NONLINEAR DYNAMIC INVERSION CONTROL OF A MINIATURE MORPHING AIRCRAFT

A Thesis by

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ABSTRACT

This thesis investigates the feasibility of using Dynamic Inversion to develop control laws for time varying systems, specifically Miniature Morphing Air Vehicles. Two Linear Time Varying systems are investigated namely a pendulum of variable length and a miniature aircraft with the ability to change wing sweep. Simulations of the pendulum and the aircraft are provided with and without the control law in the loop. These simulations show the effectiveness of using dynamic inversion to control LTV systems. The aircraft model used in the simulations is a fourth order model which contains only the longitudinal dynamics. The stability and performance of the open and closed loop is investigated using a notion of LTV poles, that employ a time-varying Ricatti equation.

This work also presents preliminary nonlinear aerodynamic model for a miniature aircraft which morphs through variable wing sweep. The estimation of the aerodynamics and the stability derivatives were conducted using Athena Vortex Lattice (AVL). Dynamic Inversion is used to design a control law using the nonlinear model of the aircrafts longitudinal dynamics. The nonlinear zero-dynamics are discussed and investigated for stability. The control law is validated by simulating the aircraft in a pull up maneuver while simultaneously morphing.
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CHAPTER 1
INTRODUCTION

Morphing Air Vehicles

The ability to change shape in flight is as old as creation itself. One only has to look at birds to observe how seamlessly they change the shape of their wings during different segments of their flight regime. Human beings have strived to mimic this ability from the early days of powered flight with the Wright brothers using wing warping to achieve lateral control of the Wright flyer. Fast forward to the modern era of flight and an aircraft that easily comes to mind as having the ability to change its shape is the F-14. The F-14 had the capability to change its wing sweep from 20 degrees to 68 degrees in flight. This wing sweep was computer controlled and was a function of speed and angle of attack. At low speeds and correspondingly high angles of attack the wings would be almost unswept providing excellent low speed performance and handling which is crucial on carrier landings. Conversely at high speeds the wings would sweep back giving the aircraft an almost blended wing lifting body configuration. This configuration provided the F-14 the ability to achieve dash speeds in excess of Mach 2 and gave it the ability to excel in its role as a carrier based interceptor. The ability to morph allowed the F-14 to optimize its aerodynamic configuration to suit its mission profile.

The human fascination with flight, especially avian flight combined with the new era of autonomous unmanned flight (drones) has led to the rise in interest in very small flight vehicles or Micro Air Vehicles (MAV). MAVs are typically on the size order of garden variety birds. Interest in the flight characteristics of MAVs led to the Air Force Research Laboratory (AFRL) performing extensive research and building a MAV to serve as a baseline and test bed for future research on MAVs[1]. The AFRL air vehicle, named the Generic Micro Aerial Vehicle (GENMAV), served as a starting point for the design of the Variable Sweep Aerial Vehicle which is presented in Chapter 5.

Small scale Morphing Aerial Vehicles with wing spans of the order of 6 inches to 12 inches have been built at the University of Florida and several aspects of morphing have been under investigation for several years. The work done in [2] presents a MAV with a wing span of 12 inches. The aircraft has a
membrane type wing and is capable of varying the wing sweep in flight. The aircraft was instrumented with sensors to gather flight data which was then used to study the flight dynamics and develop dynamic models due to the effect of inflight morphing. Flight data revealed that shape change had a direct and significant effect on the dynamic characteristics of the aircraft. The tests showed promise that morphing could be used to allow aggressive maneuvering and increase the agility of MAV’s. Biological inspired MAVs were investigated in [3] by retrofitting an existing MAV with multi-joined wings based on the plan form of a sea-gull’s wings. The mechanical construction of the wing is quite complex and allows the outboard sections to be swept independent of the inboard sections. Asymmetric wing sweep is also possible. Equations of motion for morphing aircraft which take into account the effect of morphing and associated time-varying inertias. Wind-tunnel tests of a fully actuated MAV are discussed in [4]. The aircraft was capable of several different configurations involving variable sweep, span and twist. These wind-tunnel tests quantify the performance benefit of morphing as well as provide the stability derivatives which are used to develop control systems.

The ability to morph in flight raises questions about the method of actuation. The size of the MAVs being discussed as well as the rates at which morphing occurs place severe demands on the actuators which can be used. The use of Macro Fiber Composites (MAC) to construct movable and shape changing structures such as a variable camber wing is discussed in [5]. MAC’s are a type of piezoceramic material that offers structural flexibility and high actuation authority. The material changes shape when an electric current is applied to it hence the method for actuation is intrinsic to the material. Using MAC’s for some surfaces on a MAV would provide the benefit of reducing the size and weight of the aircraft structures. Another construction technique for a morphing surface is covered in [6] which employ a flexible skin over a honey comb structure. The inclusion of new actuation methods and new materials is crucial to the development of morphing MAV’s.

Several benefits can be obtained by incooperating morphing into miniature air vehicles. Chief among these is the ability to optimize the aerodynamic configuration of the vehicle to the required mission. This could be thought of as static morphing in that the aircraft has a well-defined configuration for a given segment of the mission and it then transitions to a different configuration as the mission
requires. A second benefit is dynamic morphing in that the configuration of the vehicle changes during a
dynamic maneuver, such as a level turn or a pull up, to allow the aircraft to maneuver aggressively.
Reaping the benefits of morphing require the development of control laws which can handle the transient
inertias and aerodynamics.

When used in a mission adaptive fashion, morphing MAVs can be modeled as Time Varying
systems since because the changing external shape due to morphing results in time varying dynamics.
Non-linear time varying systems can be written in the form shown in equation (1.1) where the states of
the system are denoted by $x(t)$ and the inputs by $u(t)$. The explicit dependence of the equation on time
indicates that the dynamics vary with time.

$$\dot{x} = f(x(t), u(t), t) \quad (1.1)$$

Likewise, a linear time varying system can be written in the state space form as shown in equation (1.2),
where the system matrices $A(t)$ and $B(t)$ vary as a function of time.

$$\dot{x} = A(t)x + B(t)u \quad (1.2)$$

In Chapter 2 we discuss the concept of poles for a Linear Time Varying (LTV) system using
Kamen’s concept [7]. In [7], a factorization technique is developed which allows one to write the
governing differential equation, which characterizes a time varying system, as a function of polynomials
with time varying coefficients. A time varying Riccati equation is then solved to obtain the time varying
poles. The time varying poles and the corresponding modes can then be used to make inferences as to
the stability and performance of the LTV system.

Chapter 2 also provides an overview of Dynamic Inversion. Dynamic Inversion falls into the
category of modern control techniques and is based on feedback linearization [8]. Control laws developed
using dynamic inversion essentially “invert” a model of the system. The aerodynamics of morphing aircraft
vary with external shape and the inclusion of the full aerodynamic model which include these time varying
effects, makes Dynamic Inversion well suited for controlling these time varying systems. The concept of
control variables and stability of the zero-dynamics is also discussed in Chapter 2.
Apart from Dynamic Inversions, several techniques exist for the control of LTV systems. In [9], Dullerud develops new techniques for the analysis of LTV systems. These allow methods usually restricted to LTI systems to be used in LTV cases. The technique involves an operator based description of the state space form of the LTV system and a function which serves as a transfer function for a time varying system, which has many properties similar to the transfer function in a LTI system. Dullard then focuses on the use of Robust Control techniques on LTV systems. Nguyen [10] presents a method which uses matrix operators and canonical transformations to make a certain class LTV system (lexicographically fixed) equivalent to an LTI system. It is then possible to use state feedback for eigenvalue placement. A simple algorithm for the design of the state feedback is also provided. Valasek [11] extends this methodology to cover a general class of Multi Input Multi Output (MIMO) systems. In [12] Djouadi discusses optimal disturbance rejection for LTV systems of an infinite number of states and in [13] Hinrichsen covers the concept of stability radius for time varying system and explores the relationship between the norm of the stability radius and the solvability of a non-standard differential Riccati equation.

In Chapter 3 dynamic inversion is used to develop control laws for a pendulum of time varying length. From a conceptual, as well as mathematical standpoint several connections exist between a pendulum of time varying length and a morphing aircraft. Essentially a morphing aircraft and a variable length pendulum, both demonstrate features of shape-change that influence their respective dynamics. The open and closed loop responses of the pendulum are presented different length change rates. The zero-dynamics are analyzed using the time-varying poles discussed in Chapter 2.

In Chapter 4, a 4th order LTV model of the longitudinal dynamics of a miniature morphing aircraft is analyzed. The model of the aircraft was developed at the University of Florida and has been used extensively in morphing air vehicle research. Details of the aircraft can be found in [3],[14] and [15]. Dynamic Inversion is used to develop a control law for this aircraft for different morphing trajectories. The time-varying poles of the aircraft are calculated and the associated modes are investigated to ascertain the stability of the aircraft in the open loop as well as the closed loop.

Chapter 5 provides details of the conceptual design of a variable sweep morphing vehicle that is based on the AFRL’S GENMAV aircraft are described. This model represents a non-linear time varying
system. Dynamic Inversion for such a system is discussed. It is then used to design a control law that is capable of commanding the aircraft to perform a pull-up and morph during the pull-up.
CHAPTER 2
DYNAMIC INVERSION AND KAMEN'S CONCEPT OF POLES OF LTV SYSTEMS

In this chapter we discuss features of several tools that are used in the design and analysis of Linear Time Varying (LTV) systems. The tools discussed are dynamic inversion for control design and a method to determine the stability and performance of LTV systems by computing time varying poles and modes.

2.1 Kamen's Concept of Poles for Time Varying Systems

A LTV system can be written in state space form as shown in equation (2.1), where the A and B matrices vary with time.

\[ \dot{x} = A(t)x(t) + B(t)u(t) \]
\[ y(t) = C(t)x(t) \]

To use Kamen's concept of poles, the LTV system has to be expressed as a single ordinary differential equation. In order to do this, the system is first converted into the controllable canonical form using a Lyapunov transformation. A transformation matrix \( T(t) \) is determined such that we can define a new set of state coordinates \( z(t) \) as shown in equation (2.2).

\[ x(t) = T(t)z(t) \]

Differentiating equation (2.2) and substituting it into equation (2.1) gives equation (2.3).

\[ \dot{T}(t)z(t) + T(t)\dot{z}(t) = A(t)T(t)z(t) + B(t)u(t) \]

The system is then written in state space form in terms of \( z(t) \) as in equation (2.4).

\[ \dot{z}(t) = \left( T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t) \right)z(t) + T^{-1}(t)B(t)u(t) \]
\[ y(t) = C(t)T(t)z(t) \]

The matrix \( T^{-1} \) for SISO system is defined by equation (2.7), where \( q_1 \) through \( q_n \) are row vectors [11].

\[
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_n
\end{bmatrix}
\]
The row vectors $q_1$ through $q_n$ are obtained using the recursive formulation shown in equation (2.8) and $q_1$ is selected as shown in equation (2.9).

$$q_{i+1}(t) = q_i(t)A(t) + \dot{q}_i(t)$$

$$q_1(t) = [0 \ 0 \ 0 \ 1]R^{-1}(t)$$

Where $R$, the controllability matrix, of the system is defined in equation (2.10).

$$R(t) = [r_1(t) \ r_2(t) \ r_3(t) ... r_n(t)]$$

$$r_1(t) = B(t)$$

$$r_{i+1}(t) = A(t)r_i(t) - \dot{r}_i(t)$$

For a 2nd order system the transformation looks as follows.

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \text{(Lyapunov Transformation)} \begin{bmatrix} 0 & 1 \\ -a_0(t) & -a_1(t) \end{bmatrix}$$

For this transformed LTV system, Kamen presents a concept of time varying poles and zeros [7]. A special factorization technique is used that factorizes the polynomial (i.e the ordinary differential equation) into lower order polynomials with time varying coefficients.

The 2nd order system is written in the form shown in equation (2.11).

$$(D^2 + a_1(t)D + a_0(t))z(t) = (b_2(t)D^2 + b_1(t)D + b_0(t))u(t)$$

The left hand side of equation (2.11) is written as a non-commutative polynomial product of $p_1(t)$ and $p_2(t)$. This factorization is expressed as equation (2.12).

$$(D^2 + a_1(t)D + a_0(t))z(t) = (D - p_1(t))(D - p_2(t))z(t)$$

Expanding the right hand side of (2.12) gives equation (2.13).

$$(D - p_1(t))(D - p_2(t))z(t) = (D^2 - [p_2(t) + p_1(t)]D + p_1(t)p_2(t) - \dot{p}_2(t))z(t)$$

Setting the left hand side of equation (2.12) equal to the right hand side of equation (2.13) and comparing coefficients, $p_1(t)$ and $p_2(t)$ satisfy equation (2.14) and equation (2.15).

$$p_1(t) + p_2(t) = -a_1(t)$$

$$p_1(t)p_2(t) - \dot{p}_2(t) = a_0(t)$$
Multiplying both sides of equation (2.15) by \( p_2(t) \) and substituting into equation (2.14) results in a nonlinear first order differential equation for \( p_2(t) \) with time varying coefficients. This is shown in equation (2.16) and is a special case of a time varying Riccati Equation.

\[
p_2^2(t) + p_2(t) + a_1(t)p_2(t) + a_0(t) = 0 \tag{2.16}
\]

This equation can be solved for \( p_2(t) \), \( p_1(t) \) is then obtained from equation (2.14). \((p_1(t), p_2(t))\) form an ordered set with \( p_2(t) \) called a right pole and \( p_1(t) \) called the left pole. Only the right poles is needed to characterize the system.

The modes associated with the ordered pole set \((p_1(t), p_2(t))\) are defined by equation (2.17). For a 2\(^{nd}\) order system there are two right poles \( p_{21}(t) \) and \( p_{22}(t) \), these are obtained by solving equation (2.16) twice for two different initial conditions.

\[
\varphi_{p_{2n}} = e^{\int_0^t p_{2n}(t)\,dt}, n = 1 \text{ to } 2 \tag{2.17}
\]

The modes associated with the poles of the LTV system can be studied to determine the stability and performance of the system. If the modes approach zero as time tends to infinity then the system is asymptotically stable [7].

The time varying magnitude of the system response can be obtained using equation (2.18).

\[
\text{Envelope} = e^{\int_0^t \sum p_R} = e^{\int_0^t \frac{\text{real}(p_{21}) + \text{real}(p_{22})}{2} \,dt} \tag{2.18}
\]

The time varying natural frequency of the system can be obtained from the imaginary part of the pole as shown in equation (2.19)

\[
\omega(t) = e^{\int_0^t \frac{\text{imaginary}(p_{2n})}{2i} \,dt}, n = 1,2 \tag{2.19}
\]

The above description is provided for a second order LTV system of the form shown in equation (2.11). It can be extended for a general \( n^{th} \) order ODE with time-varying coefficients of the form given by equation (2.20).

\[
\left(D^n + a_{n-1}(t)D^{n-1} + a_{n-2}(t)D^{n-2} + \cdots + a_0(t)\right)z(t) = \left(D - p_1(t)\right)\left(D - p_2(t)\right)\left(D - p_3(t)\right)\left(D - p_4(t)\right)z(t) \tag{2.20}
\]
The operation defined by $S$ is performed on the time varying pole $p$ and is described by equation (2.21). Kamen presents a set of recursive equations based on the operator $S$ which are used to compute the right pole for an $n$th order system, these equations are shown in equation (2.22) and equation (2.23).

\[(Sp)(t) = p^2(t) + \dot{p}(t) \quad (2.21)\]

\[
\begin{align*}
(S^i p)(t) &= \left(S(S^{i-1}p)\right)(t) = p(t)(S^{i-1}p)(t) + \frac{\partial}{\partial t}(S^{i-1}p)(t) \\
(S^{n-1}p_n)(t) + \sum_{i=2}^{n-1} a_i(t)(S^{i-1}p_n)(t) + a_1(t)p_n(t) + a_0(t) &= 0
\end{align*} \quad (2.22, 2.23)\]
2.2 Dynamic Inversion Control for LTV Systems

Dynamic Inversion [8] falls under a category of control techniques that are typically used to control nonlinear systems as they intrinsically take into account the nonlinearities of a system, however, they can also be used to control linear systems. We restrict ourselves to a class of systems that are square ie: the number of inputs is equal to the number of outputs and the output matrix C is assumed to be constant ie \( C(t) = C \). The system is represented by equation (2.1).

Let \( r(t) \) denote the reference signal, the goal of the controller is to control the output \( y(t) \) such that it follows the desired reference signal by minimizing the error. The error is defined as shown in equation (2.24).

\[
e(t) = r(t) - y(t)
\]

The output \( y(t) \) is differentiated until the control input \( u(t) \) appears in the derivative. If the matrix product \( CB(t) \) is singular the expression for the output is differentiated until the matrix pre-multiplying \( u(t) \) is non-singular, such that \( u(t) \) appears in the derivative of the output.

\[
\dot{y} = C\dot{x} = CA(t)x + CB(t)u(t)
\]

Rearranging equation (2.25) and substituting the derivative of equation (2.24) for \( \dot{y} \) produces equation (2.26) which is the expression for the control input \( u(t) \).

\[
u(t) = \left( CB(t) \right)^{-1}(\dot{r} - CA(t)x - \dot{e}(t))
\]

An expression for \( \dot{e}(t) \) is selected as as shown in equation (2.27). These are the error dynamics of the system and can be stabilized selecting a positive definite gain matrix \( K \).

\[
\dot{e} = -Ke
\]

The expression for the control input \( u(t) \) can then be written as shown in equation (2.28).

\[
u(t) = \left( CB(t) \right)^{-1}(\dot{r} - CA(t)x + Ke)
\]
The complete closed loop system is represented by equation (2.29).

\[
\dot{x} = A(t)x(t) + B(t)[(CB(t))^{-1}(\dot{r} - CA(t)x + Ke)] \\
y(t) = Cx(t)
\] (2.29)

The same expression can be arrived at by selecting an intermediate auxiliary input as shown in equation (2.30) as discussed by Stevens in [8].

\[
v(t) = CB(t)u + CA(t)x - \dot{r}
\] (2.30)

### 2.2.1 Zero Dynamics for LTV Systems

The closed loop system with the dynamic inversion controller in the loop is obtained by substituting the control input \(u(t)\) into equation (2.1). The closed loop system given in equation (2.31).

\[
\dot{x} = [I - B(t)(CB(t))^{-1}C]A(t)x(t) + B(t)(CB(t))^{-1}(\dot{r}(t) - \dot{e}(t))
\] (2.31)

The zero dynamics of the system are the dynamics of the system when the output \(y(t)\) equal to 0 [8] as shown in equation (2.32).

\[
r(\dot{t}) = \dot{e}(t)
\] (2.32)

The resulting zero dynamics of the system can be written as equation (2.33).

\[
\dot{x} = \left[ I - B(t)(CB(t))^{-1}C \right]A(t)x(t)
\] (2.33)

While the error dynamics are guaranteed to be stable by the proper selection of \(K\) the rest of the dynamics, which are the zero-dynamics of the system, may or may not be stable. The zero dynamics, given by equation (2.33), need to be checked for stability. The modes corresponding to the zero dynamics are unobservable and cannot be modified by the dynamic inversion controller; therefore if some of the zero dynamic modes are unstable, the closed loop system will be unstable. The linear combination of states which constitute the output \(y(t)\) of the system, shown in equation (2.1), is called the Control Variable (CV). If the zero-dynamics are unstable a different CV is selected (i.e.: change the \(C\) matrix) to obtain stable zero-dynamics.
CHAPTER 3
DYNAMIC INVERSION CONTROL OF A TIME VARYING PENDULUM

3.1 Time Varying Pendulum

In this chapter, a controller is designed for a pendulum of time varying length and such a pendulum is depicted in Figure 3.1. \( \theta(t) \) is the angular displacement from the vertical, \( l(t) \) is the time varying length and \( \dot{l}(t) \) is the rate of change of length. The nonlinear equations of motion for the time varying pendulum can be written as shown in equation (3.1).

\[
I^2 \ddot{\theta} + 2I \dot{\theta} \dot{l} + gl \sin \theta = \frac{M_{app}}{m}
\]  

(3.1)

Where \( m \) is the mass of the bob and \( M_{app} \) is the externally applied moment.
Applying the small angle approximation to equation (3.1) and rearranging to obtain an expression in terms of angular acceleration results in equation (3.2).

\[
\ddot{\theta} = \frac{M_{\text{app}}}{ml(t)^2} - 2 \frac{l \dot{\theta}}{l(t)} - \frac{g \theta}{l(t)} \tag{3.2}
\]

Equation (3.2) is a linear equation in terms of angular velocity (\(\dot{\theta}\)) and angular displacement (\(\theta\)).

Defining \(x_1(t) = \theta(t)\) and \(x_2(t) = \dot{\theta}(t)\), the equation of motion for a LTV pendulum is presented in state space form in equation (3.3)

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{g}{l(t)} & -2l(t) \frac{l(t)}{l(t)}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{1}{ml(t)^2}
\end{bmatrix} M_{\text{app}} \tag{3.3}
\]
3.2 Open Loop System

The effect of the time varying length on the open-loop dynamics of the pendulum are investigated by considering three different pendulum growth rates i.e. three different values for \( \dot{l} \). Initially the length of the pendulum is held constant (i.e. \( \dot{l} = 0 \)), this results in the pendulum becoming linear time invariant (LTI) system.

We examine a case when the pendulum growth rate is positive. The length of the pendulum increases with time, i.e. \( \dot{l} > 0 \), which results in an increase in the moment of inertia. Since the moments acting on the pendulum due to gravity (atmospheric effects are ignored) remain constant, the increasing moment of inertia due to increasing length acts as a damping term and causes the frequency and amplitude of oscillation to decreases. The real parts of the LTV poles \( p_{21} \) and \( p_{22} \) are shown in Figures 3.2.

![Figure 3.2. LTV Poles (real part) for pendulum. \( \dot{l} = 0.2 \).](image-url)
The imaginary parts of the LTV poles $p_{21}$ and $p_{22}$ are shown in Figures 3.3.

Figure 3.3. LTV Poles (imaginary part) for pendulum. $\dot{i} = 0.2$.

Unlike LTI systems, examining the poles of LTV systems alone does not always provide a good indication of the stability of the system and the LTV modes need to be examined. The real and imaginary part of the modes $\varphi_{21}$ and $\varphi_{22}$ are presented in Figure 3.4 and 3.5 respectively.

Figure 3.4. LTV Modes (real part) for pendulum. $\dot{i} = 0.2$. 
The modes provide a better picture of the stability of the LTV pendulum as the length increases. The magnitudes of the real and imaginary parts of the modes decrease and approach zero and the frequency of oscillation decreases. Based on examination the modes, for $\dot{l} > 0$, the LTV pendulum can be said to be asymptotically stable.

Finally we examine an open loop case when the pendulum growth rate is negative, ie $\dot{l} < 0$ and the length of the pendulum decreases with time. Figure 3.6 presents the real part of the LTV poles.
The imaginary parts of the poles are shown in Figure 3.7.

![Graph showing imaginary parts of poles for pendulum](image)

**Figure 3.7. LTV Poles (imaginary part) for pendulum. \( \dot{l} = -0.4 \).**

The moment of inertia decreases as the length of the pendulum shortens while the gravitational forces remain constant. The result is that the magnitudes of the poles grow as the pendulum shortens and the frequency of oscillation increases. An examination of the open loop modes is conducted to assess the stability of the pendulum when its length decreases with time. The plots of the real and imaginary parts of open loop modes \( \phi_{21} \) and \( \phi_{22} \) are shown in Figures 3.8 and 3.9 respectively.

![Graph showing real parts of modes for pendulum](image)

**Figure 3.8. LTV Modes (real part) for pendulum. \( \dot{l} = -0.4 \)**
Figure 3.9. LTV Modes (imaginary part) for pendulum. \( \dot{l} = -0.4 \)

The frequency and magnitude of oscillation for both the real and imaginary parts of the modes increase as the length of the pendulum decreases. The LTV pendulum, with decreasing length, is unstable as an open loop system. For comparison the LTV poles discussed in the preceding section are plotted together with the LTI poles (poles when the pendulum’s length doesn’t change) in Figure 3.10.

Figure 3.10. Comparison of Poles for the Pendulum

For the cases where \( \dot{l} > 0 \), we see that the magnitude of the real part grows more negative as the length of the pendulum grows showing that a positive growth rate acts as a damping term. Conversely the
magnitude of the real part grows more positive for the case where \( \dot{i} < 0 \) and the negative \( \dot{i} \) excites the oscillations.

### 3.3 Dynamic Inversion Controller

The objective of the dynamic inversion controller is to make the pendulum's angular displacement track a reference signal and initially the C matrix (and also the CV) shown in equation (3.4) is selected.

The initial output and the reference signal are angular displacement \( \theta \).

\[
C = [1 \ 0] \tag{3.4}
\]

The error is defined as the difference between the reference signal and the desired output and is shown in equation (3.5). The dynamic inversion controller attempts to drive the error term to zero.

\[
e = r(t) - y(t) \tag{3.5}
\]

The output is differentiated until the control \( u(t) \) appears in the expression for the derivative of the output as shown in equation (3.6).

\[
\dot{y} = CA(t)x(t) + CB(t)u(t) \tag{3.6}
\]

However, the product \( CB(t) = 0 \); therefore, the second derivative of the output is calculated and shown in equation (3.7).

\[
\ddot{y} = \left[ CA(t)^2 + C \dot{A}(t) \right] x(t) + \left[ CA(t)B(t) + C \dot{B}(t) \right] u(t) + CB(t) \dot{u} \tag{3.7}
\]

With \( CB(t) = 0 \) and matrix products \( C. \dot{A}(t) \) and \( C. \dot{B}(t) \) being zero, equation (3.7) is simplified and written as shown in equation (3.8).

\[
\ddot{y} = \left[ CA(t)^2 \right] x(t) + CA(t)B(t)u(t) \tag{3.8}
\]

The auxiliary input \( v(t) \) is defined shown in equation (3.12).

\[
v(t) = CA(t)B(t)u(t) - \dot{i} + \left[ CA(t)^2 \right] x(t) \tag{3.9}
\]

Rearranging equation (3.9) to obtain the state feedback results in the expression for the control input \( u(t) \) and is shown in equation (3.10).

\[
u(t) = \left[ CA(t)B(t) \right]^{-1} \{ \dot{i} - \left[ CA(t)^2 \right] x(t) + v(t) \} \tag{3.10}
\]

The expression for controller input \( u(t) \) is substituted back into \( \ddot{y} \), as shown in equation (3.8), which simplifies to equation (3.11).
\[ \ddot{y} = \ddot{r} + v(t) \]  

Equation (3.12) is the 2\textsuperscript{nd} derivative of the error dynamics term. Substituting equation (3.11) for \( \ddot{r}(t) \) in equation (3.12) gives equation (3.13), which are the error dynamics of the closed loop system.

\[ \ddot{e} = \ddot{r}(t) - \ddot{y}(t) \]  

\[ \ddot{e} = -v(t) \]  

The auxiliary input \( v(t) \) is selected such that the error dynamics are stable and takes the form shown in equation (3.14). The error dynamics can then be written as a 2\textsuperscript{nd} order ordinary differential equation and is shown in equation (3.15) [8].

\[ v(t) = K_1 \dot{e} + K_2 e \]  

\[ \ddot{e} + K_1 \dot{e} + K_2 e = 0 \]  

The gains \( K_1 \) and \( K_2 \) are selected such that the error dynamics are stable. To satisfy the conditions for stability \( K_1 \) and \( K_2 \) must meet the conditions shown in equation (3.16) and equation (3.17).

\[ K_1 > 0 \]  

\[ K_1^2 - 4 K_2 < 0 \]  

Equation (3.18) represents the closed loop system and the block diagram is shown in Figure 3.11.

\[ \dot{x} = A(t)x + B([CA(t)B(t)]^{-1}[\ddot{r} - [CA(t)^2]x(t) + v(t)]) \]  

Figure 3.11. Block Diagram for a LTV Pendulum Dynamic Inversion Controller.
The zero-dynamics of the closed loop system are checked to ensure their stability. The zero-dynamics are defined as the dynamics of the system when the auxiliary input \( v(t) \) is selected to produce an output \( y(t) \) equal to 0. Select \( v(t) \) as shown in equation (3.19) results in \( y(t) \) being equal to 0 for the closed loop system.

\[
v(t) = -\dot{r}
\]  

(3.19)

The zero-dynamics can then be expressed as equation (3.20).

\[
\dot{x} = \left[ I - B(t)(CA(t)B(t))^{-1}CA(t) \right] A(t)x(t)
\]  

(3.20)

The poles and modes were calculated for the CV shown in equation (3.4), for several growth rates, to examine the stability of the zero dynamics.

Figure 3.12. LTV Modes (real part) for pendulum. \( \dot{i} = 0.2, \ C = [1 \ 0] \).

The real part of the modes for the zero-dynamics are shown in Figure 3.12 and the imaginary parts in Figure 3.13. The modes are divergent, hence the zero-dynamics are unstable.

Figure 3.13. LTV Modes (imaginary part) for pendulum. \( \dot{i} = 0.2, \ C = [1 \ 0] \).
Due to the instability of the zero-dynamics a new CV is required for the dynamic inversion controller to work. Through trial and error several combinations of the states were used as CV’s before settling on the one shown in equation (3.20). Using equation (3.20) as the the CV stabilizes the zero-dynamics and provides the desired state response.

\[ C = [1 - 0.35] \]  

(3.21)

The poles and modes for the zero-dynamics are presented for a case where the pendulum grows in length (\( \dot{\theta} > 0 \)). The real and imaginary parts of the poles for the zero-dynamics are shown in Figures 3.14 and 3.15 respectively and converge to zero.

Figure 3.14. LTV Poles (real) for pendulum. \( \dot{\theta} = 0.2, \ C=[1 -0.35] \).

Figure 3.15. LTV Poles (imaginary) for pendulum. \( \dot{\theta} >=0.2, \ C=[1 -0.35] \).
The real and imaginary parts of the modes are shown in Figures 3.16 and 3.17 respectively.

Figure 3.16. LTV Modes (real part) for pendulum. $\dot{\theta} = 0.2$, $C=[1 -0.35]$.

Figure 3.17. LTV Modes (imaginary part) for pendulum. $\dot{\theta} = 0.2$, $C=[1 -0.35]$.

The real and imaginary parts of the modes converge to non-zero values, hence the zero-dynamics for the closed loop system are neutrally stable.
The time varying poles are presented for a case where the pendulum's length shrinks $\dot{l} < 0$ and they converge to zero.

Figure 3.18. LTV Poles (real) for pendulum. $\dot{l} = -0.4$, $C=[1 -0.35]$.  

Figure 3.19. LTV Poles (imaginary) for pendulum. $\dot{l} = -0.4$, $C=[1 -0.35]$.  

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The time varying modes are shown in Figures 3.20 and 3.21 respectively. The modes converge to non-zero values, hence the zero-dynamics are neutrally stable.

Figure 3.20. LTV Real Modes for pendulum. $\dot{l} = -0.4$, $C=[1 -0.35]$.

Figure 3.21. LTV Imaginary Modes for pendulum. $\dot{l} = -0.4$, $C=[1 -0.35]$.
3.4 Simulations of Pendulum with Linearly Increasing Length

The time varying pendulum was simulated under several different initial conditions in the open and closed loop using MATLAB. All the closed loop simulations used the CV presented in equation (3.20). Figure 3.22 shows the open loop dynamic response of a pendulum whose length increases with time.

![Graph 1](image1)

Figure 3.22. Pendulum with Linear Increase in Length – Open Loop (\( \dot{l}(t) = 0.2 \text{ m/s} \))

The initial length of the pendulum is at 2 m and its final length at the end of the simulations is 30 m. The weight of the pendulum bob is 0.5kg. The dynamic response shows that the magnitude of angular oscillation (\( \theta \)) of the pendulum damps out over time. The goal of the dynamic inversion controller is to force the pendulum to follow a reference signal.

The closed loop response of the pendulum with the control law given by equation (3.17) is shown in Figure 3.23. The dynamic inversion controller forces the oscillation of the pendulum to follow the reference signal (\( \tau \)), which in this case is a sinusoidal wave. Figure 3.23 also shows the control torque \( \omega(t) \) required to force the pendulum to follow the reference signal. The magnitude of the torque applied to the pendulum by way of the dynamic controller increases in order to overcome the damping effect of the rate of length increase of the pendulum.
Equation (2.18) is used to calculate the open and closed loop magnitude of the modes for the pendulum and is shown in Figure 3.24.

Figure 3.24. Magnitude Envelope for a Pendulum with Increasing Length ($\dot{L}(t) = 0.2\text{m/s}$)
The open and closed loop frequency for the modes are calculated using equation (2.19) and is shown in Figure 3.25.

Figure 3.25. Time Varying Frequency for a pendulum with Increasing Length ($\dot{l}(t) = 0.2\text{m/s}$)
3.5 Simulations of Pendulum with Linearly Decreasing Length

Figure 3.24 shows a pendulum whose length decreases with time in the open loop. The pendulum starts at a length of 15 m and decreases in length at a rate of \(-4\) m/s. The magnitude of oscillation of angular displacement increases with decreasing length.

![Graphs showing pendulum's angular displacement, angular velocity, and length over time.]

Figure 3.26. Pendulum with Linear Decrease in Length – Open Loop (\(\dot{l}(t) = -0.4\) m/s)

By closing the loop as shown in Figure 3.25 with the dynamic inversion controller, torque is applied to the pendulum which causes the pendulum to follow the sinusoidal reference signal. The decreasing moment of inertia results in the controller having to command less torque to match the reference signal.

![Graphs showing pendulum's angular displacement, angular velocity, length, and control force over time.]

Figure 3.27. Pendulum with Linear Decrease in Length – Closed Loop (Rate = -0.4 m/s)
The time varying magnitude and frequency for modes of the pendulum are shown in the open and closed loop in Figure 3.28 and Figure 3.29 respectively.

Figure 3.28. Magnitude Envelope for a Pendulum with Decreasing Length ($\dot{l}(t) = -0.4$ m/s)

Figure 3.29. Time Varying Frequency for a pendulum with Decreasing Length ($\dot{l}(t) = -0.4$ m/s)
CHAPTER 4
4th ORDER MICRO MORPHING AIR VEHICLE

4.1 Longitudinal Dynamics, 4th Order Miniature Morphing Air Vehicle

In this chapter, we perform a dynamic inversion controller design for the longitudinal axis of a morphing MAV. The MAV has the capability of variable wing sweep and is modeled as a time varying system. The general form of the non-linear dynamics is given by equation (4.1).

\[ \dot{x} = f(x, u, \mu) \]  

where \( u \) indicates the conventional control surface (the elevator) and \( \mu \) indicates the wing sweep angle. Equation (4.1) can be linearized about a trim trajectory \((x_0(t), u_0(t))\) as shown in equation (4.2).

\[ \Delta \dot{x} = \frac{\partial f}{\partial x}|_{x_0,u_0} \Delta x + \frac{\partial f}{\partial u}|_{(x_0,u_0)} \Delta u \]  

The linearized equation is shown in equation (4.3).

\[ \Delta \dot{x} = A(\mu) \Delta x + B(\mu) \Delta u \]  

For a given morphing trajectory \( \mu(t) \), equation (4.3) can be written in the general form shown in equation (4.4), the composition of \( A(t) \) and \( B(t) \) is shown in equation (4.5).

\[ \dot{x} = A(t)x(t) + B(t)u(t) \]  

\[ \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} X_u(t) & X_w(t) & X_q(t) & -g \cos(\theta_0) \\ Z_u(t) & Z_w(t) & U_0 & -g \sin(\theta_0) \\ M_u(t) & M_w(t) & M_q(t) & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} Z_{de}(t) \\ M_{de}(t) \end{bmatrix} \delta e \]  

The state vector \( x \) is presented in (4.6) and forward velocity is represented by \( u \), vertical velocity by \( w \), pitch rate by \( q \) and pitch attitude is represented by \( \theta \).

\[ x = \begin{bmatrix} u \\ w \\ q \end{bmatrix} \]  

The control surface deflection is given in equation (4.6)

\[ u = \delta e \]  

A dynamic inversion controller is designed for the 4th order system with the output matrix \( C \) selected as shown in equation (4.8). Selecting the output is a crucial step in dynamic inversion design,
the linear combination of states in the output is also called the Control Variable (CV). The CV selected such that it provides control of the desired states and also results in stable zero-dynamics. However there is no defined procedure to select a CV, and is done iteratively through trial and error. Initially \( q \) was selected as the output and results in the closed loop zero-dynamics being unstable but bounded. The work done by Enns [16] and presented in Stevens [8] suggests that adding a small airspeed term to the CV’s in LTI aircraft provide stable zero-dynamics [8]. Based on this a small airspeed term was added to the CV for the LTV aircraft. The CV shown in equation (4.8) was settled on after trying several different combinations of states, the addition of the small pitch term resulted in a CV which provided the desired control over the states as well as stable zero-dynamics.

\[
y = Cx = \begin{bmatrix} -0.5 & 0 & 1 & 0.5 \\ \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ \end{bmatrix} \tag{4.8}
\]

The process defined in Chapter 2 is used to derive the control law for the dynamic inversion controller. The matrix product \( CB \) is non-singular for the entire range of wing sweep so only the first derivative of the output is required. The control law is defined by \( u(t) \) and is shown in equation (4.9).

\[
u(t) = \left( CB(t) \right)^{-1} \left( \dot{r} + Ke - CA(t)x \right) \tag{4.9}
\]

A reference signal \( r(t) \) is selected which forces the output to go to 0. Therefore reference signal is selected as in equation (4.10).

\[
r = 0 \tag{4.10}
\]

The resulting closed loop system with the dynamic inversion controller is shown in equation (4.11).

\[
\dot{x} = A(t)x(t) + B(t)\left[ \left( CB(t) \right)^{-1} \left( \dot{r} + Ke - CA(t)x(t) \right) \right] \tag{4.11}
\]

The aircraft model considered is the one used in [14] and is shown in Figure 4.1. It has two inboard and two outboard wing sections, each capable of independent sweep. For the simulations presented here the entire wing is assumed to sweep dependently i.e. the inboard and outboard sections do not sweep independently and only symmetric sweep configurations are considered.
The effect of morphing on the open loop dynamic response of the aircraft was examined by simulating the morphing trajectories in Table 4.1. The simulation assumed that perturbations in all four states hit the aircraft at the instant that morphing commenced. For the morphing trajectory labeled $\mu_1$, the wings of the MAV are initially swept forward ($-30^\circ$) and in approximately two seconds they reach the swept backward configuration ($+30^\circ$). The morphing trajectory labeled $\mu_2$ has the MAV with wings initially swept backward configuration and ends with the wings in the swept forward configuration.

Both the morphing trajectories have approximately the same average morphing rate of 60deg/second, however since the trajectories are an exponential function of time, the instantaneous morphing rate varies with time.

Table 4.1. Morphing Trajectories for Wing Sweep from $-30^\circ$ to $+30^\circ$.

<table>
<thead>
<tr>
<th>Label</th>
<th>Morphing Trajectory</th>
<th>Sweep Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>$\mu_1 = 30 - 60(0.5)^t$</td>
<td>$-30^\circ$ to $+30^\circ$</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>$\mu_2 = (60e^{-3.5t}) - 30$</td>
<td>$+30^\circ$ to $-30^\circ$</td>
</tr>
</tbody>
</table>
4.2 Simulation Results for Morphing Trajectory μ1.

The aircraft is initially trimmed, at t = 0, it begins to morph with the morphing trajectory shown in Figure 4.2 and the wings change sweep from -30 degrees to +30 degrees in approximately 2 seconds. At t = 0 a wind disturbance hits the aircraft which affects all four states. Figure 4.3 presents the open loop state response of the MAV.

Figure 4.3. Open Loop Response for μ1 Morphing Trajectory.

The vertical velocity component (w) and the pitch rate (q) show an initial very high frequency response that is heavily damped and lasts less than a second, followed by a lower frequency response which is
less damped. The forward velocity component ($u$) and the pitch attitude ($\theta$) display a low frequency response which is lightly damped. Based on the open loop response the MAV is stable for the exponential morphing trajectory $\mu_1$. The open loop LTV poles are calculated using the method developed by Kamen [8] and are shown in Figure 4.4.

![Figure 4.4. Open Loop Poles for $\mu_1$ Morphing Trajectory](image)

The starting value for the poles are the same as the LTI poles at $t=0$ and are examined to ensure that the resulting LTV poles remain bounded. This is done for the following reason: the LTV poles are calculated using a special form of the Riccati equation, there are some initial values which cause the Riccati equation to become unstable and result in unbounded poles even when the original system has a bounded response.

The poles $p_{41}$ and $p_{42}$ correspond to the short period mode, the real part of LTV poles $p_{41}$ and $p_{42}$ start at the same value as the LTI pole and show a fast decay to a value close to zero. The poles $p_{43}$ and $p_{44}$ correspond with the phugoid mode, the real part of $p_{43}$ and $4$ show an initial fast oscillation followed by a slower decay to a negative value. The overall magnitude of $p_{43}$ and $p_{44}$ is much lower than $p_{41}$ and $p_{42}$.
The LTV modes are calculated from the LTV poles and provide a better picture of the system's stability. The open loop modes are calculated for each of the open loop time varying poles using equation (4.12).

\[ \varphi_{4n} = e^{\int_0^t p_n(t) dt}, \quad n = 1 \text{ to } 4 \]  

(4.12)

The oscillatory nature of the open loop response of the aircraft is reflected in the time varying modes shown in Figure 4.5. The magnitude of the time varying modes converges to zero, thereby showing that the system is stable for the morphing trajectory. The convergence of the modes to zero correlates well with the dynamic response of the state in Figure 4.3.

![Figure 4.5. Open Loop Modes for \( \mu_1 \) Morphing Trajectory.](image)

The state response of the aircraft to the perturbation is simulated with the dynamic inversion controller in the loop and is shown in Figure 4.6. The disturbances in all the states decay to zero. Figure 4.6 also shows the open loop response for comparison. While the MAV is stable for the \( \mu_1 \) morphing trajectory, the closed loop response shows that the dynamic inversion controller is able to increase the response time of the aircraft by decreasing the time for the states to return to zero. The controller also decreases the magnitude of the initial overshoot of the states.
The LTV poles which are calculated for the closed loop system are the zero-dynamics poles and are shown in Figure 4.7.

Figure 4.6. Closed Loop Response for $\mu$1 Morphing Trajectory

Figure 4.7. Zero Dynamics Poles for $\mu$1 Morphing Trajectory
The modes for the zero-dynamics are shown in Figure 4.8 and converge to constant values, hence the zero-dynamics of the system are neutrally stable.

Figure 4.8. Zero Dynamics Modes for \( \mu_1 \) Morphing Trajectory.

The open and closed loop (zero-dynamics) time varying magnitudes for the modes are presented in Figure 4.9.

Figure 4.9. Time Varying Magnitude \( \mu_1 \) Morphing Trajectory.
The open and closed loop (zero-dynamics) time varying frequencies are presented in Figure 4.10.

![Graphs showing open and closed loop dynamics](image)

Figure 4.10. Time Varying Frequency $\mu_1$ Morphing Trajectory.

### 4.3 Simulation Results for Morphing Trajectory $\mu_2$.

![Graph showing morphing trajectory $\mu_2$](image)

Figure 4.11. Morphing Trajectory $\mu_2$, $+30^\circ$ to $-30^\circ$

A morphing trajectory was considered which causes the wing sweep to change from $+30^\circ$ (fully swept back) to $-30^\circ$ (fully swept forward) in 2 seconds. Once again a perturbation is assumed to hit the aircraft, at the same time as the commencement of morphing, that affect all four states. The morphing
trajectory is presented in Figure 4.12. The open loop dynamic response of the MAV is presented in Figure 4.12. All four states show oscillatory responses that grow in magnitude, indicating that the MAV is unstable for this morphing trajectory.

The open loop dynamic response of the MAV is presented in Figure 4.12. All four states show oscillatory responses that grow in magnitude, indicating that the MAV is unstable for this morphing trajectory.

The LTV poles were calculated for this morphing trajectory and are shown in Figure 4.13 with the corresponding modes presented in Figure 4.14.

The LTV poles were calculated for this morphing trajectory and are shown in Figure 4.13 with the corresponding modes presented in Figure 4.14.
The real and imaginary part of all four modes ($\varphi_{41}$, $\varphi_{42}$, $\varphi_{43}$ and $\varphi_{44}$) show diverging oscillatory behavior which agree with the state response and confirms that the MAV is unstable in the longitudinal axis with the wings fully swept forward.

![Image of graphs showing open loop modes](image)

Figure 4.14. Open Loop Modes for $\mu^2$ Morphing Trajectory.

The closed loop state response of the aircraft is shown in Figure 4.15. In the closed loop, with the dynamic inversion controller in the loop, the disturbance to the states decays to zero. The controller is capable of stabilizing the otherwise unstable system.

![Image of graphs showing closed loop response](image)

Figure 4.15. Closed Loop Response $\mu2$ Morphing Trajectory.
The zero-dynamics poles for the closed loop LTV system are shown in Figure 4.16.

![Figure 4.16. Zero Dynamics Poles for $\mu$2 Morphing Trajectory](image)

The zero-dynamics modes are shown in Figure 4.17 and converge to stable values.

![Figure 4.17. Zero Dynamics Modes for $\mu$2 Morphing Trajectory](image)

Their behavior is similar in behavior to the closed loop state response. The zero-dynamics for the system for morphing trajectory $\mu$2 are neutrally stable.
The open and closed loop (zero-dynamics) magnitudes for the modes are shown in Figure 4.18 and the frequencies are shown in Figure 4.19.

Figure 4.18. Time Varying Magnitude for \( \mu_2 \) Morphing Trajectory.

Figure 4.19. Time Varying Frequency for \( \mu_2 \) Morphing Trajectory.
CHAPTER 5
NONLINEAR DYNAMIC INVERSION CONTROL

In this chapter, an Athena Vortex Lattice (AVL) based aerodynamic modeling of a variable wing-sweep MAV is performed. On the basis of this analysis, a nonlinear dynamic model of the morphing MAV is determined. A dynamic inversion based control design using this model is then performed.

5.1 A Morphing Aircraft Configuration

The Air Force Research Laboratory has developed a Generic Micro Air Vehicle (GENMAV) [1] to serve as a test bed for Micro Air Vehicle research. The GENMAV consists of a circular fuselage, a thin cambered wing and a conventional tail. The wing span is 24 inches and the overall length of the fuselage is 17 inches. The base airfoil and the modified GENMAV airfoil are shown in Figure 5.1

![GENMAV airfoil section](image)

Figure 5.1. GENMAV airfoil section

The GENMAV airfoil is based on a design from the University of Florida with a lot of the reflex beyond 30% MAC removed. The GENMAV aircraft is shown in Figure 5.2. In this chapter, we use the GENMAV configuration as a starting point and we assume the existence of a variable wing sweep morphing capability.

![GENMAV General Configuration](image)

Figure 5.2. GENMAV General Configuration.
An isometric view of the aircraft is presented in Figure 5.3.

![Variable Sweep Air Vehicle (VSAV) General Configuration](image)

Figure 5.3. Variable Sweep Air Vehicle (VSAV) General Configuration

The aircraft is comprised of a circular fuselage with a tail boom and conventional empennage. The fuselage is a body of revolution with a tubular tail boom and the main section of the fuselage uses the NACA 0012 airfoil as the cross sectional profile. The wing consists of a rectangular plan form set at an incidence angle of 5 degrees and uses the modified airfoil of then GENMAV. The wing can be swept backward to +30° as well as a forward swept forward to -30°. The cross sectional profiles of the horizontal and vertical tail are flat plates. The maximum thickness of the fuselage pod (without the tail boom) occurs at 12% of the chord and is 0.0610 m. The length of the fuselage including the boom is 0.4317 m. The vertical tail has a height of 0.1140 m with a root chord of 0.0711 m and a tip chord of 0.0574 m. The horizontal tail has a span of 0.1871 m, a root chord of 0.0993 m and a tip chord of 0.0599 m. The wing span and chord are 0.3048 m and 0.1270 m respectively.
A side view of the aircraft shown in Figure 5.4 demonstrates the critical dimensions of the aircraft as well as the camber line for the airfoil, a top view of the aircraft is shown in Figure 5.5.

**Figure 5.4. VSAV Side View**

**Figure 5.5. VSAV Top View**
Two top views of the aircraft with the wings swept backward to +30° and forward to -30° are shown in Figure 5.6 and Figure 5.7 respectively.

Figure 5.6. VSAV Top View with Wings Swept backward to +30°.

Figure 5.7. VSAV Top View with Wings Swept forward to -30°.
5.2 Aerodynamic Model of A Morphing Aircraft

Athena Vortex Lattice (AVL) [17] was used to develop the aerodynamic model for this morphing aircraft. AVL is a program originally written by Harold Youngren for the MIT Athena TODOR aero software collection. The most recent version of the code has been further developed by Mark Drela and Harold Youngren and is freely available. AVL models the lifting surfaces (wings and empennage) using an extended vortex lattice model and uses a slender-body model for the fuselage. AVL models the lifting surface and the trailing wakes as a single-layer vortex sheet that is discretized into horseshoe vortex filaments. AVL models the slender bodies such as the fuselage using source and doublet filaments.

The aerodynamic outputs from AVL include forces and moments acting on the aircraft at different trimmed states as well as the stability and control derivatives in body and stability axes. The stability derivatives from AVL were used to develop the aerodynamic model for VSAV. The aircraft body axis system is defined in Figure 5.8.

![Figure 5.8. Definition of Aircraft Body Axis [18].](image)

The aircraft stability axis system is defined in Figure 5.9.

![Figure 5.9. Definition of Aircraft Stability Axis [18].](image)
AVL was used to obtain the stability and control derivatives for the aircraft in various configurations and various trimmed states. Only those derivatives which dealt with motion in the longitudinal plane were considered. Stability derivatives were calculated for three wing sweep angles, wings fully swept back (+30°), wings at neutral sweep (0°) and wings fully swept forward (-30°). For each sweep configuration stability derivatives for the longitudinal axis were calculated for 13 elevator deflections (-30° to +30°) and 8 angles of attack (-10° to 25°) for each elevator deflection. The non-dimensional stability and control derivatives are shown in Table 5.1.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{m,q} )</td>
<td>( \partial C_m / \partial q )</td>
<td>Derivative of Cm with respect to pitch rate</td>
</tr>
<tr>
<td>( C_{m,\delta e} )</td>
<td>( \partial C_m / \partial \delta_e )</td>
<td>Derivative of Cm with respect to elevator deflection</td>
</tr>
<tr>
<td>( C_{m,\alpha} )</td>
<td>( \partial C_m / \partial \alpha )</td>
<td>Derivative of Cm with respect to elevator deflection</td>
</tr>
<tr>
<td>( C_{L,q} )</td>
<td>( \partial C_L / \partial q )</td>
<td>Derivative of CL with respect to pitching moment</td>
</tr>
<tr>
<td>( C_{L,\delta e} )</td>
<td>( \partial C_L / \partial \delta_e )</td>
<td>Derivative of CL with respect to elevator deflection</td>
</tr>
<tr>
<td>( C_{L,\alpha} )</td>
<td>( \partial C_L / \partial \alpha )</td>
<td>Derivative of CL with respect to angle of attack</td>
</tr>
</tbody>
</table>

Table 5.1. Non-Dimensional Stability and Control Derivatives
The coefficient of lift ($C_L$) at $0^\circ$ elevator deflection for three wing sweep configurations (-30°, 0° and +30°) are presented in Figure 5.10.

![Lift Curve Slopes at Wing Sweep Configurations](image)

**Figure 5.10. Lift Curve Slopes at Wing Sweep Configurations**

The difference in magnitude and gradient of the $C_L$ curves for the wings fully swept forward and fully swept back in almost negligible, with the zero lift angle of attack occurring roughly at $-5^\circ$. At $0^\circ$ angle of attack the $C_L$ for both configurations is 0.2. When compared to the $0^\circ$ sweep configuration, however, there are marked differences. The $C_L$ curve for the $0^\circ$ sweep configuration has a steeper gradient and a much higher magnitude at a given angle of attack, the zero lift alpha is at $-7^\circ$.

Since AVL is a panel method code, it does not predict non-linear aerodynamics; hence the $C_L$ curves do not indicate the natural stall angle of attack for the various sweep configurations. At higher angles of attack the $C_L$ for the $0^\circ$ sweep configuration is over twice the magnitude of the $C_L$ for the -30° and +30° sweep configurations.

The coefficient of drag ($C_D$) at $0^\circ$ elevator deflection for three wing sweep configurations (-30°, 0° and +30°) are presented in Figure 5.11. The viscous component of total drag was estimated at 0.02 and occurs at $-5^\circ$ for all three sweep configurations. The $0^\circ$ sweep configuration produces the highest drag of the three and the forward sweep configuration produces the least drag.
Figure 5.11. Drag Coefficient at Different Wing Sweep Configurations

Representative pitch stability \( (C_{M\alpha}) \) for the three configurations is presented in Figure 5.12 for 0° elevator deflection. The configuration with the wings fully swept backward has the most negative values of \( C_{M\alpha} \) to about 20° alpha and would be the most stable in pitch. The forward swept wing configuration is the least stable in pitch for elevator deflection of 0°. It should be noted however that the values of \( C_{M\alpha} \) are negative for all the sweep configurations, indicating that all three sweep configurations are stable in pitch for 0° of elevator deflection. When the entire matrix of stability derivatives is investigated the 0° sweep and +30° sweep configurations are stable in pitch for a given angle of attack and elevator deflection, with the +30° sweep configuration being the most stable. The -30° sweep configuration displays the least pitch stability and is even unstable for some combinations of elevator deflection and angles of attack. The reduced pitch stability of this configuration can be attributed in part to the aerodynamic center of the wing being the closet to the center of gravity of the aircraft when compared to the other two sweep configurations. The \( C_L / C_D \) ratio and the varying pitch stability as a function of sweep can be used to tailor sweep profiles and optimize the aerodynamics and maneuverability of the aircraft to give it the ability to execute a maneuver with increased efficiency when compared to a non-morphing aircraft.
Simulations of the aircrafts' flight dynamics are conducted using MATLAB. The simulation code consists of an aerodynamic model constructed using the stability derivatives shown in Table 5.1, which are then used to formulate the nonlinear dynamic equations for an aircraft constrained to flight in the longitudinal (XZ plane). The dynamic equations of motion for the aircraft as solved using MATLAB's differential equation solver ODE45. The stability derivatives are fed into the simulation program for sweep angle, angle of attack and elevator deflection as 3-dimensional arrays. The simulation code is capable of querying the arrays and interpolating to obtain the stability derivatives for the appropriate wing sweep angle, angle of attack and elevator deflection. The form for the aerodynamic model and the nonlinear aircraft equations of motion were obtained from Frederico [19].
The $C_{L\alpha}$ for the aircraft is shown in Figure 5.13 as a function of wing sweep angle and angle of attack for 0° elevator deflection. The highest lift curve slope is for the aircraft with wings in the unwept configuration.

Figure 5.13. Variation of $C_{L\alpha}$ with Sweep angle and Angle of Attack at 0° Elevator.

Figure 5.14 presents $C_{M\alpha}$ for the aircraft as a function of wing sweep angle and angle of attack for 0° elevator deflection.

Figure 5.14. Variation of $C_{M\alpha}$ with Sweep angle and Angle of Attack at 0deg Elevator.

The small negative values for $C_{M\alpha}$ at negative sweep angles (wings forward) indicate decreased pitch stability resulting in increased longitudinal maneuverability. This indicates that it is beneficial to have the
wings swept forward during aggressive maneuvering. As shown in Figure 5.15, the magnitude of $C_{MQ}$ is smaller at the negative sweep angles; hence the aircraft displays degraded pitch damping at negative sweep angles. The opposite behavior is seen at the positive sweep angles.

The reverse of this behavior is seen at the positive sweep angles.

The total lift coefficient for the aircraft is a function of angle of attack ($\alpha$), pitch rate ($Q$) and elevator deflection ($de$) and is obtained by using equation (5.1).

$$C_L = C_{L0} + C_{L\alpha} \cdot \alpha + C_{Lq} \cdot q + C_{Lde} \cdot de$$  \hspace{1cm} (5.1)

The total drag coefficient is the sum of the viscous drag ($C_{D0}$) and induced drag ($C_{Dind}$) and is obtained using equation (5.2).

$$C_D = C_{D0} + C_{Dind}$$  \hspace{1cm} (5.2)

The total pitching moment coefficient is a function of angle of attack $\alpha$, pitch rate $q$ and elevator deflection ($de$) and is obtained by using equation (5.3).

$$C_M = C_{m0} + C_{m\alpha} \cdot \alpha + C_{mq} \cdot q + C_{mde} \cdot de$$  \hspace{1cm} (5.3)

The total force in the aircraft body $X$ direction is obtained using equation (5.4).

$$F_X = (C_L \sin(\alpha) - C_D \cos(\alpha)) \cdot \bar{q} \cdot S$$  \hspace{1cm} (5.4)
The total force in the aircraft body Z direction is obtained using equation (5.5).

\[
F_Z = (-C_L \cos(\alpha) - C_D \cos(\alpha)) \cdot \bar{q} \cdot S
\]  
(5.5)

The total pitching moment about the aircraft body Y axis is obtained using equation (5.6).

\[
M = \bar{q} \cdot S \cdot c \cdot C_M
\]  
(5.6)

The non-linear equations of motion in aircraft body axis are presented in equation (5.7) through equation (5.11).

\[
\dot{U} = -Q \cdot W - g \cdot \sin(\theta) + \frac{(F_x + \text{Thrust})}{\text{mass}}
\]  
(5.7)

\[
\dot{W} = Q \cdot U + g \cdot \cos(\theta) + \frac{(F_z)}{\text{mass}}
\]  
(5.8)

\[
\dot{Q} = \frac{M}{I_{YY}}
\]  
(5.9)

\[
\dot{\theta} = q
\]  
(5.10)

\[
d e = -40 d e + 40 u e l e v
\]  
(5.11)

Equation (5.11) is a first order model for elevator’s actuator dynamics. The dynamic equations (5.7) through equation (5.11) are in the body axes. These are transformed to the stability axes prior to being solved using ODE45 such that the states of the aircraft are in terms of total velocity, angle of attack, pitch rate and pitch attitude. The total velocity for the aircraft in the stability axis is obtained using equation (5.12).

\[
V t = \sqrt{U^2 + W^2}
\]  
(5.12)

The relationship between forward velocity, total velocity and angle of attack is shown in equation (5.13).

\[
U = V t \cdot \cos(\alpha)
\]  
(5.13)

The relationship between vertical velocity, total velocity and angle or attack is shown in equation (5.14).

\[
W = V t \cdot \sin(\alpha)
\]  
(5.14)

Using equation (5.12) through equation (5.14), the aircraft equation of motion can be written in stability axes as shown in equation (5.15) through equation (5.19)

\[
\dot{V} t = \frac{(U \cdot \dot{U} + W \cdot \dot{W})}{V t}
\]  
(5.15)
\[ \dot{\alpha} = \frac{(u\dot{w} - w\dot{u})}{u^2 + w^2} \quad (5.16) \]

\[ \dot{Q} = \frac{M}{I_{yy}} \quad (5.17) \]

\[ \dot{\theta} = Q \quad (5.18) \]

\[ \dot{de} = -40de + 40uelev \quad (5.19) \]

We then use non-linear dynamic inversion to design a control law for the aircraft which enables it to perform a pull up with the wings morphing. A brief description of dynamic inversion design for non-linear time varying systems is provided in Chapter 5.3.

### 5.3 Dynamic Inversion Control for Non-Linear Time Varying Systems

Dynamic Inversion for nonlinear systems is described in [8]. For the nonlinear system \( x(t) \) is the state vector, \( u(t) \) is the control input, \( y(t) \) is the output. It is assumed that the entire state vector \( x(t) \) is available for feedback and that the system is square with the number of control inputs \( m \) equal to the number of outputs \( p \). It is also assumed that the system is input affine. The nonlinear equations of motion presented in equation (5.15) and equation (5.19) are in input affine form for the time varying system as shown in equation (5.20).

\[ \dot{x} = f(x, t) + g(x, t)u(t) \quad (5.20) \]

The output equation is shown in equation (5.21) and is a non-linear combination of the states. It is also called the controlled variable (CV).

\[ y(t) = H(x) \quad (5.21) \]

To enable the states of the system to follow a desired trajectory \( r(t) \), the tracking error is defined as the difference between the output and the desired trajectory and is shown in equation (5.22).

\[ e(t) = r(t) - y(t) \quad (5.22) \]

As in the linear case the output \( y(t) \) is differentiated until the control input term appears as shown in equation (5.23).

\[ \dot{y} = \frac{\partial H}{\partial x} \dot{x} = \frac{\partial H}{\partial x} f(x) + \frac{\partial H}{\partial x} g(x)u(t) \quad (5.23) \]
Defining $F(x, t)$ and $G(x, t)$ as in equation (5.24) and equation (5.25).

$$F(x, t) = \frac{\partial H}{\partial x} f(x, t)$$  \hspace{1cm} (5.24)

$$G(x, t) = \frac{\partial H}{\partial x} g(x, t)$$  \hspace{1cm} (5.25)

The derivative of the output can be written as shown in equation (5.26)

$$\dot{y}(t) = F(x, t) + G(x, t)u(t)$$  \hspace{1cm} (5.26)

This can be used to determine $u(t)$ as in equation (5.27).

$$u = G^{-1}(x, t)[-F(x, t) + \dot{r}(t) + v(t)]$$  \hspace{1cm} (5.27)

Define an auxiliary input $v(t)$ such that

$$\dot{y}(t) = r(t) + v(t)$$  \hspace{1cm} (5.28)

Substituting equation (5.28) into the derivative of equation (5.22) gives an expression for the error dynamics and is shown in equation (5.29).

$$\dot{e}(t) = -v(t)$$  \hspace{1cm} (5.29)

A simple choice for the auxiliary input is shown in equation (5.30). It stabilizes the error dynamics as long as the gain matrix $K$ is position definite.

$$v(t) = Ke(t)$$  \hspace{1cm} (5.30)

The auxiliary input is substituted into the equation for the dynamic inversion controller. The control equation is shown in equation (5.31) and the closed loop can be expressed as shown in equation (5.32).

$$u = G^{-1}(x, t)[-F(x, t) + \dot{r} + Ke]$$  \hspace{1cm} (5.31)

$$\dot{x} = f(x, t) + g(x, t)G^{-1}(x, t)[-F(x, t) + \dot{r} + Ke]$$  \hspace{1cm} (5.32)

**5.4 Control Law Development for Morphing Aircraft**

The dynamic inversion design technique is used to develop a control law for the morphing aircraft using the nonlinear aerodynamic model and the aircraft equations of motion presented in earlier section. The output $(y(t))$ (Control Variable) is selected such that the zero-dynamics are stable. Initially pitch rate $(\dot{Q})$ was selected as the CV, however this resulted in unstable zero-dynamics. Reference [8] details a
control variable which has been used in LTI aircraft in the longitudinal axis with some success. This was used as a starting point and altered iteratively through trial and error to arrive at equation (5.33). A criteria which was used when altering the CV was to keep all the terms related to pitch attitude or derivatives of pitch attitude ie, \( \theta \), \( Q \) or \( \dot{Q} \). Since the CV only deals with rotation about the aircraft body Y axis, it is possible to write a function for the reference signal which commands a pull up. The output \( y(t) \) is defined as shown in equation (5.33).

\[
y(t) = H = 12Q + \frac{\dot{Q}}{g} + 0.75\theta
\]  \hspace{1cm} (5.33)

The expression for the output is differentiated using the chain rule as shown in equation (5.34).

\[
\dot{y}(t) = \frac{\partial H}{\partial x} \dot{x} \text{ where the states are } x = [V t \; \alpha \; Q \; \theta \; de]^t
\]  \hspace{1cm} (5.34)

The derivative of pitch rate (\( \dot{Q} \)) and pitch attitude (\( \theta \)) state vector \( (x) \) is shown in equation (5.35)

\[
\frac{\partial Q}{\partial x} = [0 \; 0 \; 1 \; 0 \; 0.75]
\]  \hspace{1cm} (5.35)

The partial derivative is written as equation (5.36).

\[
\frac{\partial H}{\partial x} = \left[ \frac{1}{g} \frac{\partial \dot{Q}}{\partial V t} \; \frac{1}{g} \frac{\partial \dot{Q}}{\partial \alpha} \left( 12 + \frac{1}{g} \frac{\partial \dot{Q}}{\partial Q} \right) \left( 0.75 + \frac{1}{g} \frac{\partial \dot{Q}}{\partial \theta} \right) \frac{\partial \dot{Q}}{\partial de} \right]
\]  \hspace{1cm} (5.36)

The partial derivatives \( \dot{Q} \) are as shown in equations (5.37) through equation (5.41).

\[
\begin{align*}
\frac{\partial \dot{Q}}{\partial V t} &= \frac{\rho.Vt.S.c.C_M}{I_{YY}} \\
\frac{\partial \dot{Q}}{\partial \alpha} &= \frac{\bar{q}.S.c.C_{M\alpha}}{I_{YY}} \\
\frac{\partial \dot{Q}}{\partial Q} &= \frac{\bar{q}.S.c.C_{MQ}}{I_{YY}} \\
\frac{\partial \dot{Q}}{\partial \theta} &= 0 \\
\frac{\partial \dot{Q}}{\partial de} &= \frac{\bar{q}.S.c.C_{Mde}}{I_{YY}} 
\end{align*}
\]  \hspace{1cm} (5.37-5.41)

The expression for the control input \( (u_{elev}) \) for the can be expressed as shown in equation (5.42).

\[
u_{elev}(t) = G^{-1}(x,t)[-F(x,t) + \dot{r}(t) + Ke(t)]
\]  \hspace{1cm} (5.42)

A positive value for the gain \( K \) stabilizes the system.

A value of \( K = 15 \) is used.
The block diagram of the closed loop aircraft as shown in Figure 5.16 reveals that intrinsic to the control law (expression for $u_{elev}$) are the aircraft dynamics, hence non-linear functions of the aircraft's state equations must be known.

Figure 5.16. Block Diagram of Closed Loop System with Dynamic Inversion Controller.

The inclusion of the aircraft's dynamic model in the control law makes dynamic inversion design ideally suited to systems where the dynamics vary with varying external geometry such in our current scenario. The stability derivatives, which constitute the non-linear functions of the state equations are implemented as a set of look up tables, that are a function of wing sweep, elevator deflection and angle of attack.
5.5 Simulated Pull Up with Morphing

The control law determined above is then used to perform a pull up maneuver. The reference signal was selected which commanded a varying pitch rate and pitch attitude, thereby causing the aircraft to perform a pull up. The dynamic inversion controller has access to all the aircraft states ($V t \alpha Q \theta \phi$) and the pitch acceleration ($\dot{Q}$) is obtained by employing a backward difference calculation on the pitch rate ($Q$). The reference input $r(t)$ is shown in equation (5.43) and the corresponding derivative $\dot{r}(t)$ is shown in equation (5.44).

$$r(t) = -0.1 + 0.02t + 0.005t^2 \quad (5.43)$$
$$\dot{r}(t) = 0.02 + 0.01t \quad (5.44)$$

The positive definite gain value $K$ ensures that the error dynamics are stable, i.e. the output ($y(t)$) converges to the reference value ($r(t)$). The resulting control of the pitch attitude of the aircraft provides control over horizontal and vertical displacement with respect to an earth fixed navigation frame. The relationship between the aircraft states and the aircraft inertial displacements are provided by the navigation equations which are referenced to an earth fixed inertial axis system. The differential equation which determines the horizontal displacement over ground is presented in equation (5.45). The differential equation which determines height above ground is shown in equation (5.46).

$$\dot{x}_E = U + W \sin(\theta) \quad (5.45)$$
$$\dot{h}_E = W \sin(\theta) - W \cos(\theta) \quad (5.46)$$

The morphing trajectory is designed such that the morphing begins at the 7 second mark and ends at the 9 second mark. These times were selected because it brackets the region when the aircraft changes its pitch attitude from nose down to nose up. A linear morphing trajectory is considered, where the aircraft enters the pull-up with the wings swept back to $30^\circ$ and sweeps forward ending with the wings swept forward at $-30^\circ$ and is presented in equation (5.57). The wings morph forward at a rate of 30deg/sec and lasts 2 seconds.

$$\mu(t) = 30^\circ + (-30 \, \text{deg} \, s^{-1} \, t) \quad 7 \, \text{sec} \leq t \leq 9 \, \text{sec} \quad (5.47)$$
The results of the maneuver are presented in Figure 5.17 as a plot of vertical displacement (altitude) versus horizontal displacement.

![Figure 5.17 Vertical Displacement vs Horizontal Displacement for the Pull Up Maneuver](image)

The vertical displacement is shown as a function of time in Figure 5.18.

![Figure 5.18 Vertical Displacement vs Time during the Pull Up Maneuver](image)
The total velocity of the aircraft \((Vt)\) during the pull up is presented in Figure 5.19. The total velocity of the aircraft increases during the dive portion of the pull up (from 2 seconds to 8 seconds), and then drops during the climb portion as the aircraft trades airspeed and kinetic energy for altitude (from 9 seconds to 20 seconds).

![Figure 5.19 Total Velocity (m/s) vs Time during the Pull Up Maneuver](image)

The angle of attack response is shown in Figure 5.20.

![Figure 5.20 Angle of Attack (deg) vs Time during the Pull Up Maneuver](image)
Figure 5.21 presents a comparison of the reference signal, shown in equation (5.45), to the aircraft state response which is the output shown in equation (5.33).

![Reference Signal and Output vs Time](image1)

**Figure 5.21. Reference Signal and Output vs Time during the Pull Up Maneuver.**

The controller is able to follow the reference signal very well. The elevator response which is commanded by the controller is shown in Figure 5.22.

![Elevator Deflection vs Time](image2)

**Figure 5.22 Elevator Deflection \( (\delta e) \) vs Time during the Pull Up Maneuver**
5.6 Nonlinear Zero Dynamics

The closed loop non-linear system for the aircraft is represented by equation (5.32). The selection of the stabilizing matrix $K$ ensures that the error dynamics, equation (5.22), is stable and the output converges to the reference signal. The zero-dynamics of the systems are defined as the dynamics of the system when auxiliary input $v(t)$ is selected such that the output $y(t)$ is zero [8]. Setting $y(t)$ to zero in equation (5.22), results in the relationship shown in equation (5.43).

$$e(t) = r(t)$$

(5.48)

Using the relationship in equation (5.29), an expression is obtained for $v(t)$ which results in $y(t)$ being equation to zero and is shown in equation (5.44)

$$v(t) = -\dot{y}(t)$$

(5.49)

Substituting equation (5.44) into the closed loop system produces the expression for the zero-dynamics shown in equation (5.45).

$$\dot{x} = \left[ I - g(x, t)G^{-1}(x, t) \frac{\partial H}{\partial x} \right] f(x, t)$$

(5.50)

The zero-dynamics are non-linear, their boundedness is shown by simulation. Equation (5.45) was simulated for several initial conditions. The simulation shows that for the initial conditions, the states remain bounded and do not diverge. The simulation was conducted with the aircraft morphing from 30° of wing sweep to -30° of wing sweep for 0 to 2 seconds, which is the same morphing trajectory used in the controller design.
Examining the states for total velocity ($V_t$) in Figure 5.23 reveals that for a wide range of initial conditions the total velocity reaches a bounded value.

![Graph of Total Velocity](image)

**Figure 5.23. Zero-Dynamics, Total Velocity ($V_t$) (m/s)**

A similar result is seen in Figure 5.24, where the aircraft attains a bounded angle of attack regardless of the initial value and the transient oscillation.

![Graph of Angle of Attack](image)

**Figure 5.24. Zero-Dynamics, Angle of Attack ($\alpha$) (deg)**
The response of the pitch rate for the zero-dynamics is presented in Figure 5.25 for an initial state of 0 deg/s.

![Figure 5.25. Zero-Dynamics, Pitch Rate (\(Q\)) (deg/s)](image)

The aircraft converges to stable and bounded point for the pitch rate with the oscillation damps out. The pitch attitude response is shown in Figure 5.26 shows the aircraft settling at a pitch attitude without diverging..

![Figure 5.26. Zero-Dynamics, Pitch Attitude (\(\theta\)) (deg)](image)
The elevator deflection of the aircraft is shown in Figure 5.27 for a range of initial values; it remains bounded for all the initial conditions that were considered.

![Graph](image)

**Figure 5.27.** Zero-Dynamics, Elevator Deflection (de) (deg)

The simulation shows that regardless of the initial condition, the states remain bounded. This demonstrates that the selection of the Controlled Variable in equation (5.33) ensures that the aircraft has non-diverging zero–dynamics with all the states remaining bounded.
CHAPTER 6
CONCLUSIONS

This thesis investigates the use Dynamic Inversion to develop control laws that can effectively stabilize and control a morphing aircraft modeled as a nonlinear time varying system. The inclusion of the full system model in the control law enables the controller to effectively manage the time varying influences that directly affect the system dynamics.

Kamen’s concept of poles and modes, that involve the use of a time varying Ricatti equation, were used to calculate the time varying poles for morphing aircraft in the open and closed loop. By using these modes it is possible to determine the performance of different morphing trajectories of the aircraft. The LTV modes also reveal the stability and performance of the zero-dynamics of the system in the closed loop. The control variable was suitably adjusted to ensure that the closed loop system remains stable while still allowing for the desired states in the control variable to follow a defined reference signal.

The control law was used to perform a simulated pull up maneuver which had the wings sweep for a back to front at the bottom of the pull up. Investigating the state response of the zero-dynamics showed that the selected CV results all the remaining states in being bounded as the aircraft tracks the reference signal and executes the pull up.

There remains a lot of work to be done in study of using Dynamic Inversion to control morphing aircraft. One area is in developing suitable Control Variables specifically for time varying systems and investigating their effect on the performance of the aircraft.

Dynamic Inversion depends on the inclusion of a full aerodynamic model of the aircraft in the control law and to this end it would be beneficial to move from AVL-based estimates of aerodynamic data to aerodynamic data obtained from wind tunnel testing. This would provide the most reliable system dynamics short of using actual flight data. A further step to providing a reliable aerodynamic model would be to develop and incorporate an unsteady aerodynamic model in the control law that would model the transient aerodynamics during morphing.

The ultimate goal is to develop a morphing aircraft and corresponding control laws using Dynamic Inversion which allows control of the aircraft during all segments of the flight regime and maximizes the improved agility provided by morphing.
REFERENCES


REFERENCES (continued)


