cost coefficient C_1 and the parameter λ was illustrated in an example involving the economic design of an (np) chart. Moreover, Lorenzen and Vance (1987) compared the performance of the $\bar{x}$ chart, the cumulative sum chart, and the exponentially weighted moving average chart, with a weight of 0.25 on current observations, based on a cost criterion. They used their cost model to determine the expected loss per hour for each case.

They assumed that manufacturing activities are allowed during investigating and repairing the assignable cause. Their findings were that the CUSUM chart performed best, followed by the $\bar{x}$ chart, and then the EWMA chart.

2.4.3 Knappenberger and Grandage's Model

Knappenberger and Grandage (1969) developed a model for the economic design of the $\bar{x}$ control charts. Their model was different than Duncan's model in that there is no constraint on the number of assignable causes that can occur. Specifically, the process can shift from one out-of-control state to another, as long as the shift results in further quality deterioration. It is assumed that the process is stopped while out-of-control signals are investigated and that the costs of investigating both real and false alarms are the same. The expected total cost E(C) per unit of product consists of three elements. The first element $E(C_1)$, the expected cost per unit associated with carrying out the charting procedure is $((a_1+a_2.n)/k)$, where k is the number of units produced between samples. The second element $E(C_2)$, the expected cost per unit associated with investigating and correcting the process, when the chart indicates the process is out-of-control, assuming that the process is stopped, is $((a_3/k)\ \mathbf{q}\ \boldsymbol{\alpha}^t)$, where $\mathbf{q}$ is a row vector representing values of probabilities q_i (the probability of rejecting H_0 when $\mu = \mu_i$, (i= 0, 1, 2, 3, 4, 5, 6)) and $\boldsymbol{\alpha}^t$ is the transpose of the row vector representing the steady-state probability that the process is in state (i) (that is, $\mu = \mu_i$) at the time of the test. And the third element $E(C_3)$, the expected unit of producing a defective product, is $(a_4\boldsymbol{\phi}\ \boldsymbol{\gamma}^t)$, where (a_4) is the cost of a defective unit, $\boldsymbol{\phi}$ is a row vector of the conditional probabilities of producing a defective unit given the process mean, and $\boldsymbol{\gamma}^t$ is the transpose of the row vector representing the true state of the process. Therefore, the sum of the three cost elements represents the expected total cost per unit as

$$E(C) = E(C_1) + E(C_2) + E(C_3) \tag{2.40}$$

This can be written as

$$E\,(C) = (a_1+a_2.n)/k + (a_3/k)\,\mathbf{q}\boldsymbol{\alpha}^t + a_4\boldsymbol{\phi}\boldsymbol{\gamma}^t \tag{2.41}$$

Knappenberger and Grandage used two-stage direct search method to minimize the cost function. They conducted limited sensitivity analysis and presented the solutions to 81 numerical examples of a variety of cost coefficients and process parameters. They also minimized the expected cost per unit produced rather than the expected cost per unit time, as in Duncan's model.

2.4.3.1 Montgomery and Klatt's Approach to Multivariate T^2 Chart

Montgomery and Klatt (1972) developed an optimal economic design of the T^2 control chart. Based on the structure of the Knappenberger and Grandage model, they employed a single assignable cause version. However, their results were restricted to the case of two quality characteristics. It is assumed that, when $\boldsymbol{\mu} = \boldsymbol{\mu_0}$, the process is in-control and there is only one out-of-control state, which is when the process mean vector is $\boldsymbol{\mu_1} = \boldsymbol{\mu_0} + \boldsymbol{\delta}$, where the (p x 1) vector $\boldsymbol{\delta}$ is known. It is also assumed that the time the process remains in the in-control state before going out-of-control is an exponential random variable with mean λ^{-1} hours. Moreover, when the process goes out-of-control, it stays out-of-control until detected. However, the assignable cause is detected as soon as the T^2 chart plots out-of-control, that is, when $T^2 > T^2_{\alpha,p,n-p}$. The expected total cost per unit of product E(C) consists of three terms. The first term $E(C_1)$, the expected cost per unit of sampling and for carrying out the test procedure, is $((a_1+a_2.n)/k)$, where (a_1) is the fixed cost per sample, (a_2) is the per-unit cost of sampling, and (k) is the number of units produced between successive samples. The second term $E(C_2)$ the expected cost per unit of investigating and for correcting the process, is $((a_3/k)\ \beta\ \rho^t)$, where (a_3) is the expected cost of investigating and correcting an out-of-control process. However, the cost of investigating real and false alarms is assumed to be the same. Also, β is a column vector of the probability that the process is in state μ_i (i = 0, 1) at the time the test is performed. The transpose of the row vector of conditional probability is that the test procedure indicates the process is out-of-control, given that the process is in state μ_i is ρ^t. The third element $E(C_3)$, the expected cost per unit of producing a defective product, is $(a_4\,\gamma\ \phi^t)$, where (a_4) is the cost of a defective unit, ϕ is a column vector of the conditional probabilities of producing a defective unit given that $\mu = \mu_i$ process mean, and γ is the column vector of probability that the process is in state μ_i at any point in time. Therefore, the sum of the three cost elements represents the expected total cost per unit $E(C) = E(C_1) + E(C_2) + E(C_3)$, or

$$E(C) = (a_1+a_2.n)/k + (a_3/k)\beta\rho^t + a_4.\gamma\phi^t \qquad (2.42)$$

Montgomery and Klatt also investigated the sensitivity of the model to estimates of the cost coefficients and of the population covariance matrix. They concluded that the optimum control chart parameters are relatively insensitive to errors in estimating these parameters. They also concluded that both the magnitude of the shift and the sign (+/-) of the correlation coefficient relating the two quality characteristics affect the optimum economic design. Moreover, if the shift in both quality characteristics is in the same direction, negative correlation between the quality characteristics leads to a smaller sample than would be required if the correlation were positive. This occurs because negative correlation always leads to a more powerful test if the process shift is in the same direction for both quality characteristics.

CHAPTER 3

DISCUSSION

In multivariate SPC, the focus usually is to simultaneously monitor several quality characteristics that may be correlated. Hotelling first publicized the multivariate approach to quality control in 1947 in the testing of bombsights. Hotelling introduced the T^2 control chart as a technique for monitoring the overall quality of a process. An advantage of this approach is that the T^2 statistic is a single measure of excellence. This field remained relatively undeveloped until the late 1950's with the increasing availability of computers.

Advancements in technology raised the need for simultaneous monitoring of several quality characteristics that could be correlated. With increased competition in the marketplace, many companies have utilized Six Sigma methodology as a means to reduce the cost of poor quality in order to maintain their market share. The need to apply multivariate SPC became more desirable, especially with the complexity of processes and the dependency of quality characteristics on each other. Furthermore, customer expectations require the evaluation of the product or system quality as a whole rather than the sum of many individual parts. With the availability of product alternatives, customers are becoming more demanding for higher quality. In order for companies to remain competitive, they must achieve high levels of product quality, which is becoming challenging to achieve since quality characteristics are interrelated to each other as a result of technological advancements. Additionally, management demand for implementing Six Sigma programs to achieve better quality makes it more challenging to do so. Recently this effort has made multivariate SPC more popular.

Practitioners in industry avoid using multivariate SPC because of its complex computational intensiveness. However, increased availability of high-speed computers and statistical software programs, formerly available only to very few, has made the statistical computations of multivariate SPC easier.

Many new techniques have made multivariate SPC more useful (e.g., Ghare and Torgerson, (1968); Montgomery and Wadsworth, (1972); Alt, (1973, 1982); Alt et al., (1976); Montgomery and Klatt, (1972); Jackson, (1956, 1959); Jackson and Bradley, (1966); Jackson, (1985); Tracy et al., (1992); Mason et al., (1997); and Woodall et al., (2004)). Among industries, the use of multivariate control charts to monitor manufacturing processes is increasingly popular. This is the result of many recent advances that have occurred in multivariate SPC, such as in multivariate cumulative sum control charts (e.g., Crosier, (1988); Healy, (1987); Pignatiello and Runger, (1990); Woodall and Ncube, (1985)); and multivariate exponentially weighted moving average control charts (e.g, Lowry et al., 1992). The improved effectiveness of these techniques has made it possible to identify the cause of an out-of-control signal. While it is common in industry to monitor individual process characteristics with separate univariate charts, more attention is being given to combine characteristics into a single chart.

Multivariate control charts are generally utilized in cases where the quality measurements follow the multivariate normal distribution, and the process performance is monitored over time. In addition, they can be used to indicate when quality characteristics change. In univariate SPC, a signal is produced when a sample point does not confirm to the structure that is established by the historical data. Through the use of appropriate control charts, it is possible to determine if this signal is due to a shift in the process mean and/or a shift in the process variation. Since there is only a single variable to consider, signal interpretation is relatively straightforward. However, in multivariate SPC, a signal can be caused by a variety of situations. These include out-of-control behavior of a single variable, a relationship between two or more variables, or a combination of the two situations with some variables being out-of-control and others having a counter-relationship due to the correlation between variables.

Multivariate SPC may be useful whenever there is more than one variable quantifying the quality characteristic of a product and/or process. Multivariate SPC is particularly valuable when these variables are correlated. In some cases, the true source of variation may not be recognized or may not be measurable. Multivariate SPC is more resilient to correlation whether it exists or not. Moreover, practitioners' lack of knowledge of the correlation between the quality characteristics in their processes does not mean that the correlation does not exist. It is important to recognize that almost all processes are multivariate, but multivariate SPC often is not utilized because the process characteristics are assumed to be independent. As a result practitioners tend to use traditional SPC as an alternative to multivariate SPC. However, the distortion in the process-monitoring procedure increases as the number of quality characteristic increases since some of these variables may be correlated. Consequently, the more variables there are, the more likely one of the charts will contain an out-of-control condition, even when the process has not shifted. Thus, the false alarm rate or probability of Type 1 error is increased if each variable is controlled separately.

Multivariate control charting provides a means to identifying shifts in any (p) quality characteristics by charting only one parameter such as Hotelling T^2. This simplifies the charting procedures. It is easier to monitor one chart rather than monitoring several charts for the same quality characteristics. Moreover, industries are determined to achieve higher levels of product and process quality with the least amount of resources committed. Multivariate SPC reduces the cost of process monitoring since it works with the same data collected for traditional SPC.

The increasing demand for less variability raises the need to monitor correlated variables simultaneously. A number of methodologies were developed to provide a clear procedure for interpreting out-of-control signals in the case of multivariate control charts. However, limited research has been conducted to address their implementation boundaries and utilization in practice.

A literature review of multivariate control charts was presented in Chapter 2. The literature did not provide specific guidelines of when to use multivariate statistical analysis. Industry applications require practitioners to be able to decide when it is necessary to use multivariate control charts. An important implication of this is the need for practitioners to know the level of correlation of process characteristics and the process model parameters in order to decide when to use multivariate control charts to detect the special causes and interpret out-of-control conditions.

3.1 Research Gap

Statistical process control has helped industry become more aware of the benefits of implanting statistical procedures. Companies' management requires their practitioners to use the Six Sigma methodology as a method to reducing cost. Recent development in technology has presented systems of interconnected processes. The key to the success is to understand and reduce process variation. With the recent technological advancements, almost all processes are related or dependent on common variables. Incidentally, quality practitioners often do not investigate the correlation levels between the variables they are monitoring. Their underestimation of the relationship between the variables will result in traditional SPC becoming less effective. Most quality practitioners use Shewhart charts to monitor process performance over time. Montgomery (2001) pointed out that if the (p) quality characteristics are not independent, which is usually the case if they are related to the same product, then Type I and Type II error probabilities for the traditional Shewhart charts become distorted. The literature did not show any method for measuring the distortion in a Shewhart joint control procedure. Additionally, there is a clear gap in the literature in indicating the levels of correlation that mandate the use of multivariate charts. The statistical effect of mis-specifying the process model while applying traditional Shewhart charts has not been quantified. The need to use multivariate SPC has received great attention in recent years as companies strive to be more competitive and achieve higher quality products with the lowest cost possible.

Companies are becoming more customer-focused and need to remain competitive; this makes it economically necessary to utilize the best SPC techniques. The economic consequences of using Shewhart charts as a preference procedure to multivariate charts were not presented in the literature. Practitioners need to be aware of the most economical SPC method in order to monitor the performance of the quality characteristics of any process.

3.2 Research Objectives

This research addressed the gap identified in section 3.1, based on the literature review. This was accomplished by quantifying the effect of changes in the level of correlation between variables coupled with changes in the process model and chart design parameters. Another objective was to assess the economic feasibility of utilizing Hotelling's T^2 multivariate control chart as an alternative to traditional Shewhart $\bar{x}$ charts. This investigation was undertaken for the case of two quality characteristics.

Special considerations were given to the Hotelling T^2 chart. As such, the most popular Shewhart $\bar{x}$ chart was used to provide a baseline for the performance measures. Measures of performance were selected to evaluate the statistical performance and economic feasibility for multivariate control charts. This research will help practitioners select the appropriate charting technique with a clear understanding of the statistical and economic consequences.

3.3 Research Procedures

The first stage of this research focused on the statistical performance of multivariate T^2 control charts by establishing the level of correlation and the process model parameters that mandate when it is best to use multivariate control charts as an alternative to traditional Shewhart charts.

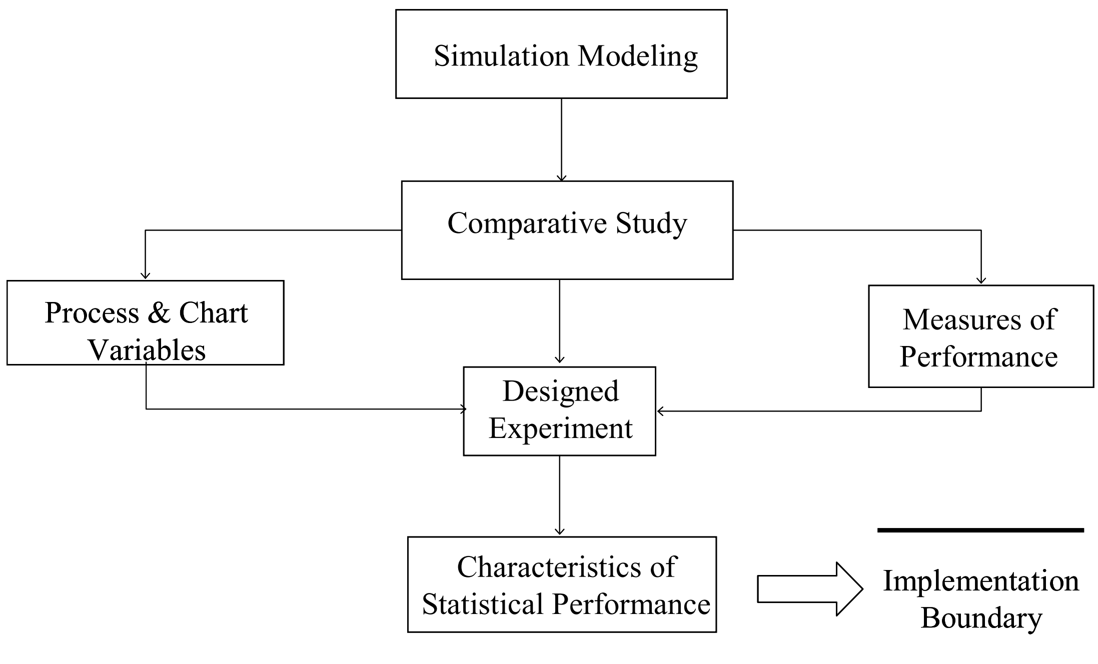


Figure 3.1 Research procedure (Stage I)

Figures 3.1 and 3.2 present the procedures used to achieve the research objectives. Simulated data was analyzed using univariate and multivariate SPC techniques. The effect of correlation was studied by generating two random variables from a bivariate normal distribution;

the variables have different levels of correlation ranging from 0 to 0.9. The effect of changes in the process model parameters was also analyzed. The average run length was used as a measure of performance to evaluate the performance of the control charts and obtain the average probability rate for Type I (α) and Type II (β) errors. Changes in the level of correlation between the variables coupled with changes in the process model and chart design parameters were analyzed using a statistical designed experiment.

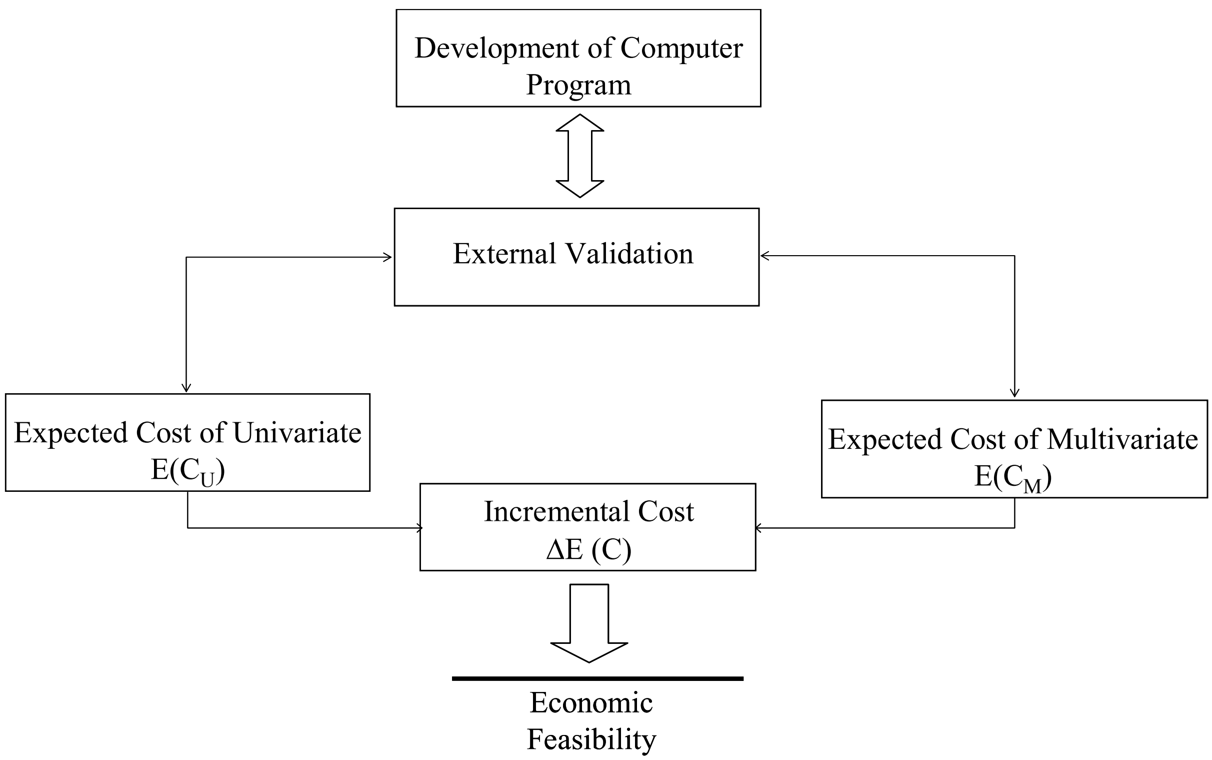


Figure 3.2 Research procedure (Stage II)

The second stage of this research was devoted to the economic feasibility of utilizing Hotelling's T^2 multivariate control chart as an alternative to the traditional Shewhart $\bar{x}$ chart. By using economic design models from Knappenberger and Grandage (1969), and Montgomery and Klatt (1972) for the traditional Shewhart $\bar{x}$ chart and the multivariate T^2 chart respectively, an incremental cost model was constructed to examine the cost and worth of switching from univariate to multivariate SPC techniques under specified levels of process and chart variables, thus determining the economic consequences of using traditional Shewhart charts.

CHAPTER 4

INITIAL INVESTIGATIONS

To determine the characteristics of statistical performance of the Shewhart $\bar{x}$ and multivariate T^2 charts, initial investigations of the charts performance were conducted to account for the Type I probability (α) and Type II probability (β) errors. Simulation modeling was developed and verified using operating characteristics (OC) curves for the Shewhart $\bar{x}$ chart. The performance of the multivariate T^2 chart was validated using the same conditions of the Shewhart $\bar{x}$ chart under (0) correlation conditions. After this process was done, the simulation was carried out to calculate Type I probability (α) and Type II probability (β) errors for the two SPC techniques.

Random variables were generated at specified levels of correlation ranging from (– 0.8) to (+ 0.8). Simulated data were analyzed using univariate and multivariate SPC techniques to study the average rate of false alarms based on simulated ARL_0. After causing a shift in the mean, ranging from 1σ to 3 σ, simulated data were analyzed to study the probability of Type II error following the shift based on simulated ARL_1.

4.1 Simulation Development and Verification

A simulation was conducted using the software @ Risk™ version 4, an add-in for Microsoft Excel software, to generate simulated data of two random variables from bivariate normal distribution using a random generating function. The levels of correlation between the two variables were then varied over a range from (- 0.8) to (+ 0.8). Equation (2.1) provides the expected value of ARL_0 for the $\bar{x}$ chart with 3 σ limits as

$$ARL_0 = E(x) = \frac{1}{\alpha} = \frac{1}{0.0027} = 370.4$$

A large number of simulated runs would be required to provide adequate indication of this measure; therefore number of runs of (N = 10,000) were conducted. A subgroup size of (n = 4) was selected for this simulation since it is the most common subgroup size in practice. The simulated data was then plotted on an $\bar{x}$ chart and T^2 chart. All points that fell outside the control limits were counted to obtain the ARL_0. This was performed using the software Statgraphics™ Centurion XV. In addition, each simulation was repeated m = 5 times to achieve a 95 percent confidence level with a targeted accuracy of (± 4) in estimating the ARL. The objective of this simulation was to obtain the average error probability (α) base on simulated ARL_0.

Figure 4.1 shows the simulation procedure, which was verified using the case where no correlation existed ($\rho = 0$) to meet the ARL_0 for the traditional Shewhart $\bar{x}$ charts based on the OC curve.

The next step was to obtain the ARL_1, which was done by causing a shift in the mean of the second variable from 1σ to $3\ \sigma$. The shift was caused for (N = 100) to ensure detecting the shift of the mean of the second variable, shown in equation (4.1) as

$$\mu_i = \mu_2 + k_i\ \sigma \tag{4.1}$$

The levels of correlation between the two variables were then varied from (-0.8) to (+0.8). The simulation was then performed in the same manner as mentioned previously and the data is plotted on an $\bar{x}$ and T^2 chart. The ARL_1 was obtained by recording the number of points falling within the control limits after causing the shift up to the charts, thus detecting an out-of-control point.

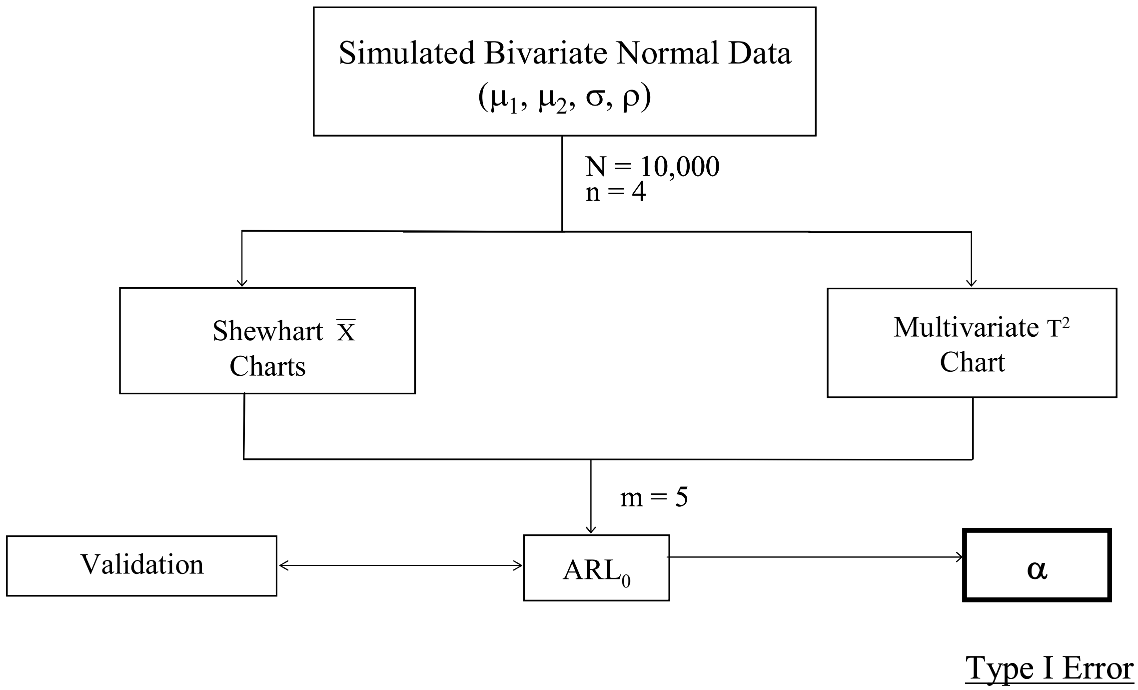


Figure 4.1 Simulation procedure (Type I error probability)

The analysis was done using the software Statgraphics™ Centurion XV. The objective of the second simulation was to find the average error probability (β) base on simulated ARL_1.

Figure 4.2 shows the simulation procedure. The simulation procedure was verified using the case where no correlation exists ($\rho = 0$) to meet the ARL_1 for the traditional Shewhart charts following the same operating characteristics from the OC curve available for different levels of shifts (Montgomery, 2001).

4.2 Data Analysis and Validation

Figure 4.3 shows the plotted Type I error probability (α) results obtained from the ARL_0. From this graph, it can be seen clearly that the average probability Type I error for the multivariate T^2 chart varied around 0.00232. However, the average probability Type I error for the Shewhart $\bar{x}$ chart increased from 0.00448 to 0.00624 as the level of correlation increases from (-0.8) to (+ 0.8).

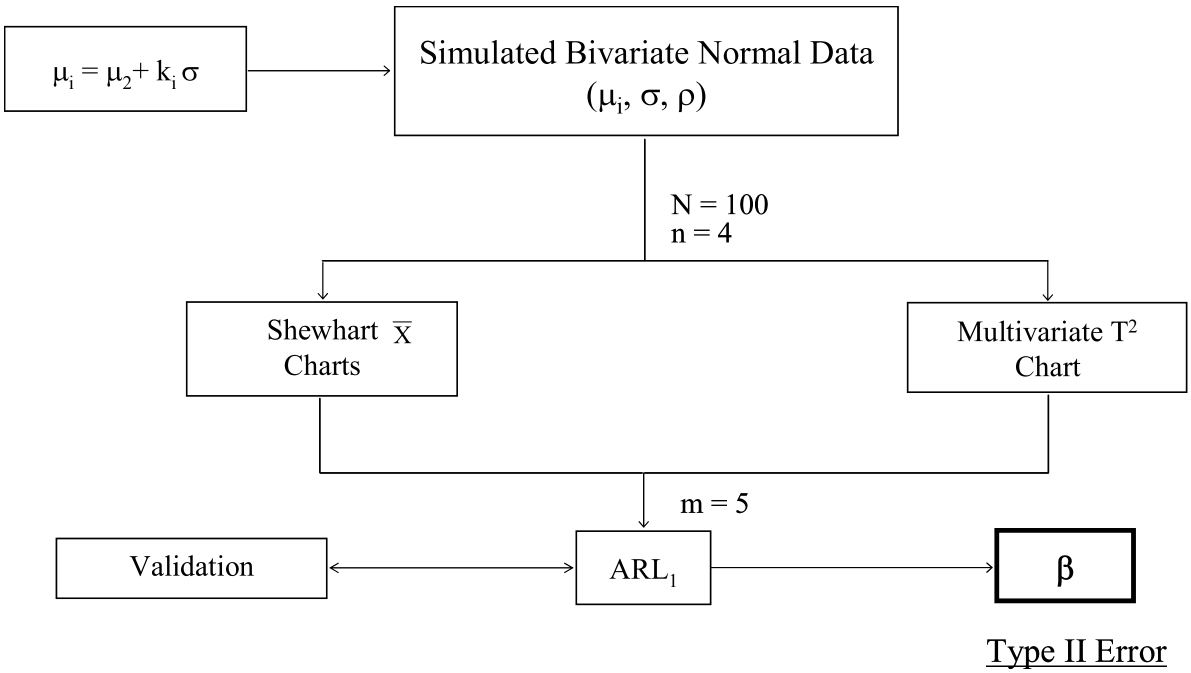


Figure 4.2 Simulation procedure (Type II error probability)

To test if the variation of the average probability error was statistically significant, an analysis of variance (ANOVA) was conducted using the software StatGraphics™ Centurion XV. Table 4.1 shows that the interaction between the chart type and the correlation level was statistically significant with p- value < 0.0001.

Figure 4.4 examines the interaction between the chart type and the correlation effect on the average probability error. It can be concluded that when using the multivariate T^2 chart, changes in the level of correlation between the two variables did not result in a significant increase in the average error probability (α). However, when using the Shewhart $\bar{x}$ chart, changes in the level of correlation resulted in a significant difference in the average probability error. When correlation was at its low level, the average probability error was 0.004624, whereas, an average probability error of 0.0061493 was observed at the high level of correlation. This amounts to a more than 32 percent increase in the average rate of false alarm.

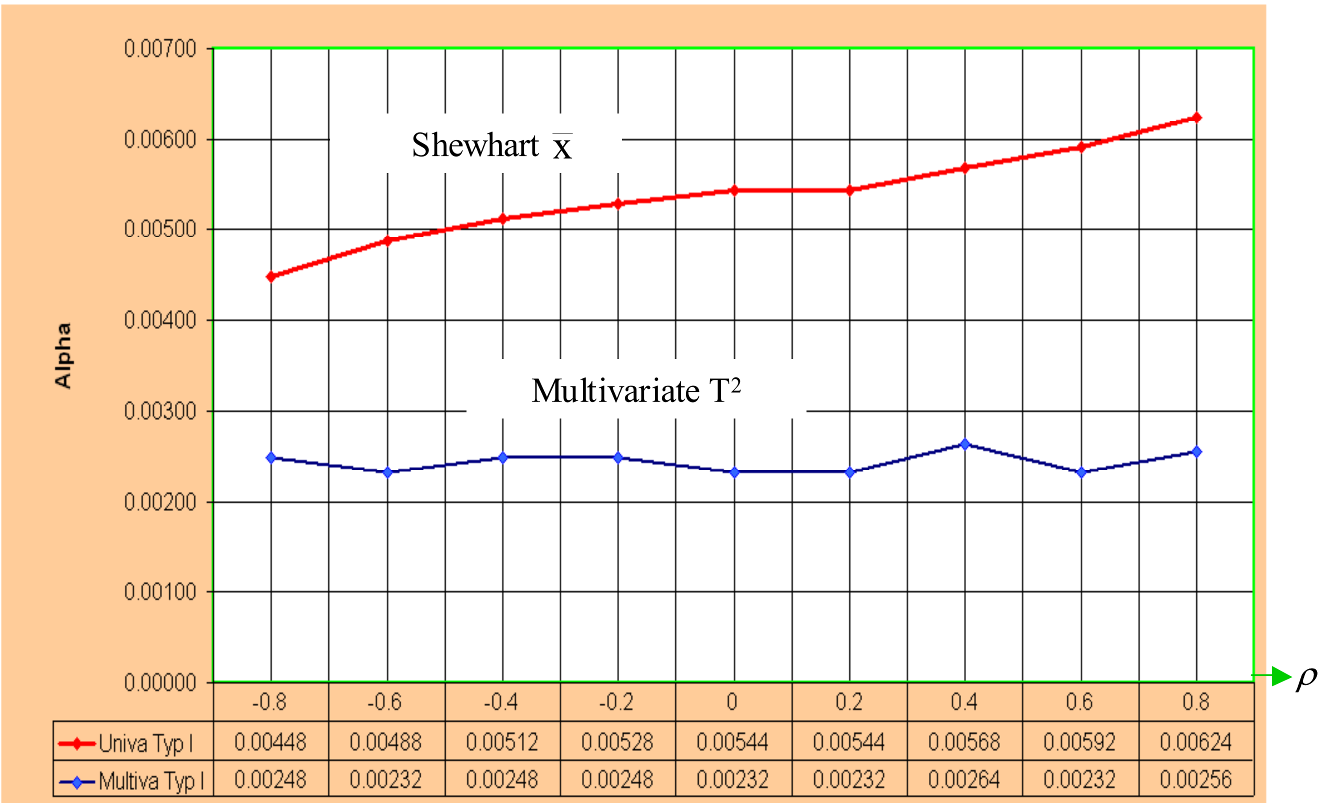


Figure 4.3 Simulated data: Type I error probability

TABLE 4.1 ANALYSIS OF VARIANCE (ANOVA): TYPE I ERROR PROBABILITY

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	4.138E-005	3	1.379E-005	979.36	< 0.0001
Chart Type(A)	3.919E-005	1	3.919E-005	2782.93	< 0.0001
Correlation (B)	1.184E-006	1	1.184E-006	84.08	< 0.0001
AB	1.001E-006	1	1.001E-006	71.08	< 0.0001
Residual	1.972E-007	14	1.408E-008		
Total	4.157E-005	17			

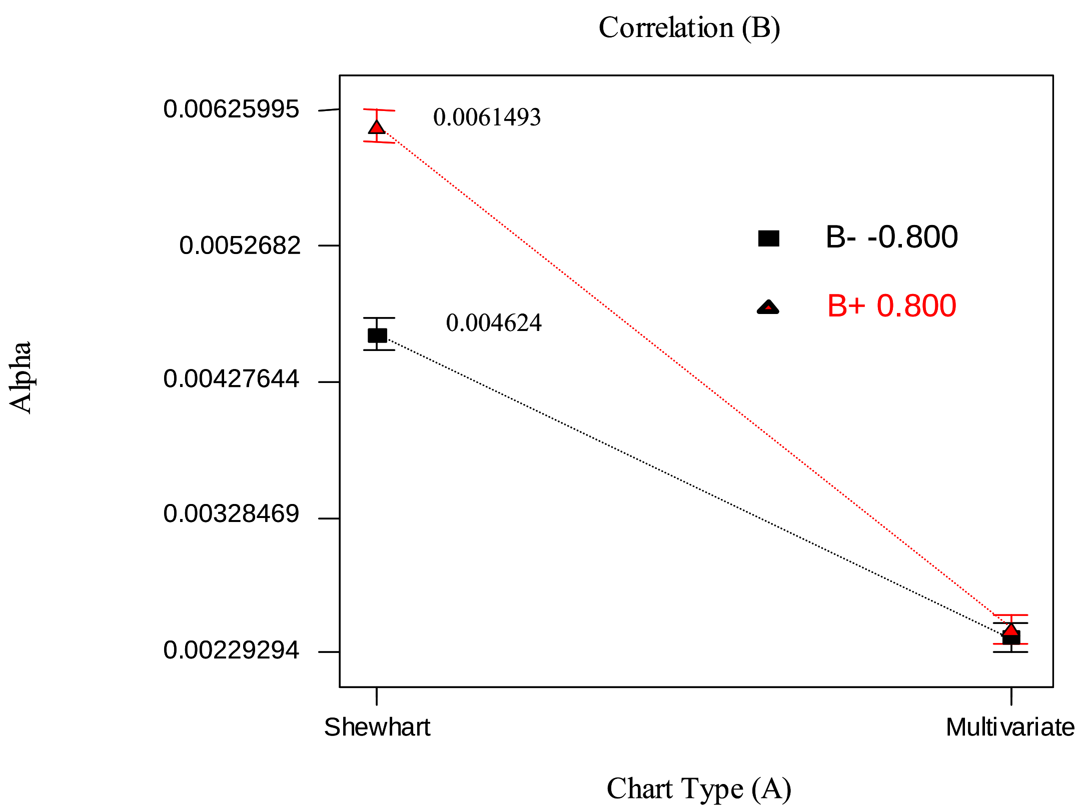
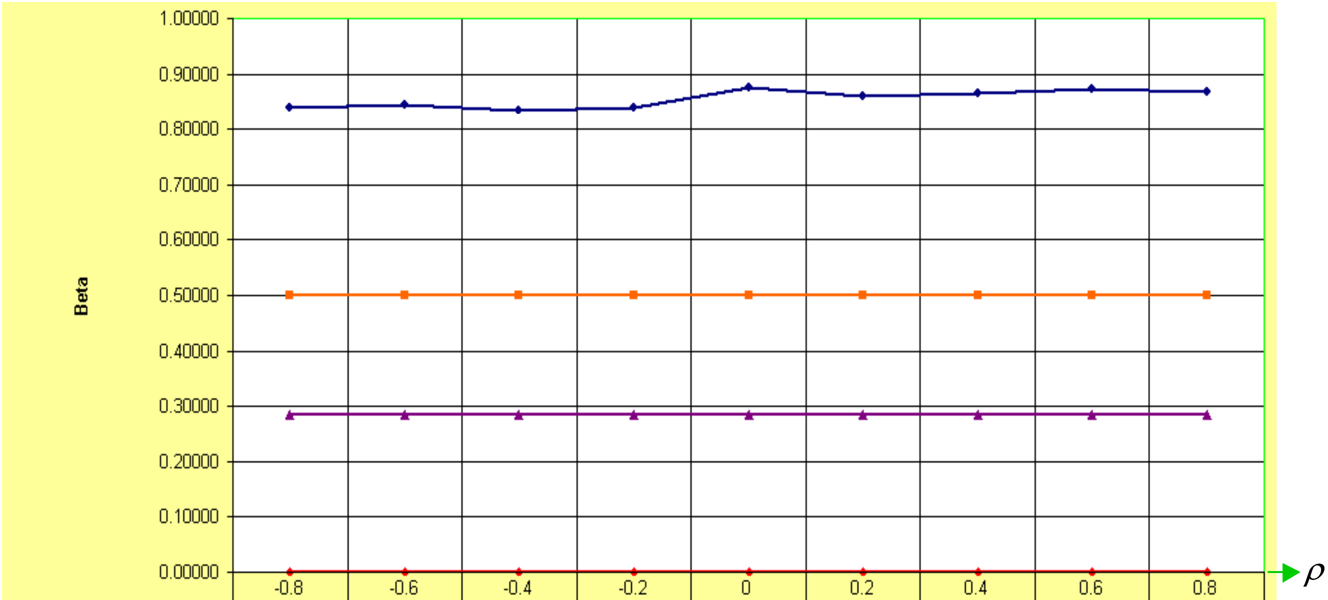


Figure 4.4 Chart type and correlation interaction plot

From the second simulation, Figure 4.5 shows the plotted Type II error probability (β) results obtained from the ARL_1 for the Shewhart $\bar{x}$ control chart. It can be seen clearly that the average probability of Type II error for the Shewhart $\bar{x}$ chart varies around the theoretical values as the level of shift changes (1, 1.5, 2, and 3 σ). The β results are plotted as 0.8750, 0.5000, 0.2875, and 0.0000, respectively. It can be conclude that when the Shewhart $\bar{x}$ chart was used, the levels of correlation between the variables did not affect the Type II error probability (β).



	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
1 Sigma Univa Typ II	0.83871	0.84375	0.83333	0.83871	0.87500	0.86111	0.86486	0.87179	0.86842
1.5 Sigma Univa Typ II	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000
2 Sigma Univa Typ II	0.28571	0.28571	0.28571	0.28571	0.28571	0.28571	0.28571	0.28571	0.28571
3 Sigma Univa Typ II	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Figure 4.5 Simulated data: Type II error probability
Shewhart $\bar{x}$ chart

Also, from the second simulation, Figure 4.6 shows the plotted Type II error probability (β) results obtained from the ARL_1 for the multivariate T^2 control chart. It can be seen that changes in the average probability of Type II error occurred as the correlation increased to the (+ 0.8) or decreased to the (– 0.8).

To test if the variation of the average probability error was statistically significant, an analysis of variance (ANOVA) was conducted using the software StatGraphics™ Centurion XV.

Table 4.2 shows that the quadratic term representing the shift level magnitude was highly influential. The interaction between the chart type and the correlation levels was also statistically significant, with a very low p-value < 0.0001.

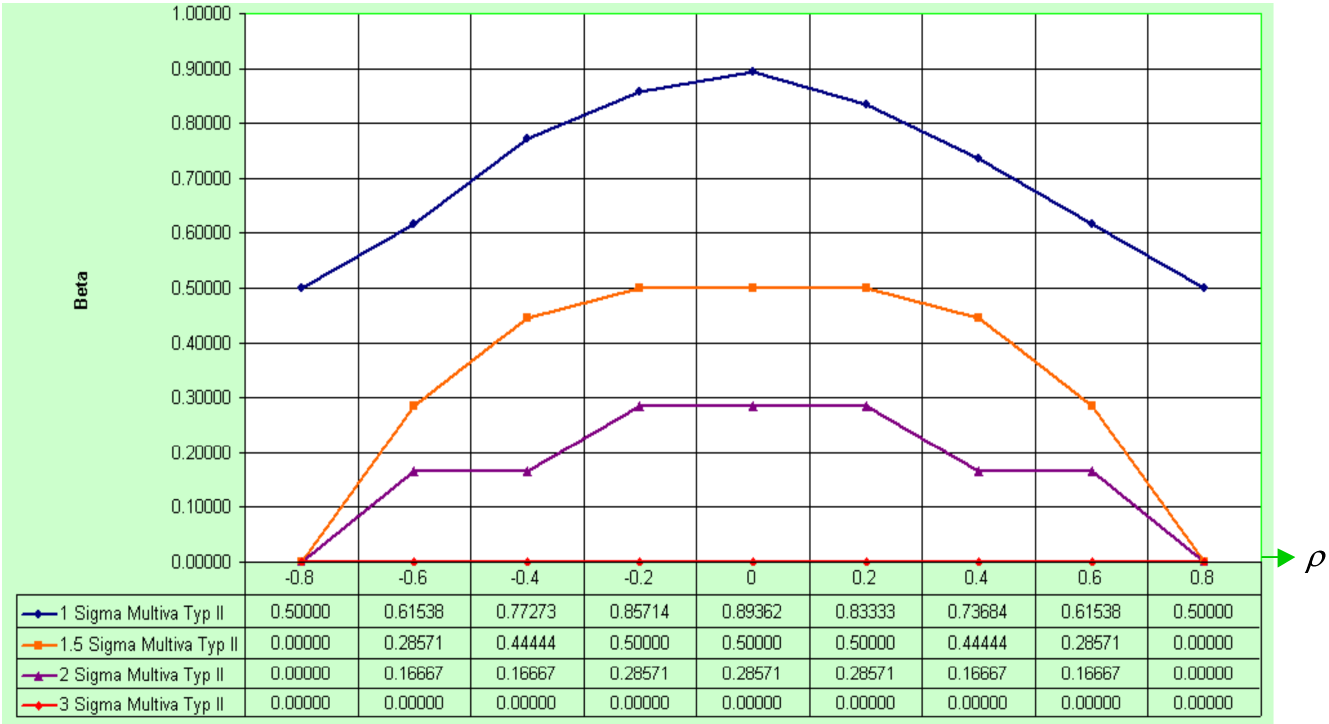


Figure 4.6 Simulated data: Type II error probability multivariate T^2 chart

TABLE 4.2 ANALYSIS OF VARIANCE (ANOVA): TYPE II ERROR PROBABILITY

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	2.5	6	0.34	163.75	< 0.0001
Chart Type(A)	0.14	1	0.14	67.53	< 0.0001
Correlation (B)	0.14	1	0.14	67.77	< 0.0001
Shift Level (C)	1.56	1	1.56	748.24	< 0.0001
B^2	0.012	1	0.012	5.71	< 0.0255
C^2	0.054	1	0.54	25.70	< 0.0001
AB	0.14	1	0.14	67.54	< 0.0001
Residual	0.048	23	2.086E-003		
Total	2.10	29			

Figure 4.7 shows the interaction between the chart type and the correlation effect on the average probability error (β). This plot indicates that when using the Shewhart $\bar{x}$ chart, there was no significant difference in the average probability error (β) as the level of correlation changed. On the other hand, when using the multivariate T^2 chart, the average probability error decreased significantly as the level of correlation increased.

When correlation was at its low level, the average probability error was 0.524684, whereas an average probability error of 0.136691 was observed at the high level of correlation. This amounts to more than 73 percent reduction in the Type II error probability (β).

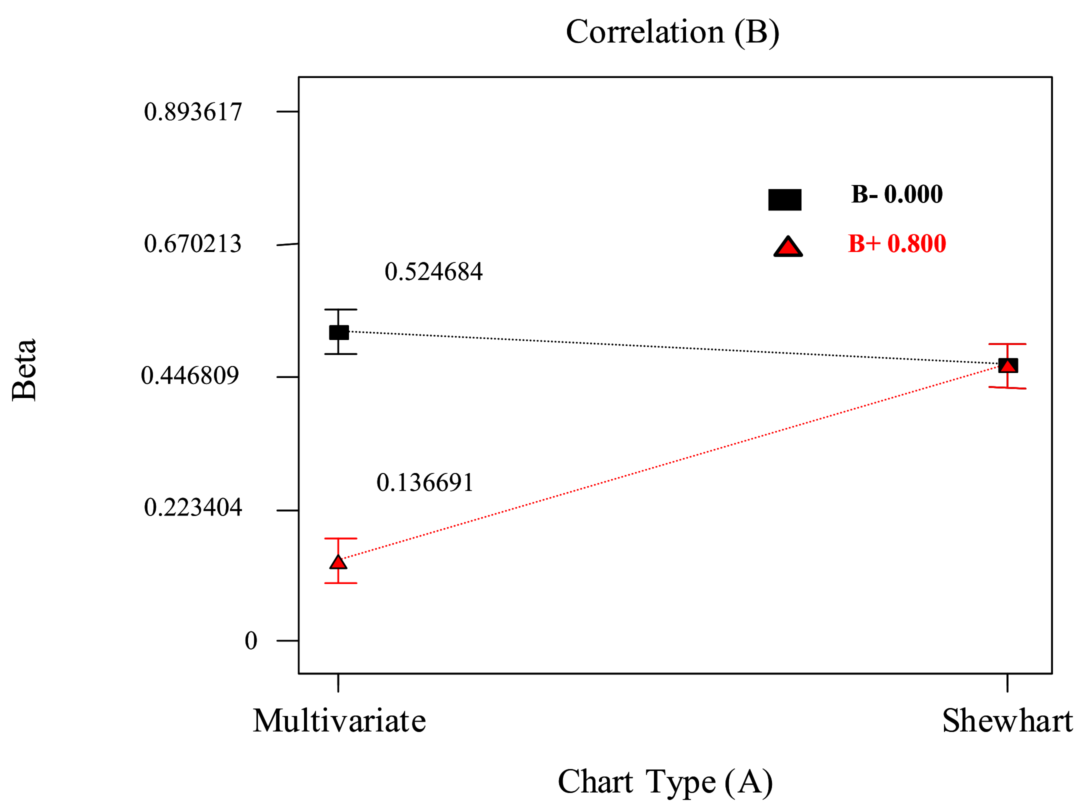


Figure 4.7 Chart type and correlation interaction plot

As a result of the initial investigation, a strong relationship between chart type and correlation levels was detected. Further in-depth investigation is required in order to characterize the statistical performance of the Shewhart $\bar{x}$ and the multivariate T^2 charts and identify any thresholds of the process and chart variables. Chapter 5 provides a designed experiment to conduct this exploration, using data that was generated utilizing the same simulation procedure. The simulation was verified and validated based on the OC curves of the Shewhart $\bar{x}$ for the case of no correlation ($\rho = 0$).

CHAPTER 5

CHARACTERISTICS OF STATISTICAL PERFORMANCE

Based on the initial performance investigation of the Shewhart $\bar{x}$ and Hotelling T^2 charts a quadratic effect was observed. The quadratic response surface model was used in this investigation and has the general form as

$$Y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i<j=2}\sum^{k} \beta_{ij} x_i x_j + \varepsilon \tag{5.1}$$

where

Y is the quantity of interest (Type II error probability)

β_i is the linear main effect coefficients

β_{ii} is the quadratic effect coefficients

$\beta_{ij.}$ is the interaction effect coefficients

ε is the random error which is assumed to have a normal distribution with mean zero and constant variance

k is the number of factors, and (i & j = 1, 2, 3, …, k)

X_j is the normalized independent variable (correlation level, subgroup size, chart type, shift size and alpha).

The level X_j of the *j*th factor is coded as

$$X_j = \frac{X_j - \frac{X_{jMAX} + X_{jMIN}}{2}}{\frac{X_{jMAX} - X_{jMIN}}{2}} \tag{5.2}$$

This coding scheme results in a coded value of -1 for the low level of factor *j*, a coded value of 1 for the high level, and a coded value of 0 for the mid level (Neter et al., 1996).

5.1 Design Selection

When designing a response surface study, a minimal requirement is that the design must be capable of providing estimates of the $p = (k+1)(k+2)/2$ parameters in the model. Any design of resolution V or higher for a two-level factorial study will provide estimates of linear main effect and all two-factor interaction effects that are confounded only with higher-order effects. However, at least three levels of each factor must be presented to obtain estimates of the k quadratic main effects.

One design that provides estimates of all parameters in the regression model shown in equation (5.1) is the full factorial design with each factor at three levels (3^k). A number of limitations are associated with this design. First, the number of treatments required by a 3^k grows rapidly with the number of factors. Second, each factor appears at exactly three levels so that it will not be possible to test for the presence of cubic or higher-order main effects (Myers and Montgomery, 2002).

A central composite design (CCD) of two-level full factorial 2^k was chosen in this study to examine the effect of changes in the level of correlation between variables coupled with changes in the process model and chart design variables for the Type II error probability. This design allows assigning a small number of carefully chosen treatments to permit estimation of the second-order response surface model.

Characteristics of CCD depend on the choice of the number of numeric factors (k), the number of center points (n_0), the number of axial points (n_a), and the distance from an axial point to the center point in coded units (θ) (Neter et al., 1996).

When choosing a CCD, a criterion that is often considered is that of rotatability. Rotatable designs have the property that the variance of the fitted values at all points equidistant from the center point is constant. The rotatability criterion is concerned with the precision of the estimator since the main purpose of the design is to estimate the response surface, i.e., to estimate the mean response at different locations (Mason et al., 2003).

Figure 5.1 shows a schematic of a central composite design in three factors (X_1, X_2, X_3) and the test locations. Note that the axial points are out of the surface, which allows an increase in the range while conducting the analysis.

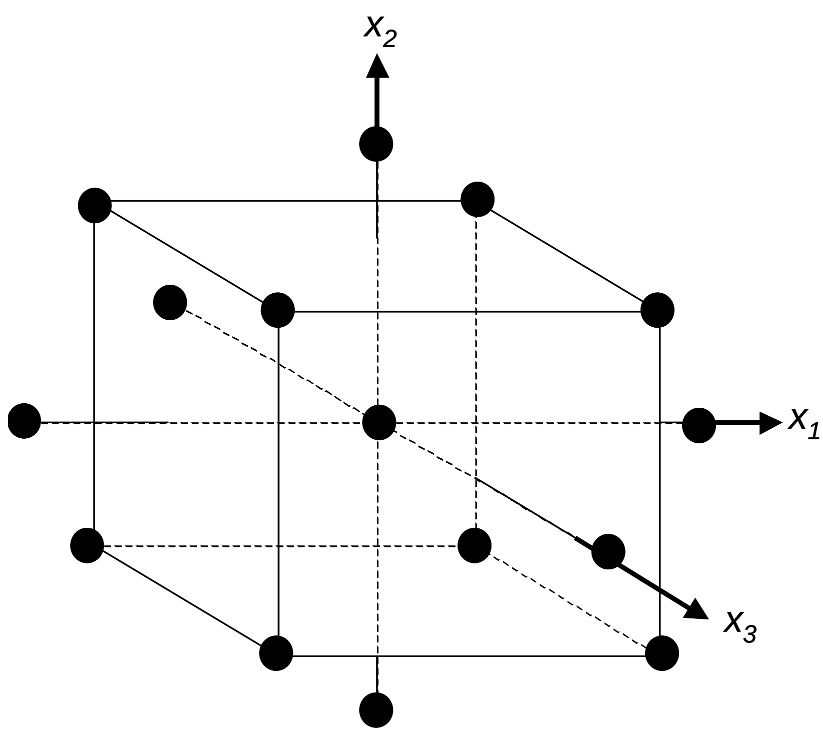


Figure 5.1 Central composite design in three Factors (X_1, X_2, X_3)

The design matrix utilized in this investigation was based on a full factorial CCD. The number of factors (k) in this study were four numeric factors (correlation level, subgroup size, shift size, and alpha), and one categorical or the chart type (the Shewhart $\bar{x}$ and Hotelling T^2

charts). The design contained twice as many axial points as there were factors in the design. Axial points, also called star points, were located such that all factors but one were set at their mid-levels. In this study, eight axial points were used ($n_a = 2k = 8$). Center points were replicated to evaluate curvature from second-order effects and to obtain an independent estimate of the error variance. Six center points ($n_0 = 6$) were replicated for each chart type, at $x_i = 0$ ($i = 1, 2 \ldots k$). The distance from an axial point to the center point in coded units was donated by $\theta = [2^k]^{1/4}$. This investigation had four numeric factors (correlation level, subgroup size, shift size, and alpha); hence, $\theta = [2^4]^{1/4} = 2$. Moreover, the number of factorial design points was 16 ($n_f = 2^4$). The total number of runs for this study were 60 runs ($n_t = n_f + n_a + n_0 = 30$ runs per chart type).

Table 5.1 summarizes the actual levels and corresponding coded levels of the process and chart factors considered in this investigation. The design allowed for the evaluation of the effect of the factors and their higher-order interactions on the response (the probability of Type II error (β)). The process factors included correlation between the pairs of variables (ρ) and the shift magnitude of the process mean (δ). The chart factors were the subgroup size (n), the probability of Type I error (alpha (α)), and chart type (Shewhart $\bar{x}$ and Hotelling T^2).

Table 5.2 shows the design matrix used in this evaluation in terms of the coded levels of selected variables as well as the simulated performance of the chart with regards to Type II error probability (β) associated with each run. Since there is a categorical factor involved in this study, thirty runs were replicated for both Shewhart $\bar{x}$ chart and Hotelling T^2 chart which resulted in total sixty runs. The center points and axial (star) points are also indicated in Table 5.2.

TABLE 5.1 ACTUAL VALUES AND CORRESPONDING CODED LEVELS OF THE PROCESS AND CHART VARIABLES

Variables	*i*	Factors	Data Type	Actual Value	Coded Level
Process	1	Correlation (ρ)	Continuous	0.1 0.5 0.9	-1 0 +1
	2	Shift Magnitude (δ)	Continuous	0.5 1.5 2.5	-1 0 +1
Chart	3	Subgroup (n)	Continuous	2.0 10.0 18.0	-1 0 +1
	4	Alpha (α)	Continuous	0.001 0.003 0.005	-1 0 +1
	5	Chart Type	Categorical	$\bar{x}$ T^2	-1 +1

TABLE 5.2 DESIGN MATRIX

Test	Point Type	Correlation (A)	Shift (B)	Subgroup Size (C)	Alpha (D)	Chart Type (E)	Response (Beta)
1	Factorial	-1	-1	-1	-1	{ -1 }	0.79167
2	Factorial	1	-1	-1	-1	{ -1 }	0.79167
3	Factorial	-1	1	-1	-1	{ -1 }	0.16667
4	Factorial	1	1	-1	-1	{ -1 }	0.16667
5	Factorial	-1	-1	1	-1	{ -1 }	0.28571
6	Factorial	1	-1	1	-1	{ -1 }	0.28571
7	Factorial	-1	1	1	-1	{ -1 }	0.00000
8	Factorial	1	1	1	-1	{ -1 }	0.00000
9	Factorial	-1	-1	-1	1	{ -1 }	0.66667
10	Factorial	1	-1	-1	1	{ -1 }	0.66667
11	Factorial	-1	1	-1	1	{ -1 }	0.16667
12	Factorial	1	1	-1	1	{ -1 }	0.16667
13	Factorial	-1	-1	1	1	{ -1 }	0.16667
14	Factorial	1	-1	1	1	{ -1 }	0.16667

TABLE 5.2 DESIGN MATRIX (Continued)

Test	Point Type	Correlation (A)	Shift (B)	Subgroup Size (C)	Alpha (D)	Chart Type (E)	Response (Beta)
15	Factorial	-1	1	1	1	{ -1 }	0.00000
16	Factorial	1	1	1	1	{ -1 }	0.00000
17	Axial	-2	0	0	0	{ -1 }	0.16667
18	Axial	2	0	0	0	{ -1 }	0.16667
19	Axial	0	-2	0	0	{ -1 }	0.89362
20	Axial	0	2	0	0	{ -1 }	0.00000
21	Axial	0	0	-2	0	{ -1 }	0.81481
22	Axial	0	0	2	0	{ -1 }	0.00000
23	Axial	0	0	0	-2	{ -1 }	0.28571
24	Axial	0	0	0	2	{ -1 }	0.00000
25	Center	0	0	0	0	{ -1 }	0.16667
26	Center	0	0	0	0	{ -1 }	0.16667
27	Center	0	0	0	0	{ -1 }	0.16667
28	Center	0	0	0	0	{ -1 }	0.28571
29	Center	0	0	0	0	{ -1 }	0.28571
30	Center	0	0	0	0	{ -1 }	0.28571
31	Factorial	-1	-1	-1	-1	{ 1 }	0.79167
32	Factorial	1	-1	-1	-1	{ 1 }	0.66667
33	Factorial	-1	1	-1	-1	{ 1 }	0.16667
34	Factorial	1	1	-1	-1	{ 1 }	0.00000
35	Factorial	-1	-1	1	-1	{ 1 }	0.28571
36	Factorial	1	-1	1	-1	{ 1 }	0.00000
37	Factorial	-1	1	1	-1	{ 1 }	0.00000
38	Factorial	1	1	1	-1	{ 1 }	0.00000
39	Factorial	-1	-1	-1	1	{ 1 }	0.66667
40	Factorial	1	-1	-1	1	{ 1 }	0.50000
41	Factorial	-1	1	-1	1	{ 1 }	0.16667
42	Factorial	1	1	-1	1	{ 1 }	0.00000
43	Factorial	-1	-1	1	1	{ 1 }	0.16667
44	Factorial	1	-1	1	1	{ 1 }	0.00000
45	Factorial	-1	1	1	1	{ 1 }	0.00000
46	Factorial	1	1	1	1	{ 1 }	0.00000
47	Axial	-2	0	0	0	{ 1 }	0.16667
48	Axial	2	0	0	0	{ 1 }	0.00000
49	Axial	0	-2	0	0	{ 1 }	0.75000
50	Axial	0	2	0	0	{ 1 }	0.00000
51	Axial	0	0	-2	0	{ 1 }	0.80000
52	Axial	0	0	2	0	{ 1 }	0.00000
53	Axial	0	0	0	-2	{ 1 }	0.16667

TABLE 5.2 DESIGN MATRIX (Continued)

Test	Point Type	Correlation (A)	Shift (B)	Subgroup Size (C)	Alpha (D)	Chart Type (E)	Response (Beta)
54	Axial	0	0	0	2	{ 1 }	0.00000
55	Center	0	0	0	0	{ 1 }	0.16667
56	Center	0	0	0	0	{ 1 }	0.16667
57	Center	0	0	0	0	{ 1 }	0.16667
58	Center	0	0	0	0	{ 1 }	0.00000
59	Center	0	0	0	0	{ 1 }	0.00000
60	Center	0	0	0	0	{ 1 }	0.00000

5.2 Statistical Analysis

A normal probability plot of the residuals (difference between observed and estimated responses) was constructed and examined to validate the underling assumptions of error normality. Plots of the residuals versus estimated response values and, plots of the residuals versus levels of the independent variables were generated and examined in order to verify that the errors had a constant variance and a zero mean. A normally distributed error with constant variance is required for testing statistical hypotheses regarding the significance of the regression model. When errors are not normally distributed with constant variance, transformation may be required in order to restore the model adequacy. In this case a square root transformation was applied $y^{*}_{ij} = \sqrt{1 + y_{ij}}$ (Neter et al., 1996).

Table 5.3, obtained by using the software Design-Expert® version 7.1.1 (2007), shows the sequential sums of squares for the linear, quadratic, and cubic terms in the model. Since the CCD did not contain enough runs to support a full cubic model, the cubic model was indicated as aliased. Based on the p-value for the quadratic term, a second-order model was fitted to the response variable. Table 5.4 indicates that the quadratic model with 30 degrees of freedom would have insignificant lack-of-fit, indicating a good approximation of the response variable.

TABLE 5.3 ANOVA FOR FITTING THE MODEL

Source	**Sum of Squares**	**df**	**Mean Square**	**F Value**	**p-value Prob > F**	
Mean vs Total	73.15478	1	73.15478			
Linear vs Mean	0.64194	5	0.12839	40.75958	< 0.0001	
2FI vs Linear	0.06755	10	0.00676	2.89859	0.0072	
Quadratic vs 2FI	0.06865	4	0.01716	20.25363	< 0.0001	*Suggested*
Cubic vs Quadratic	0.01608	18	0.00089	1.10305	0.4087	Aliased
Residual	0.01782	22	0.00081			
Total	73.96681	60	1.23278			

TABLE 5.4 ANOVA FOR LACK OF FIT

Source	**Sum of Squares**	**df**	**Mean Square**	**F Value**	**p-value Prob > F**	
Linear	0.15613	44	0.00355	2.54062	0.0574	
2FI	0.08858	34	0.00261	1.86530	0.1478	
Quadratic	0.01993	30	0.00066	0.47561	0.9434	*Suggested*
Cubic	0.00385	12	0.00032	0.22967	0.9905	Aliased
Pure Error	0.01397	10	0.00140			

Table 5.5 shows that the quadratic model had the highest adjusted R^2 and predicted R^2 and therefore, was selected.

TABLE 5.5 MODEL SUMMARY STATISTICS

Source	**Std. Dev.**	**R-Squared**	**Adjusted R-Squared**	**Predicted R-Squared**	**PRESS**	
Linear	0.05612	0.79053	0.77114	0.73887	0.21205	
2FI	0.04828	0.87372	0.83067	0.80315	0.15985	
Quadratic	0.02911	0.95826	0.93843	0.91762	0.06690	*Suggested*
Cubic	0.02846	0.97806	0.94116	0.89939	0.08170	Aliased

To test the significance of the regression model with all model terms included, an ANOVA table was then constructed. Table 5.6 provides a measure of the capability of the model to distinguish between experimental noise (random error) and the actual response. As part of this analysis, the significance of each model term was examined using forward selection procedure. Variables were added one at a time until a satisfactory fit was achieved. In this case, the F-statistics were based on a reduction in the error sums of squares attributed to the incremental contribution of a selected variable (Mason et al., 2003). When the addition of a predictor did not result in a statistically significant F-statistic, the procedure was terminated.

Table 5.6 shows that the quadratic model was significant with p-value < 0.0001. Also, it can be seen that the lack-of-fit was not significant. This means that the model was an acceptable approximation to the relationship between the probability of committing a Type II error (β) and the independent process and chart variables. The F-value of 0.47 implies that the lack-of-fit was not significant relative to the pure error. Table 5.6 also shows that the two-factor interactions involving the level of correlation and chart type (AE), shift magnitude and subgroup size (BC), and the shift magnitude and probability of Type I error, alpha (BD) were significant.

TABLE 5.6 ANOVA FOR QUADRATIC MODEL

Source	Sum of Squares	df	Mean Square	F Value	p-value Prob > F	
Model	0.772	10	0.077	95.64	< 0.0001	*significant*
Correlation (A)	0.009	1	0.009	10.84	0.0018	
Shift (B)	0.327	1	0.327	405.23	< 0.0001	
Subgroup Size (C)	0.273	1	0.273	338.14	< 0.0001	
Alpha (D)	0.013	1	0.013	16.43	0.0002	
Chart Type (E)	0.020	1	0.020	24.17	< 0.0001	
AE	0.009	1	0.009	10.84	0.0018	
BC	0.051	1	0.051	63.54	< 0.0001	
BD	0.004	1	0.004	5.30	0.0256	
B^2	0.038	1	0.038	46.64	< 0.0001	
C^2	0.036	1	0.036	44.37	< 0.0001	
Residual	0.040	49	0.001			
Lack of Fit	0.026	39	0.001	0.47	0.9545	*not significant*
Pure Error	0.014	10	0.001			
Cor Total	0.812	59				

The coefficient of determination (R^2) for this model was 0.95, which implies a good fit for the quadratic model. Also, the adjusted R^2 value of 0.94 and the predicted R^2 value of 0.93 suggest that the prediction equation fit the observed response well.

5.3 Residual Analysis

Residual analysis was used for checking the underlying assumption of the AVOVA procedure. A normal probability plot of the residuals is shown in Figure 5.2. Each residual was plotted against its expected value when the distribution is normal. A plot in which the scatter of the data points follows a straight line, suggests error normality; on the other hand, a plot in which the data points depart substantially from linearity, suggests that the error distribution is not normally distributed (Myers and Montgomery, 2002).

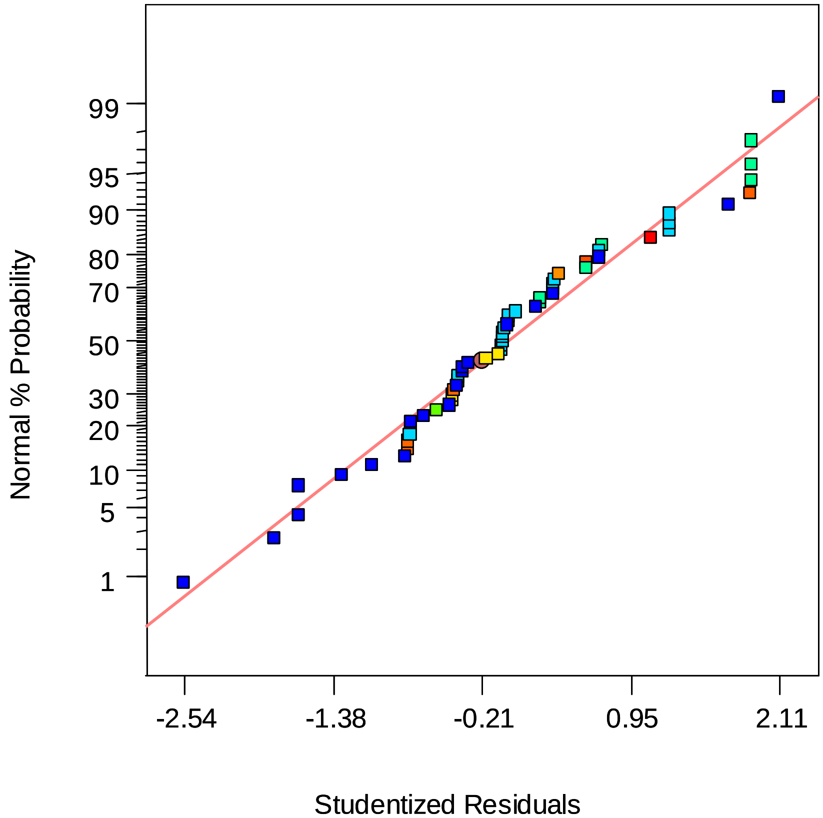


Figure 5.2 Normal probability plot of residuals for Type II error

In constructing the plot, the residuals were organized in ascending order, and the expected value of each, assuming normality, was determined using an expected value of zero and a constant variance. The normal probability plot shown in Figure 5.2 indicates no violation of the normality assumption.

As shown in Figure 5.3, the plot of residuals versus predicted values of the response supports the assumption of constant variance. Plots of studentized residuals versus levels of each of the independent variables were generated. The plots of residuals versus the chart and process variables, presented in Figures 5.4 to 5.8, indicate no relationship between the factors and the model residuals. Note that an increasing or decreasing trend (e.g., a funneling shape) of the plot is a sign that the variance is not constant. Consequently, the fitted model provided adequate approximation of the average response.

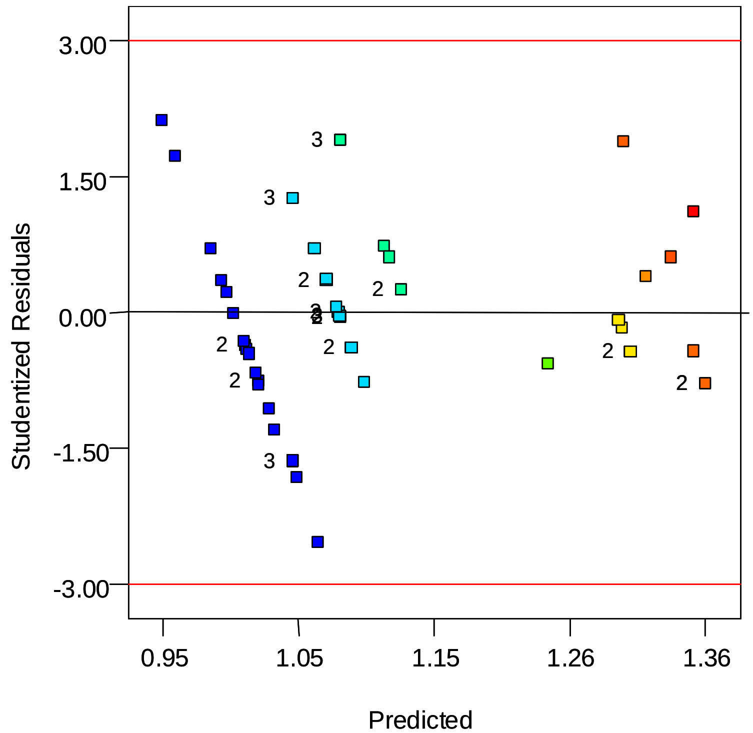


Figure 5.3 Residuals vs. predicted values

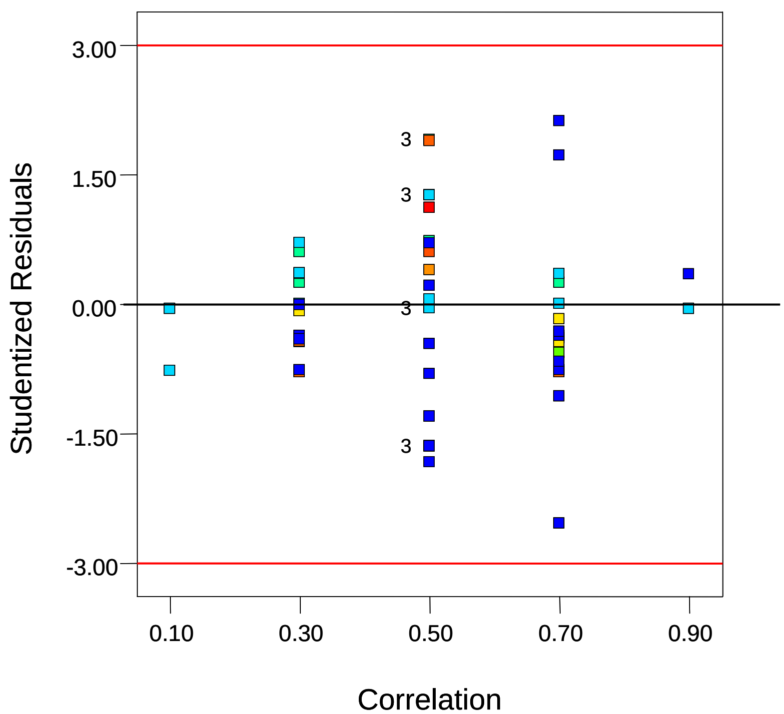


Figure 5.4 Residuals vs. correlation level

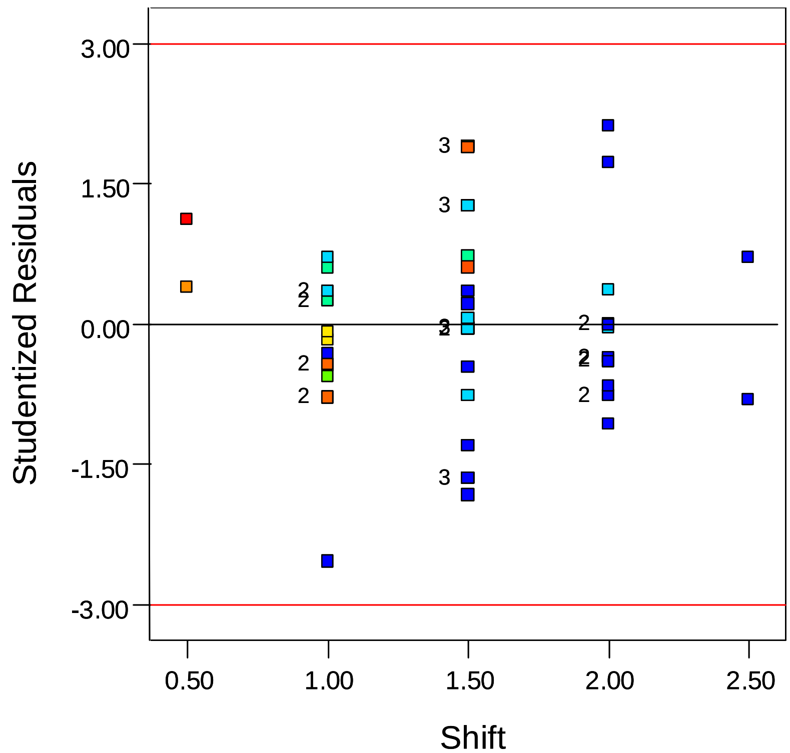


Figure 5.5 Residuals vs. shift magnitude

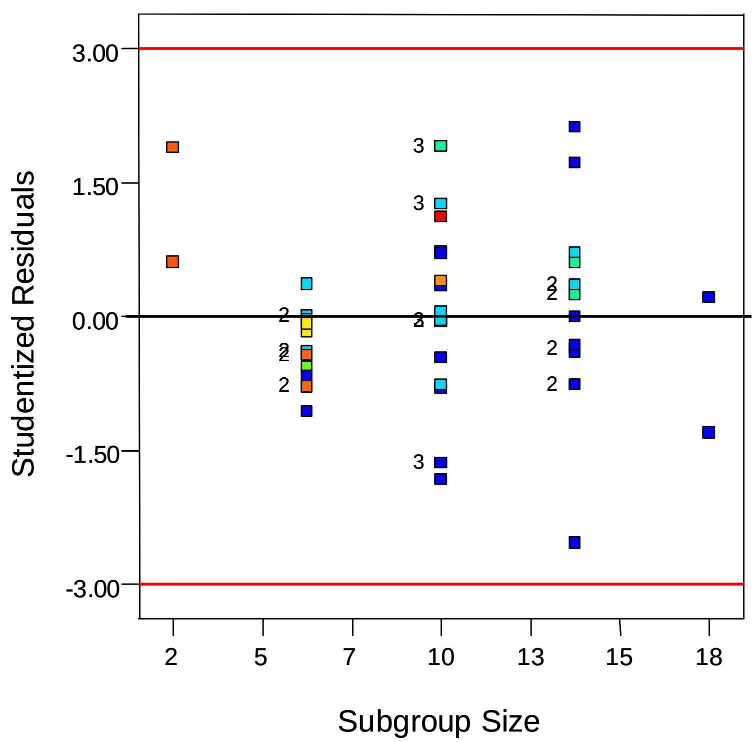


Figure 5.6 Residuals vs. subgroup size

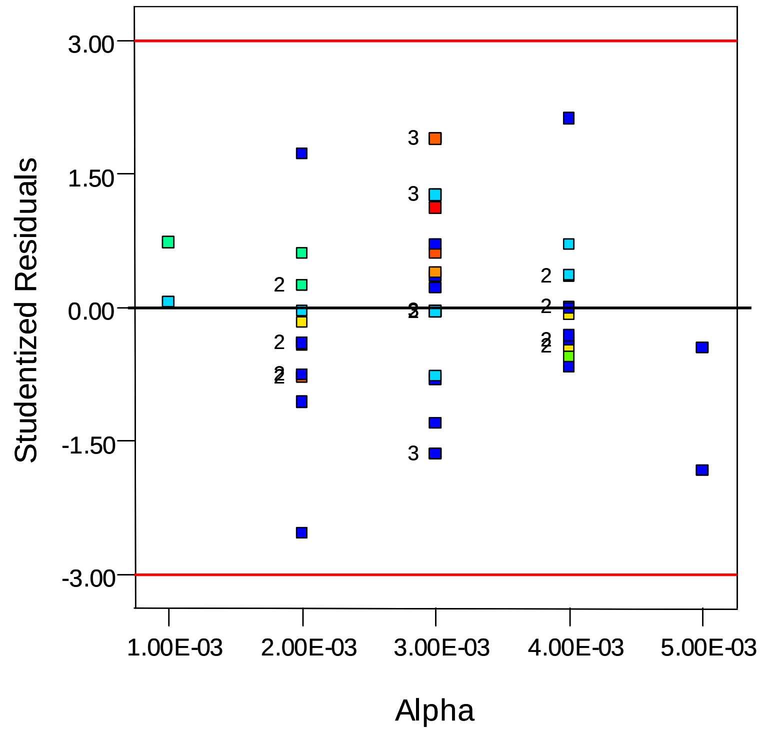


Figure 5.7 Residuals vs. alpha levels

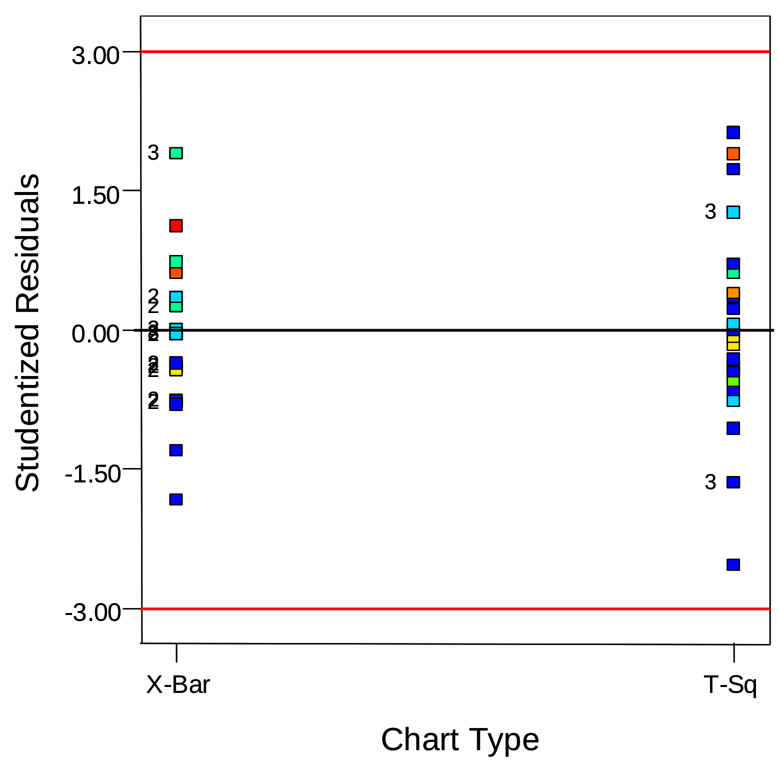


Figure 5.8 Residuals vs. chart type

5.4 Interpretation

To investigate the statistically significant interactions in the model, two-factor interaction plots were generated. Figure 5.9 depicts the interaction between the shift magnitude and the subgroup size and their effect on the average error probability (β).

As mentioned in the literature, it was revealed, as expected, that at low levels of subgroup size, the higher the shift in the process mean, the lower the average error probability. However, at high levels of subgroup size, changes in the shift magnitude did not result in significant changes in the average error probability.

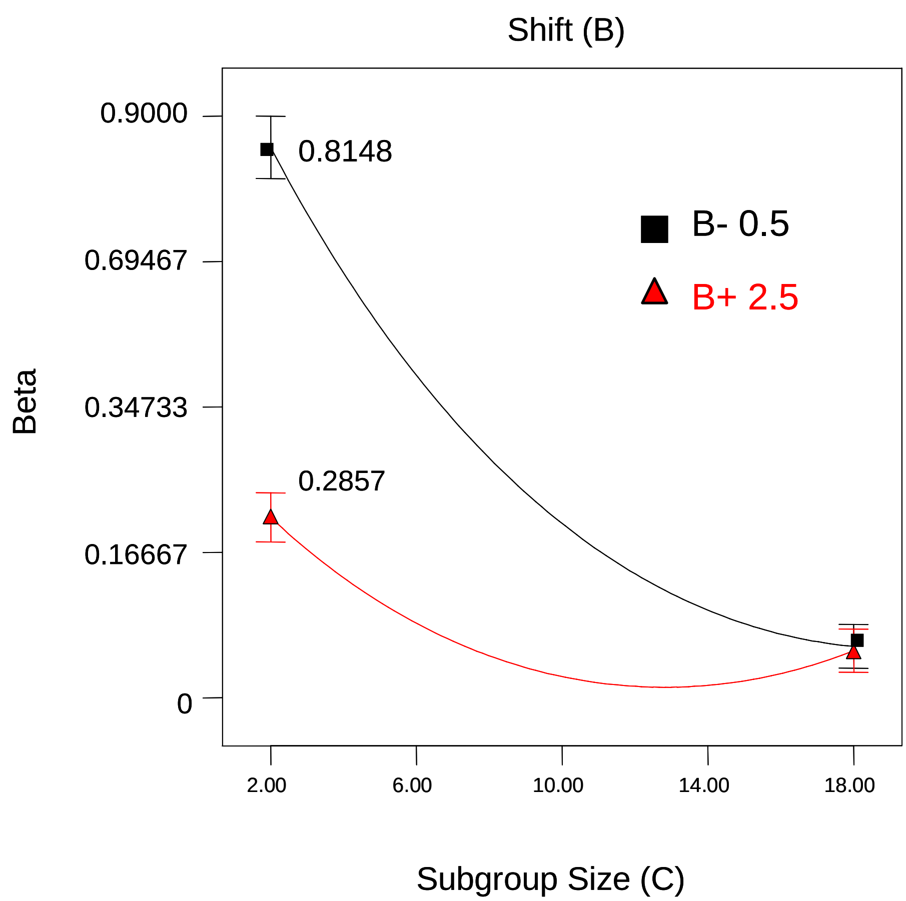


Figure 5.9 Subgroup size and shift interaction plot

Figure 5.10 depicts the interaction between the alpha levels and the shift magnitudes in the process mean and their effect on the average error probability (β). As would be expected, at the low levels of the shift magnitude, the higher the alpha levels, the lower the average error probability (β). At high levels of shift magnitude in the process mean, changes in the alpha levels did not result in significant changes in the average error probability.

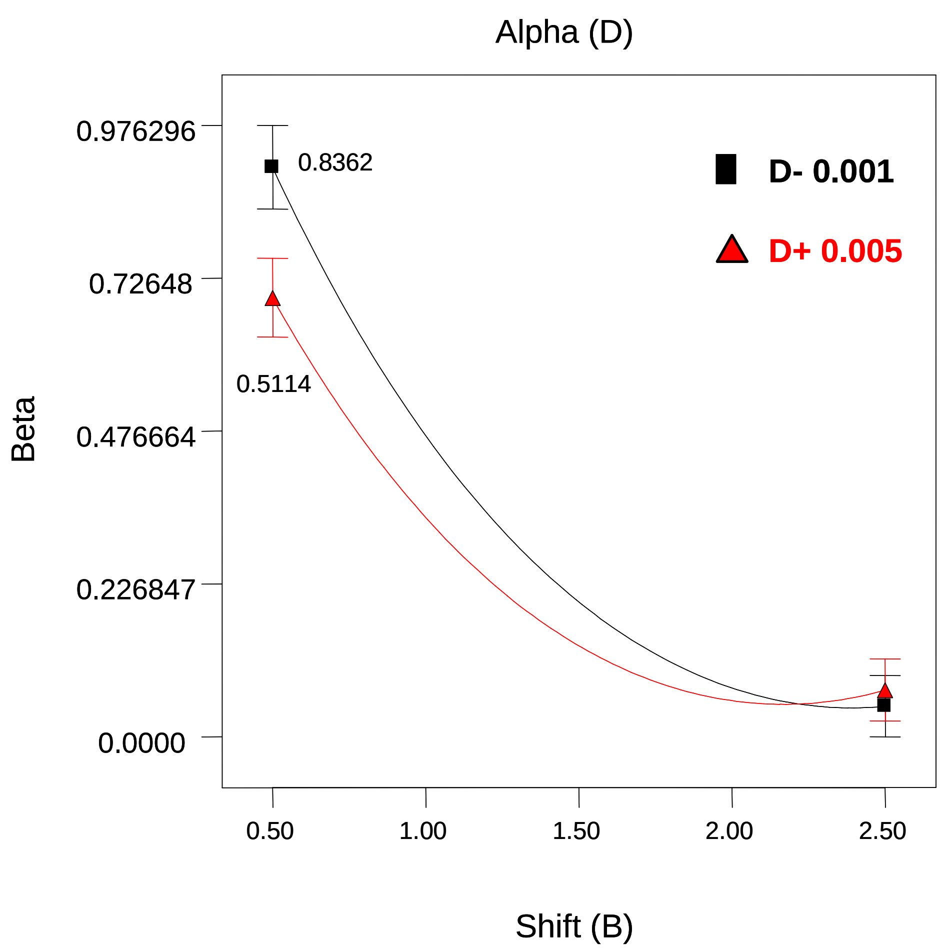


Figure 5.10 Alpha and shift interaction plot

Figure 5.11 shows the interaction between the chart type and correlation and their effect on the average error probability (β). The interaction plot indicates that Shewhart $\bar{x}$ charts lack sensitivity towards correlation among variables. On the contrary, when multivariate T^2 charts were used, the average error probability (β) decreased significantly as the level of correlation increased.

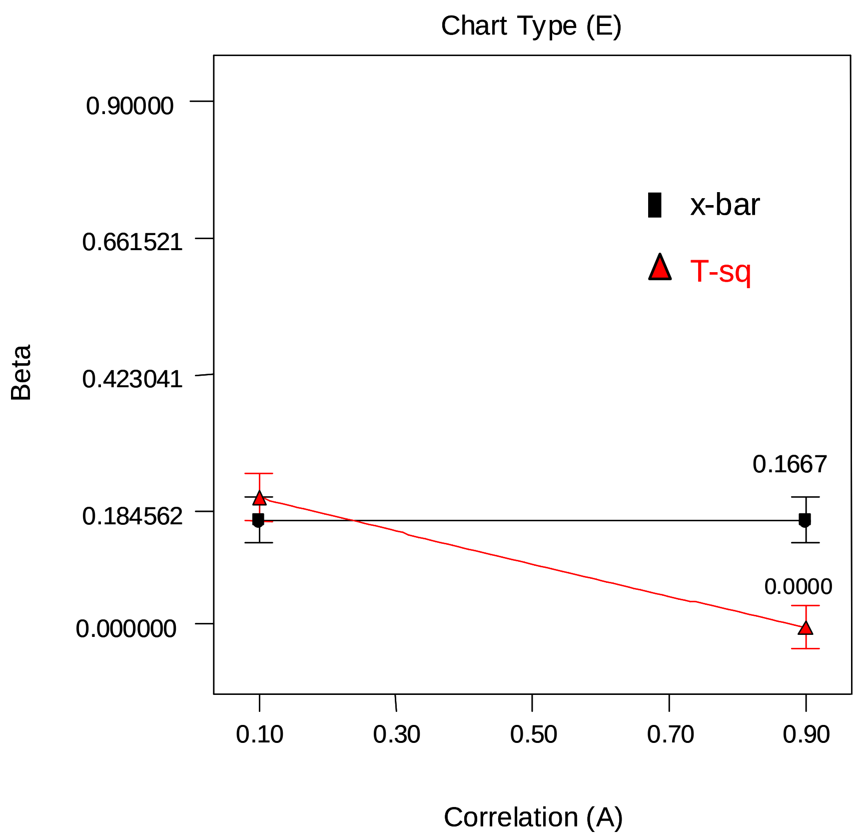


Figure 5.11 Chart type and correlation interaction plot

Using the Shewhart $\bar{x}$ chart at 0.90 correlation level, the average error rate was 0.1667, whereas an average probability tended to asymptote to 0.00 when using the multivariate T^2 chart. By switching from the Shewhart $\bar{x}$ chart to multivariate T^2 chart at the high correlation level of 0.90, there was about 100 percent reduction in the average probability (β). However, at correlation levels of 0.46 and below, using either Shewhart $\bar{x}$ charts or multivariate T^2 charts did not result in significant changes in the average error probability (β). Tables 5.7 and 5.8 provide the confidence intervals (CI) for the response (β) at correlation 0.48, shift magnitude of 1.5, subgroup size 10, and alpha 0.003. It is observed that at 0.48 correlation level and above, switching from the Shewhart $\bar{x}$ chart to the multivariate T^2 chart resulted in a significant change in the average error probability (β).

The average error probability (β) at the specified levels for each factor changed significantly by switching from $\bar{x}$ to T^2 charts. Distinctively, a threshold is pointed out in Tables 5.7 and 5.8. The average error probability (β) was reduces from 0.17 to 0.10 with a 95 percent CI of {0.14, 0.20} and {0.07, 0.13}, respectively, by switching from $\bar{x}$ charts to T^2 chart.

TABLE 5.7 CONFIDENCE INTERVAL - T^2

Factor	**Name**	**Level**	**Low Level**	**High Level**
A	Correlation	0.48	0.3	0.7
B	Shift	1.5	1	2
C	Subgroup	10	6	14
D	Alpha	0.003	0.002	0.004
E	Chart type	T^2		
	Prediction		95% CI low	95% CI high
Beta	0.10		0.07	0.13

TABLE 5.8 CONFIDENCE INTERVAL - $\bar{x}$

Factor	**Name**	**Level**	**Low Level**	**High Level**
A	Correlation	0.48	0.3	0.7
B	Shift	1.5	1	2
C	Subgroup	10	6	14
D	Alpha	0.003	0.002	0.004
E	Chart type	$\bar{x}$		
	Prediction		95% CI low	95% CI high
Beta	0.17		0.14	0.20

CHAPTER 6

ESTIMATION OF INCREMENTAL COST

In order to estimate the economic feasibility of using multivariate T^2 control charts as an alternative to the traditional univariate $\bar{x}$ charts, an incremental cost model was constructed to determine the cost or savings resulting from using multivariate T^2 control charts instead of traditional univariate $\bar{x}$ charts. Only two quality characteristics were of interest. It was assumed that the user of the $\bar{x}$ charts would not have knowledge of the relationship between the two variables and, thus, would use two separate charts to conduct the SPC tests, hence, assuming their independence. However, the lack of the user's knowledge of the relationship between the two variables did not indicate that there was no correlation between them. The variables could have had a strong dependency between them, resulting in high correlation levels (i.e. r = 0.9).

The cost model consisted of two terms. The first term was the cost of using two separate $\bar{x}$ charts to monitor the two quality characteristics of interest. The second term was the cost of utilizing the T^2 chart to monitor the two quality characteristics simultaneously. Since the interest was to determine the cost or savings of using T^2 chart as an alternative to the traditional $\bar{x}$ charts, the first term $E(C_U)$, the cost of using the $\bar{x}$ chart, was multiplied by two to include the cost associated with operating both $\bar{x}$ charts. The second term $E(C_M)$ was the cost of using one T^2 chart to monitor the same two variables. The result $\Delta E(C)$ was calculated by subtracting the second term from the first term, which provided the cost or savings of switching from $\bar{x}$ charts to the T^2 chart. Equation (6. 1) represents the cost model as

$$\Delta E(C) = 2E(C_U) - E(C_M) \tag{6.1}$$

In the univariate case, the first term E (C_U) was calculated using Knappenberger and Grandage's (1969) economic model presented in equation (2.41). A computer program was developed using the software MathCAD® 11 to estimate the total cost. The model was validated and verified for five values. An illustrative example is presented in Appendix B providing all the cost terms.

Similarly, for the multivariate case, Montgomery and Klatt's (1972) economic model presented in equation (2.42) was used to estimate the second term E (C_M). In order to calculate the cost of using a T^2 chart, a computer subprogram was developed to calculate the probability of Type II error (β) in the multivariate case. MathCAD®11 was utilized to develop the program, which is presented in Figure 6.1. The sample size (N), number of variables (p), and the non-centrality parameter (τ^2) are required to calculate the probability of Type II error for the T^2 chart.

To verify that the program shown in Figure 6.1 was correct, an example by Anderson (1958) is presented in Figure 6.2. In this example, the degrees of freedom were (p = 4, and ν_2 = 10). Also, the non-centrality parameter was given as ($\tau^2 = 31.25$). The null hypothesis $\boldsymbol{\mu_0} = 0$ was tested at 1 percent level of significance ($\alpha = 0.01$). Figure 6.2 shows that the probability of Type II error (β) is calculated to be 0.227. With the test of the hypothesis conducted at 5 percent level of significance ($\alpha = 0.05$), the probability of Type II error (β) droped significantly to 0.043.

Tables have been provided by Tang (1938) for the probability of Type II error (β) for various values of (τ^2) for significant levels of α = 0.01 and 0.05. These tables are provided for a non-central parameter ϕ, which has a relation to τ^2 as

$$\phi = \frac{\tau}{\sqrt{p+1}} \tag{6.2}$$

The Probability of Type II Error on Multivariate Tsq Charts

Chart Parameters

Number of Variables p

Sample Size N

Significance Level α

$$\nu 2 := \frac{n - p + 1}{2} \qquad n := N - 1$$

The Inverse Cumulative Probability of the F-distribution

$$f := qF\left[(1 - \alpha), p, \nu 2\right]$$

Process Parameters

Non-centrality Parameter τ^2

The Probability Density Function of the Non-central F-distribution

$$P(f) := \sum_{i=0}^{100} \left[\frac{e^{\frac{-\tau^2}{2}} \cdot \left(\frac{\tau^2}{2}\right)^i}{\frac{\Gamma\left(\frac{\nu 2}{2}\right) \cdot \Gamma\left(\frac{p}{2} + i\right)}{\Gamma\left[\frac{\nu 2}{2} + \left(\frac{p}{2} + i\right)\right]} \cdot i!} \cdot \left(\frac{p}{\nu 2}\right)^{\frac{p}{2} + i} \cdot \left(\frac{\nu 2}{\nu 2 + p \cdot f}\right)^{\frac{(p + \nu 2)}{2} + i} \cdot f^{\frac{p}{2} - 1 + i} \right]$$

The Probability of Type II Error

$$\beta := \int_0^f P(f)\, df$$

Figure 6.1 Program listing: calculation of Type II error (β)

The Probability of Type II Error on Multivariate Tsq Charts

Based on an Example Presented by T. W. Anderson (1958):

Chart Parameters

Number of Variables $p := 4$

Sample Size $N := 24$

Significance Level $\alpha := 0.01$

$$\nu 2 := \frac{n - p + 1}{2} \qquad n := N - 1$$

$$\nu 2 = 10$$

The Inverse Cumulative Probability of the F-distribution

$$f := qF[(1 - \alpha), p, \nu 2]$$

$$f = 5.994$$

Process Parameters

Non-centrality Parameter $\phi := 2.5$

$$\tau := \phi \cdot \sqrt{p + 1}$$

$$\tau^2 = 31.25$$

The Probability of Type II Error

$$\beta := \int_0^f P(f)\, df$$

$$\beta = 0.227$$

Figure 6.2 Illustrative example by Anderson (1958) to verify Type II error (β) calculation

Also, Tang tables are provided for p = 2, 3 …8, and ν_2 = 2, 4, 6, 7… 30, and 60, however, they do not provide the probability of Type II error (β) when $\nu_2 = 3$ and $\nu_2 = 5$. To validate the program provided in Figure 6.1, the Tang tables were regenerated. Appendix A provides two tables for two levels of significance ($\alpha = 0.01$ and $\alpha = 0.05$). Moreover, these tables present the probability of Type II error (β) for $\nu_2 = 3$ and $\nu_2 = 5$. The program could be used to calculate the probability of Type II error (β) under any significance levels of α.

A computer program was developed using the software MathCAD® 11 for Montgomery and Klatt's (1972) economic model presented in equation (2.42) to estimate the second term $E(C_M)$. The model was validated and verified for five values. An illustrative example providing all cost terms is presented in Appendix C.

6.1 Model Performance

To investigate the effect of the process and chart variables on the cost model, a 2^k factorial-designed experiment was performed. Since the cost estimates of this experiment were computed analytically, a single replication of this study was required. The two levels for this design were coded using equation (5.2). The levels were selected carefully to allow investigation around the median values of the process and chart variables of interest. A total of six variables were selected to investigate their effect on the response (ΔE(C)).

Four process variables and two chart variables were used in this study. The process variables were the level of correlation (ρ) between the two variables, the shift magnitude (δ) in σ units, the sampling frequency (K), and the cost coefficients (A_1, A_2, A_3), where $A_i = (a_i.\lambda`)/a_4$, and $\lambda` = \lambda / R$, R is the production rate per hour, and λ^{-1} is the mean time between shifts to the out-of-control state. The parameters a_1, a_2, a_3, and a_4 are defined as follows:

a_1 = fixed cost per sample

a_2 = per-unit cost of sampling

a_3 = mean cost of investigating and correcting a process which is out-of-control

a_4 = penalty cost of producing a defective units

The chart variables were subgroup size (n) and the probability of Type I error (alpha (α)). Table 6.1 summarizes the actual values and corresponding coded values of process and chart variables that were investigated.

TABLE 6.1 ACTUAL VALUES AND CORRESPONDING CODED LEVELS OF THE PROCESS AND CHART VARIABLES

Variables	*i*	Factors	Data Type	Actual Value	Coded Level
Process	1	Correlation (ρ)	Continuous	0 0.9	-1 1
	2	Shift Magnitude (δ)	Continuous	1 3	-1 1
	3	Sampling Frequency (K)	Continuous	0.02 0.1	-1 1
	4	Cost Coefficients (A_1, A_2, A_3)	Categorical	(0.004, 0.0004, 0.004) (0.020, 0.0020, 0.020)	-1 1
Chart	5	Subgroup (n)	Continuous	4 20	-1 1
	6	Alpha (α)	Continuous	0.001 0.005	-1 1

The first factor, the correlation levels (ρ) between pairs of variables, was selected to be (0.0 and 0.9). This allowed for calculating the expected cost or savings as the outcome of switching from univariate to multivariate SPC in two different scenarios. The first scenario was when there was no relationship between the two variables, and the second scenario was when a strong relationship existed. The second factor to be considered was the shift magnitude (δ). The shift magnitude levels were selected in increments of 0.5 σ at (1 and 3) to assess the effect on the response of small and large shifts in the process mean. The third factor was the sampling frequency (K), where levels were selected to be multiples of five at (0.02 and 0.10) to assess the effect of sampling every (k = 20 and k = 100 units). The fourth factor was the cost coefficients (A_1, A_2, A_3). The cost coefficients were selected to be a categorical factor, in order to assess the cost coefficients (A_1, A_2, A_3) effect on the response $\Delta E(C)$ at low and high levels. The levels were multiples of five, ranging around the median optimal cost selected by Knappenberger and Grandage's (1969) and Montgomery and Klatt's (1972) economic cost models. The cost coefficients low level was (0.004, 0.0004, and 0.004) and their high level was (0.020, 0.0020, and 0.020). The fifth factor was the subgroup size (n), which was selected at (4 and 20) in multiples of five to assess the effect of choosing a small as opposed to large subgroup size when determining the expected economic cost. The sixth factor was the probability of Type I error (alpha), which was selected in multiples of five to range around 3σ, which corresponds to 0.0027. Therefore, the alpha levels are selected at (0.001 and 0.005). In the univariate case, alpha (α) was transformed to L = 3.29 for alpha (α) = 0.001, and to L = 2.81 for alpha (α) = 0.005.

In conducting this study, the assumptions made by Knappenberger and Grandage's (1969) model were used for calculating both terms of the incremental cost model. The first assumption was that, on the average, the process shifts out-of-control after every 1,000 units (λ` = 0.001). The second assumption was that, the penalty cost of producing a defective product (a_4 = \$10). For this investigation, a 2^6 factorial design was employed. The total number of tests was 64. Table 6.2 provides the design matrix used in this evaluation in terms of the coded levels of the process and chart variables, as well as the $\Delta E(C)$ associated with each cost estimate.

TABLE 6.2 DESIGN MATRIX

Test	Cost (A)	Correlation (B)	Subgroup (C)	Shift (D)	Alpha (E)	Sampling Frequency (F)	Response (ΔE(C))
1	{ -1 }	-1	-1	-1	-1	-1	28.50
2	{ 1 }	-1	-1	-1	-1	-1	142.86
3	{ -1 }	1	-1	-1	-1	-1	25.51
4	{ 1 }	1	-1	-1	-1	-1	138.00
5	{ -1 }	-1	1	-1	-1	-1	60.11
6	{ 1 }	-1	1	-1	-1	-1	301.47
7	{ -1 }	1	1	-1	-1	-1	59.74
8	{ 1 }	1	1	-1	-1	-1	301.11
9	{ -1 }	-1	-1	1	-1	-1	30.24
10	{ 1 }	-1	-1	1	-1	-1	143.47
11	{ -1 }	1	-1	1	-1	-1	27.80
12	{ 1 }	1	-1	1	-1	-1	141.00
13	{ -1 }	-1	1	1	-1	-1	62.34
14	{ 1 }	-1	1	1	-1	-1	303.76
15	{ -1 }	1	1	1	-1	-1	62.14
16	{ 1 }	1	1	1	-1	-1	303.55
17	{ -1 }	-1	-1	-1	1	-1	28.30
18	{ 1 }	-1	-1	-1	1	-1	142.27
19	{ -1 }	1	-1	-1	1	-1	25.23
20	{ 1 }	1	-1	-1	1	-1	139.48
21	{ -1 }	-1	1	-1	1	-1	60.12
22	{ 1 }	-1	1	-1	1	-1	301.54
23	{ -1 }	1	1	-1	1	-1	59.77
24	{ 1 }	1	1	-1	1	-1	301.19
25	{ -1 }	-1	-1	1	1	-1	30.26
26	{ 1 }	-1	-1	1	1	-1	143.43
27	{ -1 }	1	-1	1	1	-1	27.05
28	{ 1 }	1	-1	1	1	-1	142.65
29	{ -1 }	-1	1	1	1	-1	62.34
30	{ 1 }	-1	1	1	1	-1	303.74
31	{ -1 }	1	1	1	1	-1	62.13
32	{ 1 }	1	1	1	1	-1	303.54

TABLE 6.2 DESIGN MATRIX (Continued)

Test	Cost (A)	Correlation (B)	Subgroup (C)	Shift (D)	Alpha (E)	Sampling Frequency (F)	Response (ΔE(C))
33	{ -1 }	-1	-1	-1	-1	1	7.24
34	{ 1 }	-1	-1	-1	-1	1	31.18
35	{ -1 }	1	-1	-1	-1	1	1.51
36	{ 1 }	1	-1	-1	-1	1	25.49
37	{ -1 }	-1	1	-1	-1	1	12.55
38	{ 1 }	-1	1	-1	-1	1	62.01
39	{ -1 }	1	1	-1	-1	1	11.49
40	{ 1 }	1	1	-1	-1	1	60.98
41	{ -1 }	-1	-1	1	-1	1	17.24
42	{ 1 }	-1	-1	1	-1	1	41.15
43	{ -1 }	1	-1	1	-1	1	3.78
44	{ 1 }	1	-1	1	-1	1	28.96
45	{ -1 }	-1	1	1	-1	1	23.05
46	{ 1 }	-1	1	1	-1	1	72.56
47	{ -1 }	1	1	1	-1	1	22.73
48	{ 1 }	1	1	1	-1	1	72.24
49	{ -1 }	-1	-1	-1	1	1	6.73
50	{ 1 }	-1	-1	-1	1	1	31.04
51	{ -1 }	1	-1	-1	1	1	1.68
52	{ 1 }	1	-1	-1	1	1	26.11
53	{ -1 }	-1	1	-1	1	1	12.51
54	{ 1 }	-1	1	-1	1	1	61.97
55	{ -1 }	1	1	-1	1	1	11.53
56	{ 1 }	1	1	-1	1	1	60.98
57	{ -1 }	-1	-1	1	1	1	16.65
58	{ 1 }	-1	-1	1	1	1	40.48
59	{ -1 }	1	-1	1	1	1	3.30
60	{ 1 }	1	-1	1	1	1	27.90
61	{ -1 }	-1	1	1	1	1	22.93
62	{ 1 }	-1	1	1	1	1	72.43
63	{ -1 }	1	1	1	1	1	22.61
64	{ 1 }	1	1	1	1	1	72.11

TABLE 6.3 ANOVA FOR THE MODEL

Source	Sum of Squares	df	Mean Square	F Value	p-value Prob > F	
Model	545638.4	34	16048.19	166711.7	< 0.0001	*significant*
Cost (A)	183990.5	1	183990.5	1911329	< 0.0001	
Correlation (B)	166.4611	1	166.4611	1729.23	< 0.0001	
Subgroup (C)	57407.65	1	57407.65	596361.7	< 0.0001	
Shift (D)	448.429	1	448.429	4658.366	< 0.0001	
Alpha (E)	0.047991	1	0.047991	0.498541	0.4858*	
Sampling Frequency (F)	168052.4	1	168052.4	1745761	< 0.0001	
AB	0.142158	1	0.142158	1.476767	0.2341*	
AC	23357.08	1	23357.08	242637.8	< 0.0001	
AD	0.023013	1	0.023013	0.239066	0.6286*	
AE	0.239758	1	0.239758	2.490656	0.1254*	
AF	79198.55	1	79198.55	822729.8	< 0.0001	
BC	121.0473	1	121.0473	1257.462	< 0.0001	
BD	7.561049	1	7.561049	78.54564	< 0.0001	
BE	0.277459	1	0.277459	2.882297	0.1003*	
BF	44.68851	1	44.68851	464.2328	< 0.0001	
CD	26.07753	1	26.07753	270.8984	< 0.0001	
CF	22063.72	1	22063.72	229202.1	< 0.0001	
DF	158.9149	1	158.9149	1650.839	< 0.0001	
EF	0.33842	1	0.33842	3.515572	0.0709*	
ABC	0.13558	1	0.13558	1.408426	0.2449*	
ABD	0.536313	1	0.536313	5.57132	0.0252	
ABE	0.274779	1	0.274779	2.854458	0.1018*	
ABF	0.031787	1	0.031787	0.330208	0.5700*	
ACD	0.008542	1	0.008542	0.088737	0.7679*	
ACF	10486.21	1	10486.21	108932.8	< 0.0001	
AEF	0.216312	1	0.216312	2.247093	0.1447*	
BCD	12.89707	1	12.89707	133.9772	< 0.0001	
BCF	34.94203	1	34.94203	362.9846	< 0.0001	
BDF	16.80874	1	16.80874	174.6124	< 0.0001	
BEF	0.042949	1	0.042949	0.446162	0.5094*	
CDF	19.60343	1	19.60343	203.6442	< 0.0001	
ABCD	0.54935	1	0.54935	5.70675	0.0236	
ABEF	0.394744	1	0.394744	4.100673	0.0522	
BCDF	21.57727	1	21.57727	224.1488	< 0.0001	
Residual	2.791631	29	0.096263			
Cor Total	545641.2	63				

* The term included to maintain hierarchy

6.2 Statistical Analysis

An analysis of variance was constructed in order to determine the significance of the main effects and interactions. The model was refined by removing all non-significant variables from the full model. Table 6.3 depicts the ANOVA table for the final model and all the main effects and interactions that are significant. Note that the terms included with an (*) are done in order to maintain hierarchy.

A normal probability plot of the residuals was constructed and examined to validate the underling assumptions of error normality. Figure 6.3 shows the residuals plotted against their expected value. The normal probability plot indicated no violation of the normality assumption.

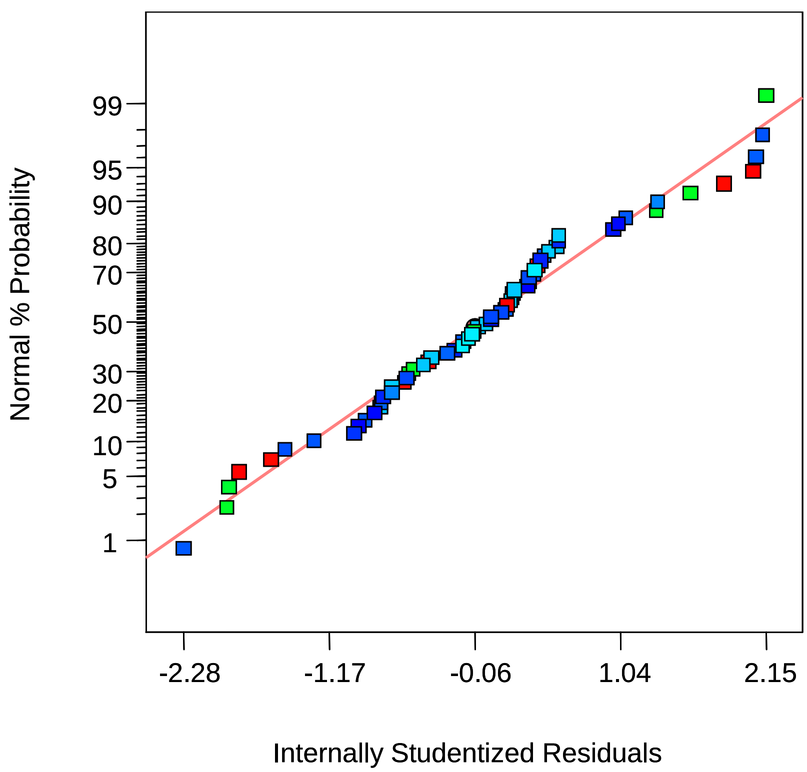


Figure 6.3 Normal probability plot of residuals for (ΔE(C))

Figure 6.4 shows a plot of residuals against their predicted values of the response indicating no violation of the normality assumption.

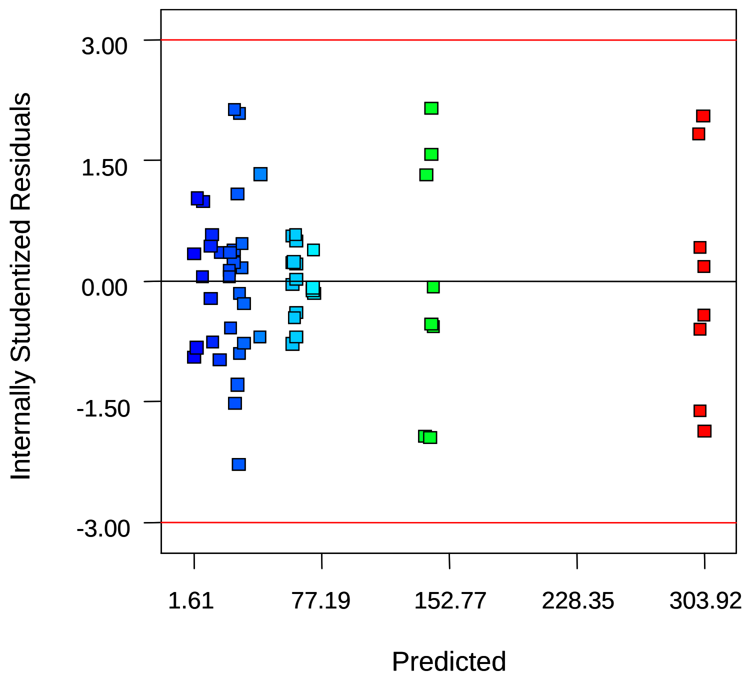


Figure 6.4 Residuals vs. predicted values

Table 6.3 shows that all three four-factor interactions including cost coefficients, correlation, subgroup size, and shift magnitude (ABCD); cost coefficients, correlation, probability of Type I error (alpha), and sampling frequency (ABEF); and correlation, subgroup size, shift magnitude, and sampling frequency (BCDF) were all significant, with p-value < 0.0001. In addition, the three-factor interaction including cost coefficients, subgroup size, and sampling frequency (ACF) was significant with p-value < 0.0001. All other significant main effects and interactions were included to maintain hierarchy.

6.3 Interpretation

To investigate statistically significant interactions in the model, interaction plots were constructed. Figure 6.5 shows a four-factor interaction plot between the cost coefficients, correlation, subgroup size and the shift magnitude. To examine the effect of correlation on the response ΔE(C), this figure provides the response behavior under two levels of correlation. It was noted that all values of ΔE(C) were positive, which indicated net savings in switching from the traditional $\bar{x}$ charts to the T^2 chart. At no correlation between the charted variables (ρ = 0.0), minimum savings ($17.68) were observed at low-cost coefficients, small shift magnitude, and small subgroup size. However, maximum savings ($188.12) were observed at high-cost coefficients, large shift magnitude, and large subgroup size.

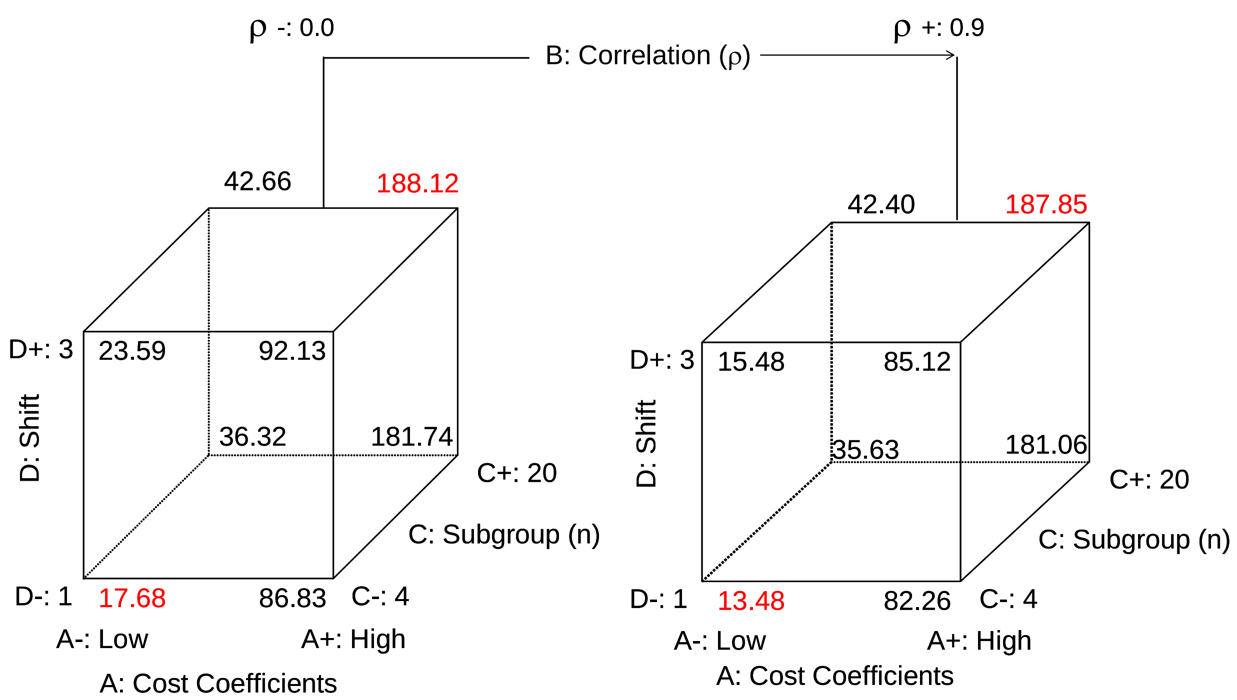


Figure 6.5 Cost Coefficients, correlation, subgroup size, and shift interaction plot

As the correlation increased between the charted variables to ($\rho = 0.9$), minimum savings decreased to \$13.48 and maximum savings decreased to \$187.85 at the same levels of cost coefficients, shift magnitude, and subgroup size in the case of ($\rho = 0.0$), respectively. This amounted to a 27.76 percent reduction in minimum savings and 0.15 percent reduction in maximum savings.

Figure 6.6 depicts the four-factor interaction plot between cost coefficients, correlation, alpha, and sampling frequency. To examine the effect of correlation on the response $\Delta E(C)$, this figure provides the response behavior under the two levels of correlation. It was noted that all values of $\Delta E(C)$ were positive, which indicates net savings in switching from traditional $\bar{x}$ charts to the T^2 chart. At no correlation between the charted variables ($\rho = 0.0$), minimum savings (\$14.57) were observed at low-cost coefficients, large alpha level, and high sampling frequency or sampling every 100 units. However, maximum savings (\$222.84) were observed at high-cost coefficients, small alpha level, and low sampling frequency or sampling every 20 units.

As the correlation between the charted variables increased to ($\rho = 0.9$), minimum savings decreased to \$9.77 at the same levels of cost coefficients, alpha, and sampling frequency in the case of ($\rho = 0.0$). Maximum savings decreased to \$221.71 at high-cost coefficients, high alpha level, and low sampling frequency. This amounted to a 32.95 percent reduction in minimum savings, and 0.51 percent reduction in maximum savings.

Figure 6.7 presents the four-factor interaction plot between the correlation, subgroup size, shift magnitude, and sampling frequency. To examine the effect of correlation on the response $\Delta E(C)$, this figure provides the response behavior under the two levels of correlation.

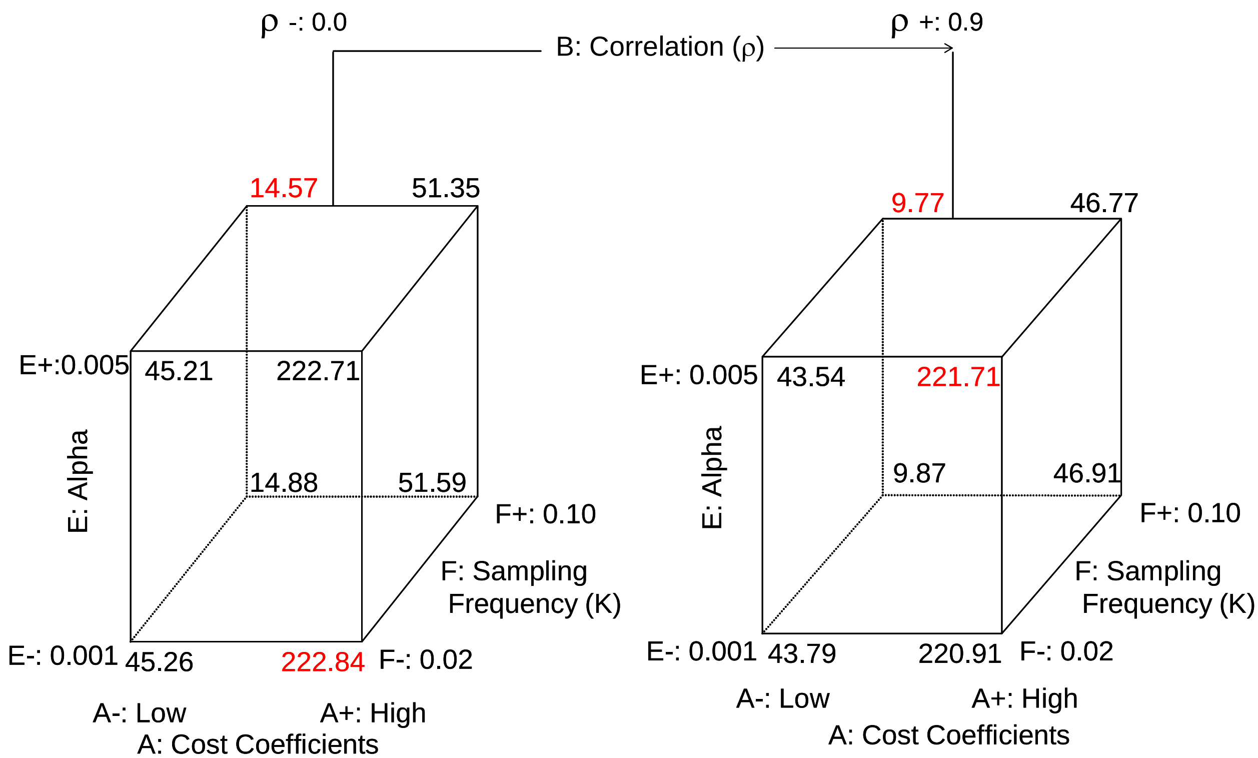


Figure 6.6 Cost coefficients, correlation, alpha, and sampling frequency interaction plot

It was noted that all values of ΔE(C) were positive, which indicates net savings in switching from the traditional $\bar{x}$ charts to the T^2 chart. At no correlation between the charted variables (ρ = 0.0), minimum savings ($6.87) were observed at high sampling frequency or sampling every 100 units, small shift magnitude, and small subgroup size. However, maximum savings ($62.32) were observed at large shift magnitude, large subgroup size, and low sampling frequency or sampling every 20 units.

As the correlation between the charted variables increased to (ρ = 0.9), minimum savings decreased to $1.65, and maximum savings decreased to $62.15 at the same levels of subgroup size, shift magnitude, and sampling frequency, respectively. This amounted to a 76 percent reduction in minimum savings, and 0.27 percent reduction in maximum savings.

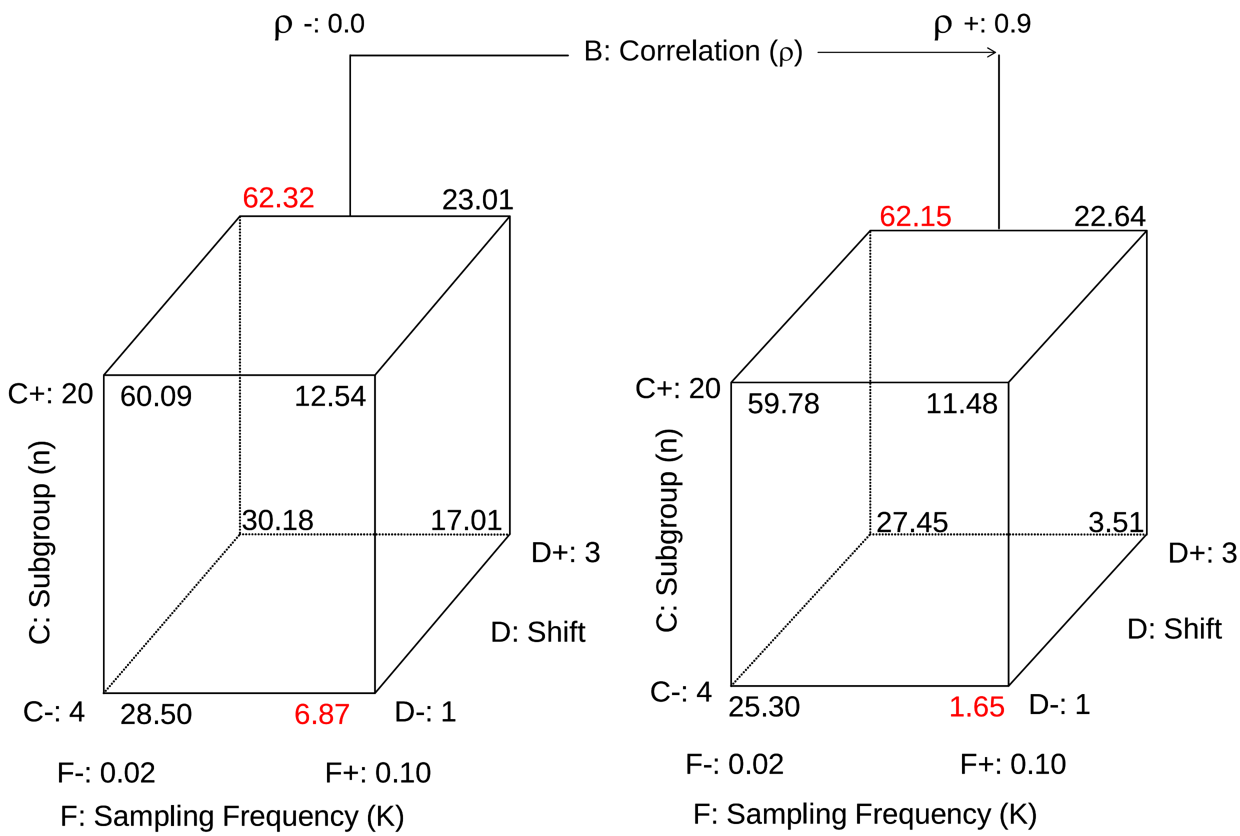


Figure 6.7 Correlation, subgroup size, shift magnitude, and sampling frequency interaction plot

Figure 6.8 shows the three-factor interaction plot between the cost coefficients, subgroup size, and sampling frequency.

Minimum savings ($11.95) occurred at low-cost coefficients, high sampling frequency or sampling every 100 units, and small subgroup size. However, maximum savings ($302.65) were observed at high-cost coefficients, large subgroup size, and low sampling frequency or sampling every 20 units. From figure 6.8, it is noted that all values of $\Delta E(C)$ are positive, which indicates net savings in switching from the traditional $\bar{x}$ charts to the T^2 chart.

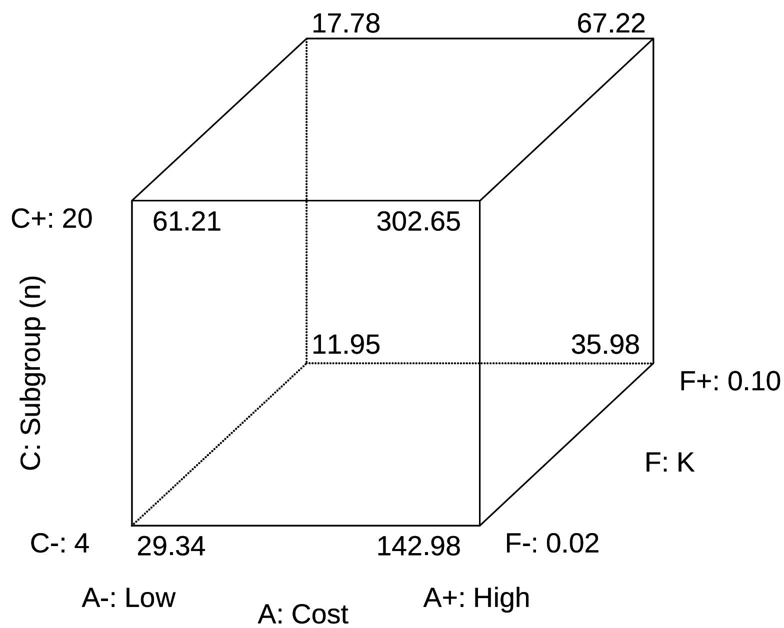


Figure 6.8 Cost coefficients, subgroup size, and sampling frequency interaction plot

Results of this study are based on the assumptions made for rate of production, penalty of producing a defective unit, and levels of process and chart variables specified in this investigation.

CHAPTER 7

SUMMARY AND CONCLUSIONS

This research offered a comprehensive review of the literature pertaining to multivariate control charts and their underlying assumptions. It also reviewed the literature concerning the economic design of control charts for both univariate and multivariate applications. Levels of correlation that mandate the use of multivariate charts and the statistical effect of mis-specifying the process model while applying traditional Shewhart charts was not provided in the literature. In addition, estimates of the economic feasibility of using multivariate T^2 control charts as an alternative to traditional univariate $\bar{x}$ charts was not supplied.

Advancements in technology have raised the need for simultaneous monitoring of several quality characteristics that could be correlated. This need is served best by utilizing multivariate SPC techniques. However, multivariate control charting procedures are computationally intensive, which make multivariate SPC less popular. Increased availability of high-speed computers and statistical software programs, formerly available only to very few, has made the statistical computations of multivariate SPC easier. The lack of knowledge of the correlation between quality characteristics being charted does not mean that the relationship is absent.

7.1 Summary and Results

The objective of this research was twofold. The first objective was to identify levels of correlation between quality characteristics at which the statistical performance of multiple $\bar{x}$ control charts deteriorate in comparison to an equivalent T^2 chart. In achieving this objective, simulated data was analyzed using $\bar{x}$ and T^2 charts for the case of two quality characteristics. Chapters 4 and 5 present the simulation analyses and results. A central composite design was chosen to examine the effect of changes in the level of correlation between characteristics coupled with changes in the process model and chart design variables. The process variables included correlation between the pairs of characteristics (ρ) and the shift magnitude of the process mean (δ). The chart variables were comprised of the subgroup size (n) and the probability of Type I error (α). This study was conducted for the two chart types (Shewhart $\bar{x}$ and Hotelling T^2). It was concluded that when using the T^2 chart, changes in the level of correlation between the characteristics did not result in a significant increase in the average probability error (α). On the contrary, while using the $\bar{x}$ chart, changes in the level of correlation resulted in a significant difference in the average probability of Type I error. As the correlation level increased to 0.8 a 32 percent increase in the average rate of false alarms (α) occurred. In addition, a threshold was identified at ($\rho \geq 0.48$), indicating deterioration in the performance of the $\bar{x}$ charts in comparison to the T^2 chart. Due to the quadratic effect of the correlation coefficient ρ, it could be concluded that a similar threshold exists at ($\rho \leq - 0.48$). Using the T^2 chart as an alternative SPC technique to the $\bar{x}$ chart resulted in a significant reduction in the average probability of Type II error (β).

The second objective was to assess the economic feasibility of utilizing the T^2 chart as an alternative to the $\bar{x}$ chart. This investigation was conducted for the case of two quality characteristics, by utilizing the economic design models developed by Knappenberger and Grandage's (1969), and Montgomery and Klatt's (1972). An incremental cost model was constructed to examine the cost and worth of switching from univariate to multivariate SPC techniques under specified levels of process and chart variables. Chapter 6 provides the economic benefits of using a T^2 chart instead of two $\bar{x}$ charts. A computer program was created using MathCAD® 11 software for this application. A 2^6 factorial-design was performed. Four process variables and two chart variables were considered. The level of correlation (ρ) between the two characteristics varied from 0.0 to 0.9. The process variables included the shift magnitude, sampling frequency, and the cost coefficients. The chart variables included the subgroup size and the probability of Type I error. The results indicated that using a T^2 chart for monitoring two characteristics whether they are correlated or not, will result in significant net savings. However, when the characteristics are highly correlated, the net savings were reduced due to the increased power of the T^2 chart.

7.2 Future Research

Future research in this area could be extended to include the statistical performance of the $\bar{x}$ chart in comparison to the T^2 chart for more than two variables under different levels of correlation. Moreover, future research on the statistical performance of control charts could include more sensitive charting techniques such as the CUSUM and EWMA and their multivariate direct analog charts.

The incremental cost model constructed in this research could also be extended to examine the cost and worth of switching from univariate to multivariate SPC techniques in the case of more than two quality characteristics. In addition, the economic cost models that were employed are traditional economic models that are mainly utilized to maintain current quality levels. Further investigation could be conducted employing proactive economic models that are designed to achieve improved levels of performance.

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APPENDICES

APPENDIX A

PROBABILITY OF TYPE II ERROR (β) FOR MULTIVARIATE T^2 CONTROL CHART

Table A.1 Probability of Type II Error (β) for Multivariate T^2 Control Chart
P = 4, α = 0.01

	ϕ									
ν_2	**1**	**1.5**	**2**	**2.5**	**3**	**4**	**5**	**6**	**7**	**8**
2	0.978	0.962	0.942	0.915	0.884	0.810	0.724	0.631	0.536	0.444
3	0.969	0.938	0.888	0.820	0.735	0.539	0.346	0.176	0.003	0.000
4	0.960	0.909	0.822	0.700	0.557	0.280	0.102	0.027	0.005	0.001
5	0.952	0.879	0.752	0.580	0.397	0.125	0.023	0.002		
6	0.943	0.849	0.685	0.475	0.277	0.053	0.005			
7	0.936	0.821	0.624	0.389	0.191	0.018				
8	0.928	0.796	0.571	0.322	0.136	0.010				
9	0.922	0.773	0.526	0.269	0.098	0.003				
10	0.916	0.752	0.487	0.227	0.073	0.002				
11	0.911	0.733	0.453	0.195	0.055	0.001				
12	0.906	0.716	0.424	0.169	0.042	0.001				
13	0.901	0.700	0.389	0.148	0.034					
14	0.897	0.687	0.376	0.131	0.028					
15	0.893	0.674	0.357	0.117	0.022					
16	0.890	0.662	0.340	0.106	0.018					
17	0.886	0.652	0.325	0.096	0.015					
18	0.883	0.642	0.312	0.088	0.013					
19	0.880	0.633	0.301	0.081	0.011					
20	0.878	0.625	0.290	0.075	0.010					
21	0.876	0.618	0.280	0.070	0.009					
22	0.873	0.611	0.272	0.066	0.008					
23	0.871	0.604	0.264	0.062	0.007					
24	0.869	0.598	0.257	0.059	0.006					
25	0.867	0.593	0.250	0.056	0.006					
26	0.865	0.588	0.244	0.053	0.005					
27	0.864	0.583	0.239	0.050	0.005					
28	0.862	0.578	0.234	0.048	0.005					
29	0.861	0.574	0.229	0.046	0.004					
30	0.860	0.570	0.225	0.044	0.004					
60	0.837	0.509	0.165	0.024	0.001					

Table A.2 Probability of Type II Error (β) for Multivariate T^2 Control Chart

P = 4, α = 0.05

	ϕ									
ν_2	**1**	**1.5**	**2**	**2.5**	**3**	**4**	**5**	**6**	**7**	**8**
2	0.892	0.824	0.738	0.640	0.537	0.345	0.195	0.097	0.043	0.017
3	0.861	0.744	0.592	0.429	0.283	0.091	0.019	0.003		
4	0.833	0.673	0.471	0.279	0.139	0.020	0.001			
5	0.810	0.615	0.381	0.186	0.070	0.004				
6	0.791	0.567	0.314	0.128	0.038	0.001				
7	0.774	0.529	0.265	0.092	0.022					
8	0.760	0.497	0.229	0.069	0.013					
9	0.748	0.471	0.201	0.054	0.008					
10	0.738	0.449	0.179	0.043	0.006					
11	0.729	0.430	0.161	0.035	0.004					
12	0.721	0.414	0.148	0.030	0.003					
13	0.714	0.401	0.136	0.025	0.002					
14	0.708	0.389	0.127	0.022	0.002					
15	0.702	0.378	0.119	0.019	0.002					
16	0.697	0.369	0.112	0.017	0.001					
17	0.693	0.361	0.106	0.016	0.001					
18	0.689	0.354	0.101	0.014	0.001					
19	0.685	0.347	0.097	0.013	0.001					
20	0.681	0.341	0.093	0.012	0.001					
21	0.678	0.335	0.089	0.011	0.001					
22	0.675	0.331	0.086	0.010	0.001					
23	0.672	0.326	0.083	0.010						
24	0.670	0.322	0.080	0.009						
25	0.668	0.318	0.078	0.009						
26	0.665	0.315	0.076	0.008						
27	0.663	0.312	0.074	0.008						
28	0.661	0.309	0.072	0.008						
29	0.660	0.306	0.071	0.007						
30	0.658	0.303	0.069	0.007						
60	0.632	0.265	0.049	0.004						

APPENDIX B

ILLUSTRATIVE EXAMPLE OF ECONOMIC COST MODEL FOR UNIVARIATE $\bar{X}$ CHARTS BASED ON KNAPPENBERGER AND GRANDGE'S MODEL (1969)

Sample Size **Critical Region Parameter** **Interval Between Samples**

$N := 4$ $L := 3$ $K := 0.04$

Process Shifts Out-of-Control Every 1,000 Units **Process Mean**

$\lambda := 0.001$ $\mu 0 := 0$

Number of Units Produced Between Samples

$k := \frac{K}{\lambda}$ $k = 40$

Number of Out-of-Control States

$s0 := 0$ $s1 := 1$ $s2 := 2$ $s3 := 3$ $s4 := 4$ $s5 := 5$ $s6 := 6$

Conditional Probability of Process Shifting

Priori Distribution Parameter

$pai := 0.376$

$$\mu 1 := \frac{s6 \cdot pai}{1 - (1 - pai)^{s6}}$$

Average Shift in σ Units

$\mu 1 = 2.4$

Conditional Probability of Producing a Defective Unit Given $\mu = \mu_i$ (i = 1, 2... 6)

$\phi0 := \text{cnorm}(-3)\cdot 2$ $\phi1 := \text{cnorm}(-2)$ $\phi2 := \text{cnorm}(-1)$ $\phi3 := \text{cnorm}(0)$

$\phi0 = 0.003$ $\phi1 = 0.0228$ $\phi2 = 0.1587$ $\phi3 = 0.5$

$\phi4 := \text{cnorm}(1)$ $\phi5 := \text{cnorm}(2)$ $\phi6 := \text{cnorm}(3)$

$\phi4 = 0.8413$ $\phi5 = 0.9772$ $\phi6 = 0.9987$

$\phi := (\phi0\ \ \phi1\ \ \phi2\ \ \phi3\ \ \phi4\ \ \phi5\ \ \phi6)$ $\phi = (0.003\ \ 0.023\ \ 0.159\ \ 0.5\ \ 0.841\ \ 0.977\ \ 0.999)$

Conditional Probability of Rejecting H_0 Given that $\mu = \mu_i$

$q0 := \text{cnorm}(s0\cdot\sqrt{N} - L)\cdot 2$ $q1 := \text{cnorm}(s1\cdot\sqrt{N} - L)$ $q2 := \text{cnorm}(s2\cdot\sqrt{N} - L)$

$q0 = 0.003$ $q1 = 0.1587$ $q2 = 0.8413$

$q3 := \text{cnorm}(s3\cdot\sqrt{N} - L)$ $q4 := \text{cnorm}(s4\cdot\sqrt{N} - L)$ $q5 := \text{cnorm}(s5\cdot\sqrt{N} - L)$

$q3 = 0.9987$ $q4 = 1$ $q5 = 1$

$q6 := \text{cnorm}(s6\cdot\sqrt{N} - 3)$

$q6 = 1$

$q := (q0\ \ q1\ \ q2\ \ q3\ \ q4\ \ q5\ \ q6)$ $q = (0.003\ \ 0.159\ \ 0.841\ \ 0.999\ \ 1\ \ 1\ \ 1)$

Probability that Process is in State μ_i at Time Test is Performed

$p0 := \exp(-\lambda \cdot k)$

$$p1 := \frac{(1 - \exp(-\lambda \cdot k)) \cdot s6! \cdot pai^{1} \cdot (1 - pai)^{s6-1}}{\left[1 - (1 - pai)^{s6}\right] \cdot 1! \cdot (s6 - 1)!}$$

$p0 = 0.9608$

$p1 = 0.009$

$$p2 := \frac{(1 - \exp(-\lambda \cdot k)) \cdot s6! \cdot pai^{2} \cdot (1 - pai)^{s6-2}}{\left[1 - (1 - pai)^{s6}\right] \cdot 2! \cdot (s6 - 2)!}$$

$$p3 := \frac{(1 - \exp(-\lambda \cdot k)) \cdot s6! \cdot pai^{3} \cdot (1 - pai)^{s6-3}}{\left[1 - (1 - pai)^{s6}\right] \cdot 3! \cdot (s6 - 3)!}$$

$p2 = 0.013$

$p3 = 0.011$

$$p4 := \frac{(1 - \exp(-\lambda \cdot k)) \cdot s6! \cdot pai^{4} \cdot (1 - pai)^{s6-4}}{\left[1 - (1 - pai)^{s6}\right] \cdot 4! \cdot (s6 - 4)!}$$

$$p5 := \frac{(1 - \exp(-\lambda \cdot k)) \cdot s6! \cdot pai^{5} \cdot (1 - pai)^{s6-5}}{\left[1 - (1 - pai)^{s6}\right] \cdot 5! \cdot (s6 - 5)!}$$

$p4 = 0.005$

$p5 = 0.001$

$$p6 := \frac{(1 - \exp(-\lambda \cdot k)) \cdot s6! \cdot pai^{6} \cdot (1 - pai)^{s6-6}}{\left[1 - (1 - pai)^{s6}\right] \cdot 6! \cdot (s6 - 6)!}$$

$p6 = 0$

$p := (p0 \;\; p1 \;\; p2 \;\; p3 \;\; p4 \;\; p5 \;\; p6)$

$p = (0.961 \; 0.009 \; 0.013 \; 0.011 \; 0.005 \; 0.001 \; 0)$

$b1 := p1$ $b2 := p2$ $b3 := p3$ $b4 := p4$ $b5 := p5$ $b6 := p6$

$b1 = 0.0089$ $b2 = 0.0134$ $b3 = 0.0108$ $b4 = 0.0049$ $b5 = 0.0012$ $b6 = 0.0001$

$$b12 := q1 \cdot p2 + \frac{(1 - q1) \cdot p2}{1 - p0}$$

$b12 = 0.2896$

$$b15 := q1 \cdot p5 + \frac{(1 - q1) \cdot p5}{1 - p0}$$

$b15 = 0.0253$

$$b13 := q1 \cdot p3 + \frac{(1 - q1) \cdot p3}{1 - p0}$$

$b13 = 0.2327$

$$b16 := q1 \cdot p6 + \frac{(1 - q1) \cdot p6}{1 - p0}$$

$b16 = 0.0025$

$$b14 := q1 \cdot p4 + \frac{(1 - q1) \cdot p4}{1 - p0}$$

$b14 = 0.1052$

$b21 := q2 \cdot p1$

$b21 = 0.00748$

$b31 := q3 \cdot p1$

$b31 = 0.00888$

$b32 := q3 \cdot p2$

$b32 = 0.0134$

$b42 := q4 \cdot p2$

$b42 = 0.0134$

$$b23 := q2 \cdot p3 + \frac{(1 - q2) \cdot p3}{1 - p0}$$

$b23 = 0.0526$

$b43 := q4 \cdot p3$

$b43 = 0.0107641$

$$b24 := q2 \cdot p4 + \frac{(1 - q2) \cdot p4}{1 - p0}$$

$b24 = 0.0238$

$$b25 := q2 \cdot p5 + \frac{(1 - q2) \cdot p5}{1 - p0}$$

$b25 = 0.0057$

$$b26 := q2 \cdot p6 + \frac{(1 - q2) \cdot p6}{1 - p0}$$

$b26 = 0.0006$

$$b34 := q3 \cdot p4 + \frac{(1 - q3) \cdot p4}{1 - p0}$$

$b34 = 0.005$

$$b35 := q3 \cdot p5 + \frac{(1 - q3) \cdot p5}{1 - p0}$$

$b35 = 0.00121$

$$b36 := q3 \cdot p6 + \frac{(1 - q3) \cdot p6}{1 - p0}$$

$b36 = 0.000121643588$

$b41 := q4 \cdot p1$

$b41 = 0.00889388$

$b52 := q5 \cdot p2$

$b52 = 0.01339784$

$b53 := q5 \cdot p3$

$b53 = 0.01076408$

$b54 := q5 \cdot p4$

$b54 = 0.004864534$

$b51 := q5 \cdot p1$

$b51 = 0.00889388$

$b62 := q6 \cdot p2$

$b62 = 0.01339784$

$b63 := q6 \cdot p3$

$b63 = 0.01076408$

$b64 := q6 \cdot p4$

$b64 = 0.00486453$

$b61 := q6 \cdot p1$

$b61 = 0.00889388$

$$b45 := q4 \cdot p5 + \frac{(1 - q4) \cdot p5}{1 - p0}$$

$b45 = 0.001172$

$$b46 := q4 \cdot p6 + \frac{(1 - q4) \cdot p6}{1 - p0}$$

$b46 = 0.000118$

$b65 := q6 \cdot p5$

$b65 = 0.00117248$

$$b56 := q5 \cdot p6 + \frac{(1 - q5) \cdot p6}{1 - p0}$$

$b56 = 0.0001$

$$x11 := q1 \cdot p1 + \frac{(1 - q1) \cdot (p1)}{(1 - p0)}$$

$$x33 := q3 \cdot p3 + \frac{(1 - q3) \cdot (p1 + p2 + p3)}{(1 - p0)}$$

$b11 := x11 - 1$ $\quad b11 = -0.8078$

$b33 := x33 - 1$ $\quad b33 = -0.9881$

$$x22 := q2 \cdot p2 + \frac{(1 - q2) \cdot (p1 + p2)}{(1 - p0)}$$

$$x44 := q4 \cdot p4 + \frac{(1 - q4) \cdot (p1 + p2 + p3 + p4)}{(1 - p0)}$$

$b22 := x22 - 1$ $\quad b22 = -0.8985$

$b44 := x44 - 1$ $\quad b44 = -0.9951$

$$x55 := q5 \cdot p5 + \frac{(1 - q5) \cdot (p1 + p2 + p3 + p4 + p5)}{(1 - p0)}$$

$b55 := x55 - 1$ $\quad b55 = -0.9988$

$$x66 := q6 \cdot p6 + \frac{(1 - q6)\cdot(p1 + p2 + p3 + p4 + p5 + p6)}{(1 - p0)}$$

$b66 := x66 - 1 \qquad b66 = -0.9999$

$$B := \begin{pmatrix} b11 & b12 & b13 & b14 & b15 & b16 & 1 \\ b21 & b22 & b23 & b24 & b25 & b26 & 1 \\ b31 & b32 & b33 & b34 & b35 & b36 & 1 \\ b41 & b42 & b43 & b44 & b45 & b46 & 1 \\ b51 & b52 & b53 & b54 & b55 & b56 & 1 \\ b61 & b62 & b63 & b64 & b65 & b66 & 1 \\ b1 & b2 & b3 & b4 & b5 & b6 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} -0.80775 & 0.2896 & 0.23267 & 0.10515 & 0.02534 & 0.00255 & 1 \\ 0.00748 & -0.89853 & 0.05261 & 0.02378 & 0.00573 & 0.00058 & 1 \\ 0.00888 & 0.01338 & -0.98811 & 0.00503 & 0.00121 & 0.00012 & 1 \\ 0.00889 & 0.0134 & 0.01076 & -0.99514 & 0.00117 & 0.00012 & 1 \\ 0.00889 & 0.0134 & 0.01076 & 0.00486 & -0.99883 & 0.00012 & 1 \\ 0.00889 & 0.0134 & 0.01076 & 0.00486 & 0.00117 & -0.99988 & 1 \\ 0.00889 & 0.0134 & 0.01076 & 0.00486 & 0.00117 & 0.00012 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} -1.224 & -0.371 & -0.287 & -0.13 & -0.031 & -0.003 & 2.046 \\ 0.002 & -1.096 & -0.045 & -0.021 & -0.005 & -0 & 1.166 \\ 0 & 0 & -1.001 & -0 & -0 & -0 & 1.001 \\ 0 & 0 & 0 & -1 & -0 & -0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0.011 & 0.018 & 0.014 & 0.006 & 0.002 & 0 & 0.949 \end{pmatrix}$$

Probability that Process is in State i ($\mu = \mu_i$) at Time Sample is Selected

$\alpha 1 := 0.011 \quad \alpha 2 := 0.018 \quad \alpha 3 := 0.014 \quad \alpha 4 := 0.006 \quad \alpha 5 := 0.002 \quad \alpha 6 := 0.00 \quad \alpha 0 := 0.949$

$\alpha := (\alpha 0 \;\; \alpha 1 \;\; \alpha 2 \;\; \alpha 3 \;\; \alpha 4 \;\; \alpha 5 \;\; \alpha 6) \qquad \alpha = (0.949 \;\; 0.011 \;\; 0.018 \;\; 0.014 \;\; 0.006 \;\; 0.002 \;\; 0)$

Conditional Expectation of Occurrence of Assignable Cause Within an Interval of Sampling

$$F := \frac{\left[1 - (1 + \lambda \cdot k) \cdot \exp(-\lambda \cdot k)\right]}{\lambda \cdot k \cdot (1 - \exp(-\lambda \cdot k))}$$

$F = 0.4967$

$\gamma 0 := \alpha 0 \cdot p0 + F \cdot \alpha 0 \cdot (1 - p0)$

$\gamma 0 = 0.9303$

$$x1 := \alpha 1 \cdot \frac{(p1)}{1 - p0} \qquad x1 = 0.002$$

$$x2 := \alpha 0 \cdot (1 - F) \cdot p1 \qquad x2 = 0.004$$

$$x3 := \alpha 1 \cdot \left(\frac{p0}{1 - p0}\right) \cdot (1 - F) \qquad x3 = 0.1357$$

$$x4 := \left(\frac{\alpha 1 \cdot F}{1 - p0}\right) \cdot (p2 + p3 + p4 + p5 + p6) \qquad x4 = 0.004$$

$\gamma 1 := x1 + x2 + x4$

$\gamma 1 = 0.011$

$$y1 := \alpha 2 \cdot \frac{(p1 + p2)}{1 - p0} \qquad y1 = 0.01$$

$$y2 := \alpha 0 \cdot (1 - F) \cdot p2 \qquad y2 = 0.006$$

$$y3 := \left[\alpha 1 \cdot \left(\frac{p2}{1 - p0}\right) \cdot (1 - F)\right]$$

$y3 = 0.002$

$$y4 := \left(\frac{\alpha 2 \cdot F}{1 - p0}\right) \cdot (p3 + p4 + p5 + p6)$$

$y4 = 0.004$

$\gamma 2 := y1 + y2 + y3 + y4$

$\gamma 2 = 0.0224$

$$z1 := \alpha 3 \cdot \frac{(p1 + p2 + p3)}{1 - p0} \qquad z1 = 0.012$$

$$z2 := \alpha 0 \cdot (1 - F) \cdot p3 \qquad z2 = 0.005$$

$$z3 := \left[\alpha 1 \cdot \left(\frac{p3}{1 - p0}\right) \cdot (1 - F)\right] + \alpha 2 \cdot \left(\frac{p3}{1 - p0}\right) \cdot (1 - F)$$

$$z3 = 0.004$$

$$z4 := \left(\frac{\alpha 3 \cdot F}{1 - p0}\right) \cdot (p4 + p5 + p6)$$

$$z4 = 0.001$$

$$\gamma 3 := z1 + z2 + z3 + z4$$

$$\gamma 3 = 0.022$$

$$h1 := \alpha 4 \cdot \frac{(p1 + p2 + p3 + p4)}{1 - p0} \qquad h1 = 0.006$$

$$h2 := \alpha 0 \cdot (1 - F) \cdot p4 \qquad h2 = 0.002$$

$$h3 := \left[\alpha 1 \cdot \left(\frac{p4}{1 - p0}\right) \cdot (1 - F)\right] + \left[\alpha 2 \cdot \left(\frac{p4}{1 - p0}\right) \cdot (1 - F)\right] + \left[\alpha 3 \cdot \left(\frac{p4}{1 - p0}\right) \cdot (1 - F)\right] \qquad h3 = 0.003$$

$$h4 := \left(\frac{\alpha 4 \cdot F}{1 - p0}\right) \cdot (p5 + p6) \qquad h4 = 0$$

$$\gamma 4 := h1 + h2 + h3 + h4$$

$$\gamma 4 = 0.011$$

$$j1 := \alpha5 \cdot \frac{(p1 + p2 + p3 + p4 + p5)}{1 - p0}$$ $$j1 = 0.002$$

$$j2 := \alpha0 \cdot (1 - F) \cdot p5$$ $$j2 = 0.001$$

$$j3 := \left[\alpha1 \cdot \left(\frac{p5}{1 - p0}\right) \cdot (1 - F)\right] + \left[\alpha2 \cdot \left(\frac{p5}{1 - p0}\right) \cdot (1 - F)\right] + \left[\alpha3 \cdot \left(\frac{p5}{1 - p0}\right) \cdot (1 - F)\right] + \left[\alpha4 \cdot \left(\frac{p5}{1 - p0}\right) \cdot (1 - F)\right]$$

$$j3 = 0.001$$

$$j4 := \left(\frac{\alpha5 \cdot F}{1 - p0}\right) \cdot (p6)$$ $$j4 = 0$$

$$\gamma5 := j1 + j2 + j3 + j4$$

$$\gamma5 = 0.003$$

$$l1 := \alpha6 \cdot \frac{(p1 + p2 + p3 + p4 + p5 + p6)}{1 - p0}$$ $$l1 = 0$$

$$l2 := \alpha0 \cdot (1 - F) \cdot p6$$ $$l2 = 0$$

$$l3 := \left[\alpha1 \cdot \left(\frac{p6}{1 - p0}\right) \cdot (1 - F)\right] + \left[\alpha2 \cdot \left(\frac{p6}{1 - p0}\right) \cdot (1 - F)\right] + \left[\alpha3 \cdot \left(\frac{p6}{1 - p0}\right) \cdot (1 - F)\right] + \left[\left[\alpha4 \cdot \left(\frac{p6}{1 - p0}\right) \cdot (1 - F)\right] + \left[\alpha5 \cdot \left(\frac{p6}{1 - p0}\right) \cdot (1 - F)\right]\right]$$

$$l3 = 0$$

$$l4 := \left(\frac{\alpha4 \cdot F}{1 - p0}\right) \cdot (p6)$$ $$l4 = 0$$

$$\gamma6 := l1 + l2 + l3$$

$$\gamma6 = 0$$

$$\gamma := (\gamma0 \;\; \gamma1 \;\; \gamma2 \;\; \gamma3 \;\; \gamma4 \;\; \gamma5 \;\; \gamma6)$$ $$\gamma = (0.93 \;\; 0.011 \;\; 0.022 \;\; 0.022 \;\; 0.011 \;\; 0.003 \;\; 0)$$

Expected Total Cost per Unit for the Univariate x-Bar Chat

Fixed Cost per Sampling

$a1 := 10$

Cost per unit sampled

$a2 := 1$

Cost of Investigating and Correcting a Process

$a3 := 100$

Penalty Cost of Producing Defects

$a4 := 10$

$$A1 := \frac{a1 \cdot \lambda}{a4} \qquad A2 := \frac{a2 \cdot \lambda}{a4} \qquad A3 := \frac{a3 \cdot \lambda}{a4}$$

$A1 = 0.001$ $A2 = 0.0001$ $A3 = 0.01$

Expected Cost per Unit of Sampling and Testing

$$E1 := \frac{a1}{k} + \frac{a2 \cdot N}{k} \qquad E1 = 0.35$$

Expected Cost per Unit of Investigating and Correcting the Process

$$E2 := \frac{a3}{k} \cdot q \cdot \alpha^{T} \qquad E2 = 0.104$$

Expected Cost per Unit Associated with Producing Defectives

$$E3 := a4 \cdot \phi \cdot \gamma^{T} \qquad E3 = 0.2986$$

Total Expected Cost per Unit

$$E := E1 + E2 + E3 \qquad E = 0.7522$$

Total Expected Cost per Unit Associated With Optimal Testing Procedure

$$EC := E \cdot a4 \qquad EC = 7.5223$$

APPENDIX C

ILLUSTRATIVE EXAMPLE OF ECONOMIC COST MODEL FOR MULTIVARIATE T^2 CHARTS BASED ON MONTGOMERY AND KLATT'S MODEL (1972)

$a1 := 1$ $\quad a2 := 10$ $\quad a3 := 1000$ $\quad a4 := 1$ $\quad \lambda' := .0001$

$$A1 := \frac{[a1\cdot(\lambda')]}{a4} \qquad A2 := \frac{[a2\cdot(\lambda')]}{a4} \qquad A3 := \frac{[a3\cdot(\lambda')]}{a4}$$

$A1 = 0.0001$ $\quad A2 = 0.001$ $\quad A3 = 0.1$

$N := 10$ $\quad K := .15$ $\quad k := \frac{K}{\lambda'}$ $\quad k = 1500$

Probability of Remaining in the State of μ_0, while k Units are Produced

$P0 := \exp(-\lambda'\cdot k)$ $\quad P0 = 0.861$

Probability of out-of-control State of μ_1, while k Units are Produced

$P1 := 1 - P0$ $\quad P1 = 0.139$

Subgroup Size

$n := N - 1$

Delta Shift in the Mean

$$\delta := \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$S1sq := 2$

$S12 := 1$

$S2sq := 2.5$

Variance Covariance Matrix

$$\Sigma := \begin{pmatrix} S1sq & S12 \\ S12 & S2sq \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 2.5 \end{pmatrix}$$

Correlation Coefficient

$$r := \frac{S12}{\sqrt{S1sq}\cdot\sqrt{S2sq}} \qquad r = 0.447$$

Non-centrality Parameter

$\tau sq := n \cdot \delta^T \cdot \Sigma^{-1} \cdot \delta$ $\tau sq = 167.625$

Level of Significance

$\alpha 1 := 0.005$

Number of Variables

$p := 2$ $\nu 1 := p$

$\nu 2 := \frac{n - \nu 1 + 1}{2}$ $f := qF[(1 - \alpha 1), \nu 1, \nu 2]$

$\nu 2 = 4$ $f = 26.28$

Probability of Type II Error

$$w(f) := \sum_{k=0}^{100} \left[\frac{e^{\frac{-\tau sq}{2}} \cdot \left(\frac{\tau sq}{2}\right)^k}{\frac{\Gamma\left(\frac{\nu 2}{2}\right) \cdot \Gamma\left(\frac{\nu 1}{2} + k\right)}{\Gamma\left[\frac{\nu 2}{2} + \left(\frac{\nu 1}{2} + k\right)\right]} \cdot k!} \cdot \left(\frac{\nu 1}{\nu 2}\right)^{\frac{\nu 1}{2} + k} \cdot \left(\frac{\nu 2}{\nu 2 + \nu 1 \cdot f}\right)^{\frac{(\nu 1 + \nu 2)}{2} + k} \cdot f^{\frac{\nu 1}{2} - 1 + k} \right]$$

$$\beta := \int_0^f w(f)\, df$$

$\beta = 0.01617$

Power of the Test

$\rho 1 := 1 - \beta$

$\rho 1 = 0.983832$

Probability of Type I Error

Upper Control Limit = $Tsq := 19.64$

$$q(x) := \frac{\Gamma\left(\frac{p+\nu 2}{2}\right)\cdot\left(\frac{p}{\nu 2}\right)^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)\cdot\Gamma\left(\frac{\nu 2}{2}\right)}\cdot\frac{x^{\left(\frac{p}{2}\right)-1}}{\left[\left(\frac{p}{\nu 2}\right)\cdot x+1\right]^{\frac{p+\nu 2}{2}}}$$

$b := \frac{p\cdot(n-1)}{(n-p)}$ $\qquad$ $a := \frac{Tsq}{b}$

$\alpha := 1 - \int_0^a q(x)\, dx$ $\qquad$ $a = 8.593$

$\alpha = 0.03565$

$\rho 0 := \alpha$

$$\rho := \begin{pmatrix} \rho 0 \\ \rho 1 \end{pmatrix}$$

Probability that Process Shifting from In-Control to Out-of-Control During the Production of k Units

$$G := \begin{bmatrix} P0 & P1 \\ \rho 1 \cdot P0 & \rho 1 \cdot P1 + (1 - \rho 1) \end{bmatrix} \qquad G = \begin{pmatrix} 0.861 & 0.139 \\ 0.847 & 0.153 \end{pmatrix}$$

$$\beta 0 := \frac{(\rho 1 \cdot P0)}{(P1 + \rho 1 \cdot P0)} \qquad \beta 1 := \frac{(P1)}{(P1 + \rho 1 \cdot P0)}$$

$$\beta a := \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} \qquad \beta a = \begin{pmatrix} 0.859 \\ 0.141 \end{pmatrix}$$

Conditional Expectation of Occurrence of Assignable Cause within an Interval of Sampling

$$\Delta := \frac{1 - (1 + \lambda' \cdot k) \cdot \exp(-\lambda' \cdot k)}{(1 - \exp(-\lambda' \cdot k)) \cdot \lambda' \cdot k} \qquad \Delta = 0.488$$

$$\gamma 0 := \beta 0 \cdot P0 + \Delta \cdot \beta 0 \cdot P1 \qquad \gamma 0 = 0.797$$

$$\gamma 1 := \beta 1 + (1 - \Delta) \cdot \beta 0 \cdot P1 \qquad \gamma 1 = 0.203$$

$$\gamma := \begin{pmatrix} \gamma 0 \\ \gamma 1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 0.797 \\ 0.203 \end{pmatrix}$$

Conditional Probability of Producing a Defective Unit Given that Process is in State μ_0

$\mu 0 := \begin{pmatrix} 50 \\ 60 \end{pmatrix}$ $\quad l1 := 50 - 4$ $\quad l2 := 60 - 4$

$\mu 1 := \begin{pmatrix} 55 \\ 66 \end{pmatrix}$ $\quad u1 := 50 + 4$ $\quad u2 := 60 + 4$

$\delta := \mu 1 - \mu 0$ $\quad \delta = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Transformation of the Variables to Standard Normal Random Variables (μ_0)

Positive Values

$h11 := \frac{(l1) - 50}{\sqrt{S1sq}}$ $\quad h22 := \frac{(u1) - 50}{\sqrt{S1sq}}$

$h11 = -2.8$ $\quad h22 = 2.8$ $\quad hxx := 2.8$

$k11 := \frac{(l2) - 60}{\sqrt{S2sq}}$ $\quad k22 := \frac{(u2) - 60}{\sqrt{S2sq}}$

$k11 = -2.5$ $\quad k22 = 2.5$ $\quad kxx := 2.5$

Bivariate Normal Distribution Function

$$B0 := \frac{1}{2\cdot\pi\cdot\sqrt{1-r^2}}\cdot\int_{hxx}^{\infty}\int_{kxx}^{\infty}\exp\left[-\left(\frac{1}{2}\right)\cdot\left(\frac{x^2 + y^2 - 2\cdot r\cdot x\cdot y}{1-r^2}\right)\right]dx\,dy$$

$B0 = 0.000277$

Negative Values

$x1 := 2.5$

$y1 := 2.8$

$$z2(x1) := \frac{\exp\left[-\left(\frac{1}{2}\right)\cdot\left(x1^2\right)\right]}{\sqrt{2\pi}}$$

$$z3(y1) := \frac{\exp\left[-\left(\frac{1}{2}\right)\cdot\left(y1^2\right)\right]}{\sqrt{2\pi}}$$

$$\psi(x1) := \int_{-x1}^{x1} z2(t)\,dt$$

$$\psi1(y1) := \int_{-y1}^{y1} z3(t)\,dt$$

$\psi(x1) = 0.987581$

$\psi1(y1) = 0.99489$

$$\phi0 := B0 + \frac{1}{2}\cdot\left[1 - \psi1(y1) + \left(1 - \psi(x1)\right)\right]$$

$\phi0 = 0.009042$

Conditional Probability of Producing a Defective Unit Given that Process is in State μ_1

Transformation of the Variables to Standard Normal Random Variables (μ_1)

Positive Values

$$h1 := \frac{(l1) - 55}{\sqrt{S1sq}} \qquad h2 := \frac{(u1) - 55}{\sqrt{S1sq}}$$

$h1 = -6.4$ $\quad h2 = -0.7$ $\quad hx := 0.7$

$$k1 := \frac{(l2) - 66}{\sqrt{S2sq}} \qquad k2 := \frac{(u2) - 66}{\sqrt{S2sq}}$$

$k1 = -6.3$ $\quad k2 = -1.3$ $\quad kx := 1.3$

Bivariate Normal Distribution Function

$$B := \frac{1}{2\cdot\pi\cdot\sqrt{1-r^2}}\cdot\int_{hx}^{\infty}\int_{kx}^{\infty}\exp\left[-\left(\frac{1}{2}\right)\cdot\left(\frac{x^2+y^2-2\cdot r\cdot x\cdot y}{1-r^2}\right)\right]dx\,dy$$

$B = 0.052205$

Negative Values

$x := 1.3$

$y := 0.7$

$$z(x) := \frac{\exp\left[-\left(\frac{1}{2}\right)\cdot\left(x^2\right)\right]}{\sqrt{2\pi}}$$

$$z1(y) := \frac{\exp\left[-\left(\frac{1}{2}\right)\cdot\left(y^2\right)\right]}{\sqrt{2\pi}}$$

$$\omega(x) := \int_{-x}^{x} z(t)\,dt$$

$$\omega1(y) := \int_{-y}^{y} z1(t)\,dt$$

$\omega(x) = 0.806399$

$\omega1(y) = 0.516073$

$$\phi1 := B + \frac{1}{2}\cdot\left[1-\omega1(y)+\left(1-\omega(x)\right)\right]$$

$\phi1 = 0.390969$

$$\phi := \begin{pmatrix}\phi0\\ \phi1\end{pmatrix} \qquad \phi = \begin{pmatrix}0.009042\\ 0.390969\end{pmatrix}$$

Expected Total Cost per Unit for the Multivariate T^2 Chart

Expected Cost per Unit of Sampling and Testing

$$E1 := \frac{(a1 + a2 \cdot n)}{k} \qquad E1 = 0.061$$

Expected Cost per Unit of Investigating and Correcting the Process

$$E2 := \frac{a3}{k} \cdot \rho^T \cdot \beta a \qquad E2 = 0.113$$

Expected Cost per Unit Associated with Producing Defectives

$$E3 := a4 \cdot \phi^T \cdot \gamma \qquad E3 = 0.0864$$

Total Expected Cost per Unit

$$ET := E1 + E2 + E3$$

$$ET = 0.26013$$

Total Expected Cost per Unit Associated With the Optimal Testing Procedure

$$EC := ET \cdot a4$$

$$EC = 0.26013$$