ON CONVERGENCE SETS OF FORMAL POWER SERIES

A Dissertation by

Basma Al-Shutnawi

Bachelor of Science, Yarmouk University, 2000

Master of Science, Yarmouk University, 2005

Submitted to the Department of Mathematics and Statistics
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

December 2013

The following faculty members have examined the final copy of this dissertation for form and content, and recommended that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy with a major in Applied Mathematics.

Daowei Ma, Committee Chair

Thomas K. Delillo, Committee Member

Buma L. Fridman, Committee Member

Kenneth G. Miller, Committee Member

Christopher M. Rogers, Committee Member

Accepted for the College of Liberal Arts and Sciences

Ron Matson, Interim Dean

Accepted for the Graduate School

Abu S.M. Masud, Interim Dean

DEDICATION

To my parents

ACKNOWLEDGEMENTS

This dissertation would not have been possible without the guidance and help of several individuals who in one way or another contributed and extended their valuable assistance in the preparation and completion of this study. My first debt of gratitude must go to my advisor, Professor Daowei Ma, who was truly an inspiration. Without his guidance, this dissertation would not be possible. Professor Buma Fridman patiently provided the vision, encouragement and advice, necessary for me to proceed in the doctoral program and complete my dissertation. I would like to thank the graduate coordinators of the Mathematics Department, Professor Kirk Lancaster and Professor Kenneth G. Miller, who have been kind to all graduate students. I would like to thank Professor Hua Liu for his help. The complex analysis seminar was instrumental towards the completion of this dissertation. I would like to thank my officemates in 315 Jabara, B. Gort, Q. Zou, and Y. Wang, for their friendship and warmth. Last but not the least, I would like to thank my family: my husband Yaser, for his patience and support, and my lovely children Salma, Karam, and Yousef, for their excellent behavior in the period when I prepared this dissertation.

ABSTRACT

In this thesis we consider the convergence sets of formal power series of the form $f(z,t) = \sum_{j=0}^{\infty} p_j(z)t^j$, where $p_j(z)$ are polynomials. A subset E of the complex plane $\mathbb{C}$ is said to be a convergence set if there is a series $f(z,t) = \sum_{j=0}^{\infty} p_j(z)t^j$ such that E is exactly the set of points z for which $f(z,t)$ converges as a power series in t. A quasi-simply-connected set is defined to be the union of a countable collection of polynomially convex compact sets. We prove that a subset of $\mathbb{C}$ is a convergence set if and only if it is a quasi-simply-connected set. We also give an example of a compact set which is not a convergence set.

TABLE OF CONTENTS

Chapter **Page**

CHAPTER 1

Introduction

An important problem in applied mathematics is the problem of restoration of a function from its integrals over a given collection of sets. An example of such a problem is tomography. In $\mathbb{R}^3$ the integrals of a function over a set of one dimensional lines allow one to restore the function by using the inverse of the Radon transform. This of course has wide applications in medicine, namely CAT scan. If a function represents an unknown density, then the Radon transform represents the scattering data obtained as the output of a tomographic scan. Hence the inverse of the Radon transform can be used to reconstruct the original density from the scattering data, and thus it forms the mathematical underpinning for tomographic reconstruction.

So the above problems allow the **full reconstruction** of a function by knowing integrals or the properties of that function on a collection of smaller sets. Another group of mathematical problems concern the retrieval of **properties** for a function on the entire domain from the known properties on a given collection of sets. The general description of these problems is given in K. Spallek, P. Tworzewski, and T. Winiarski [16]: "Osgood-Hartogs-type problems ask for properties of objects whose restrictions to certain test-sets are well known". Their article has a number of examples of such problems. Here are two classical examples (we present these examples following [4]). The famous Hartogs theorem states that a function f in $\mathbb{C}^n$, $n > 1$, is holomorphic if it is holomorphic in each variable separately, that is, f is holomorphic in $\mathbb{C}^n$ if it is holomorphic on complex line parallel to an axis. So, one can test the holomorphy of a function in $\mathbb{C}^n$ by examining if it is holomorphic on each of the above mentioned complex lines. Bochnak-Siciak theorem in [3] is another example. That theorem states that for $f \in \mathbb{C}^\infty(\mathbb{R}^n)$ if f is real-analytic on a line segment through 0 in each direction, then f is real-analytic in a neighborhood of 0 in $\mathbb{R}^n$.

The "objects" from the quote above are not necessarily functions. There is a large body of research concerning these kinds of problems when the "object" is a power series of two or more variables, and the property to inspect is convergence of such a series. For example the papers ([14], [1], [5], [4], [8], [10], [9]) consider the convergence of formal power series of several variables provided the restriction of such a series on each element of a sufficiently large family of linear subspaces is convergent. Let $f = f(z_1, ..., z_n)$ be a power series. Consider $F(z,t) = f(tz_1, \ldots, tz_n) = \sum_{m=0}^{\infty} P_m(z)t^m$. Suppose that $F(z,t)$ as a power series in t converges for z in a set $E \subset \mathbb{C}^n$. If the capacity of E is positive, then f converges. Note that $\deg P_m = m$. This restriction is very often used in similar investigations.

A natural new question is: given arbitrary $P_m(z)$ what is the necessary and sufficient condition for a set E to form a set of convergence for the above series in t. This thesis answers the question completely for $n = 1$, namely we provide the full description of such sets in $\mathbb{C}$.

In Chapter 2 we define the **convergence set** of a formal power series f and show that a polynomially convex compact set in $\mathbb{C}^n$ is a convergence set. Then we define the class $C(\delta, A, B)$ and study the convergence sets for this class.

In Chapter 3 we define a **quasi-simply-connected** set and show that every convergence set is a quasi-simply-connected set. At the end of this chapter we prove that every quasi-simply-connected set in $\mathbb{C}$ is a convergence set.

CHAPTER 2

Convergence Sets

2.1 Convergence Sets

Let $\mathbb{C}[z]$ be the set of polynomials in z, $\mathbb{C}[[z]]$ be the set of formal power series, and $\mathbb{C}[z][[t]]$ be the set of formal power series in t with coefficients being polynomials in z.

We consider the formal series,

$$\begin{aligned} f(z) &= f(z_1, z_2, \ldots, z_n) \\ &= a_0 + \sum_{|\alpha_1|=1} a_{\alpha_1} z^{\alpha_1} + \sum_{|\alpha_2|=2} a_{\alpha_2} z^{\alpha_2} + \cdots \\ &\quad + \sum_{|\alpha_n|=n} a_{\alpha_n} z^{\alpha_n} + \cdots , \end{aligned} \tag{2.1}$$

where $\alpha_j = (\alpha_{j1}, \cdots, \alpha_{jn})$ are the n-multiple index and $|\alpha_j| = |\alpha_{j1}| + \cdots + |\alpha_{jn}|$.

A classical result of Hartogs states that $f(z)$ converges if and only if $f_\xi(t) := f(t\xi)$ converges as a series in t for ξ on every complex line through the origin. This can be interpreted as a formal analog of Hartogs's separate analyticity theorem.

Definition 2.1.1. *A power series $f \in \mathbb{C}[[z_1, \ldots, z_n]]$ is said to be convergent* if there is a constant $C = C_f$ such that $|a_{k_1 \ldots k_n}| \leq C^{k_1 + \cdots + k_n}$ for all $(k_1, \ldots, k_n) \neq (0, \ldots, 0)$.

A convergent formal power series $f(z_1, \ldots, z_n)$ with coefficients in $\mathbb{C}$ is absolutely convergent in some neighborhood of the origin in $\mathbb{C}^n$. A power series f is called divergent if it is not convergent. A power series equals 0 if all of its coefficients $a_{k_1 \ldots k_n}$ are equal to 0. If f is convergent, then it represents a holomorphic function in some neighborhood of 0 in $\mathbb{C}^n$.

Definition 2.1.2. Let $f(z,t) \in \mathbb{C}[z][[t]]$. We define the *convergence set* of f to be

$$\mathrm{Conv}(f) = \{z \in \mathbb{C} : f(z,t) \text{ converges in } t\}.$$

Definition 2.1.3. A subset $E \subset \mathbb{C}$ is said to be a *convergence set* if there exists an $f \in \mathbb{C}[z][[t]]$ such that $E = \mathrm{Conv}(f)$.

Let $\mathbb{C}[[z_1, \ldots, z_n]]$ denote the set of (formal) power series

$$f(z_1 \ldots, z_n) = \sum_{k_1,\ldots,k_n \geq 0} a_{k_1 \ldots k_n} z_1^{k_1} \cdots z_n^{k_n},$$

of n variables with complex coefficients. Let $f(0) = f(0, \ldots, 0)$ denote the coefficient $a_{0,\ldots,0}$.

Let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, \ldots\}$. For a subset S of $\mathbb{C}^n$ and a function h on S, $||h||_S$ denotes the supremum of $|h(z)|$ on S.

Definition 2.1.4. ([6, page 62]) Let K be a non-empty compact subset of $\mathbb{C}$. The *polynomial hull* of K is the set

$$\hat{K} = \Big\{ z \in \mathbb{C}^n : |p(z)| \leq ||p||_K, \ \text{ for every polynomial } \ p \Big\}.$$

A compact set K is called *polynomially convex* if $\hat{K} = K$.

A countable set is a set that admits an injective map to $\mathbb{N}$. Thus a finite set is countable, and a countable infinite set is a set that admits a bijective map to $\mathbb{N}$.

Proposition 2.1.5. *Let K be a polynomially convex compact set in $\mathbb{C}^n$. Then K is a convergence set.*

Proof. Let m be a positive integer and let $y \in \mathbb{C}^n \setminus K$. There exists a polynomial $P_y(z)$ such that $|P_y(y)| > m$, and $|P_y|_K \leq 1$.

Set $U_y = \{x : |P_y(x)| > m, x \in \mathbb{C}^n \backslash \mathrm{K}\}$. Then, U_y is an open neighborhood of y. The open cover $\{U_y : y \in \mathbb{C}^n \setminus K\}$ of the set $\mathbb{C}^n \setminus K$ contains a countable subcover $\{U_{y_k} : k = 1, 2, \ldots\}$. Now write $P_{mk}(z) = P_{y_k}(z)$. For each m we get a sequence $\{P_{mk}\}_{k=1}^{\infty}$. Since the set $\{P_{mk}\}$ is countable we can arrange it as a sequence $\{h_j(z)\}_{j=1}^{\infty}$. Set

$$f(z, t) = \sum h_j^j(z) t^j.$$

Suppose that $z \in K$. Then for each j, $|h_j(z)| \leq 1$. Hence $z \in \mathrm{Conv}(f)$. Consequently, $K \subset \mathrm{Conv}(f)$.

Now suppose that $z \in \mathbb{C}^n \setminus K$. Then for each $m \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $|P_{mk}(z)| \geq m$. It follows that the sequence $\{|h_\ell(z)|\}$ is unbounded. So the formal power series

$f(z,t)$ is divergent at this point z. Consequently, $\text{Conv}(f) \subset K$. Therefore $K = \text{Conv}(f)$.

□

Using the fact that every convex compact set in $\mathbb{C}^n$ is polynomially convex set we get the following corollary.

Corollary 2.1.6. *Every convex compact set in $\mathbb{C}^n$ is a convergence set.*

2.2 Class (δ, A, B)

Definition 2.2.1. Let $\delta \geq -1$, $A > 0$, and $B \geq 0$. A series $f(z,t) = \sum P_n(z)t^n$ is said to be of *class* $C(\delta, A, B)$ if $\deg P_n \leq An^{1+\delta}+B$ for n sufficiently large.

Definition 2.2.2. A subset $E \subset \mathbb{C}$ is said to be a $C(\delta, A, B)$ *convergence set* if there exists an $f \in C(\delta, A, B)$ such that $E = \text{Conv}(f)$.

Let $\lfloor X \rfloor$ denote the greatest integer that is less than or equal to X.

Proposition 2.2.3. *For any fixed $\delta \geq 0$, every $C(\delta, A, B)$ convergence set is a $C(\delta, 1, 0)$ convergence set.*

Proof. Let E be a $C(\delta, A, B)$ convergence set. Then there is an $f \in C(\delta, A, B)$,

$$f(z,t) = p_0(z) + p_1(z)t + \cdots + p_n(z)t^n + \cdots, \tag{2.2}$$

with $E = \text{Conv}(f)$ and $\deg p_n \leq An^{1+\delta}+B$.

Let

$$\tilde{f}(z,t) = \sum_{j=0}^{\infty} p_j(z)t^{N_j}, \tag{2.3}$$

where $N_j = \lfloor (A+B)^{\frac{1}{1+\delta}} + 1 \rfloor j$. Then, for $j \geq 1$,

$$\begin{aligned}
\deg(p_j) &\leq Aj^{1+\delta} + B \\
&\leq (A+B)j^{1+\delta} \\
&= ((A+B)^{\frac{1}{1+\delta}} j)^{1+\delta} \\
&\leq (\lfloor (A+B)^{\frac{1}{1+\delta}} + 1 \rfloor j)^{1+\delta} \\
&= N_j^{1+\delta}.
\end{aligned}$$

It follows that $\tilde{f}(z,t) \in C(\delta,1,0)$. We now prove that $E = \mathrm{Conv}(\tilde{f})$. For $z \in E$ there exists a positive number c such that $|p_n(z)| < c^n$. Let $\tilde{c} = c^{1/\lfloor (A+B)^{\frac{1}{1+\delta}}+1\rfloor}$. Then

$$|p_n(z)| < \tilde{c}^{\lfloor (A+B)^{\frac{1}{1+\delta}}+1\rfloor n} = \tilde{c}^{N_n},$$

which shows that $z \in \mathrm{Conv}(\tilde{f})$, and so $\mathrm{Conv}(f) \subset \mathrm{Conv}(\tilde{f})$.

On the other hand, for $z \in \mathrm{Conv}(\tilde{f})$ we can reverse the steps above to conclude that $\mathrm{Conv}(\tilde{f}) \subset \mathrm{Conv}(f)$. □

In the next example we will construct a $C(1,1,0)$ formal power series which defines the closed unit disc as a $C(1,1,0)$ convergence set.

Example 2.2.4. The formal power series

$$f(z,t) = 1 + zt + \frac{z^4}{4}t^2 + \cdots + \frac{z^{n^2}}{n^2}t^n + \cdots \tag{2.4}$$

is in $C(1,1,0)$ and $\mathrm{Conv}(f) = \bar{\Delta}$. Hence the closed unit disc $\bar{\Delta}$ is a $C(1,1,0)$ convergence set.

Proof. For $|z| \leq 1$,

$$\left|\frac{z^{n^2}}{n^2}\right| \leq 1/n^2,$$

so $f_z(t) = f(z,t)$ converges uniformly as the series of t in $\bar{\Delta}$. While for $|z| > 1$,

$$\lim_{n\to\infty} \left|\frac{z^{n^2}}{n^2}\right|^{\frac{1}{n}} = \infty,$$

so $f_z(t)$ is divergent. □

Proposition 2.2.5. Let $0 < \lambda < \delta$. Then the collection of $C(\delta,1,0)$ convergence sets, and the collection of $C(\lambda,1,0)$ convergence sets are the same.

Proof. It is clear that each $C(\lambda,1,0)$ convergence set is a $C(\delta,1,0)$ convergence set. Suppose that E is a $C(\delta,1,0)$ convergence set. Then there is a series

$$f(z,t) = \sum p_n(z)t^n$$

in $C(\delta,1,0)$ such that $E = \mathrm{Conv}(f)$. Let $\ell = \delta/\lambda$, and

$$g(z,t) = \sum_{n=0}^{\infty} p_n(z)^{\lfloor n^{\ell} \rfloor} t^{\lfloor n^{l+1} \rfloor}.$$

Then

$$\begin{aligned}
\deg\ p_n(z)^{\lfloor n^{\ell} \rfloor} &\le n^{1+\delta} \lfloor n^{\ell} \rfloor \\
&\le n^{1+\delta} n^{\ell} \\
&\le (n^{\ell+1})^{\lambda+1} \\
&< (\lfloor n^{\ell+1} \rfloor + 1)^{\lambda+1} \\
&\le (\lfloor n^{\ell+1} \rfloor + \lfloor n^{\ell+1} \rfloor)^{\lambda+1} \\
&= (2\lfloor n^{\ell+1} \rfloor)^{\lambda+1} \\
&= 2^{\lambda+1}(\lfloor n^{\ell+1} \rfloor)^{\lambda+1}.
\end{aligned}$$

It follows that $g(z,t) \in C(\lambda, 2^{\lambda+1}, 0)$. We now prove that $E = \mathrm{Conv}(g)$. For $z \in E$ there exists a positive number c such that $|p_n(z)| < c^n$ for all $n \ge 1$. Hence,

$$|p_n(z)|^{\lfloor n^{\ell} \rfloor} < c^{\lfloor n^{\ell} \rfloor n} \le c^{\lfloor n^{\ell+1} \rfloor},$$

which shows that $z \in \mathrm{Conv}(g)$, and so $E \subset \mathrm{Conv}(g)$. The reversed inclusion is similarly proved. Thus $E = \mathrm{Conv}(g)$. □

Recall that an F_σ set is the union of a countable collection of closed sets. Now we discuss some properties for the convergence sets in the complex plane.

Theorem 2.2.6. *Let A be a convergence set. Then A is an F_σ set.*

Proof. Suppose that $A = \mathrm{Conv}(f)$ and

$$f(z,t) = f_0(z) + f_1(z)t + \cdots + f_n(z)t^n + \cdots .$$

Now we prove that

$$A = \bigcup_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \{z \in \mathbb{C} :\ |f_n(z)| \le j^n\}. \tag{2.5}$$

For $z \in A$, by the definition of convergence set, there exists a positive integer j such that

$$|f_n(z)| < j^n,\ n \in \mathbb{N}.$$

So $z \in \bigcup_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \{z \in \mathbb{C} :\ |f_n(z)| \leq j^n\}$. Hence $A \subset \bigcup_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \{z \in \mathbb{C} :\ |f_n(z)| \leq j^n\}$. On the other hand, assume that $z \in \bigcup_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \{z \in \mathbb{C} :\ |f_n(z)| \leq j^n\}$. Then there exist a positive integer j such that $|f_n(z)| \leq j^n$, $\forall n$. So $z \in A$.

It is clear that for each n, the set $\{z \in \mathbb{C} :\ |f_n(z)| \leq j^n\}$ is closed. Therefore $\bigcap_{n=1}^{\infty} \{z \in \mathbb{C} :\ |f_n(z)| \leq j^n\}$ is closed as the intersection of closed sets. By (2.5), A is an F_σ set. □

The converse of this theorem is not true; we will give a counterexample in the next chapter.

We now discuss the intersection of two convergence sets, and state the following theorem.

Theorem 2.2.7. *A finite intersection of convergence sets is a convergence set.*

Proof. It suffices to prove that the intersection of two convergence sets is a convergence set. Suppose we have two formal power series

$$f(z,t) = f_0(z) + f_1(z)t + \cdots + f_n(z)t^n + \cdots ,$$

and

$$g(z,t) = g_0(z) + g_1(z)t + \cdots + g_n(z)t^n + \cdots ,$$

and $A = \mathrm{Conv}(f)$ and $B = \mathrm{Conv}(g)$. Let

$$\begin{aligned} F(z,t) &= f(z,t^2) + tg(z,t^2) \\ &= f_0(z) + g_0(z)t + f_1(z)t^2 + g_1(z)t^3 \cdots . \end{aligned}$$

Then $\mathrm{Conv}(F) = A \cap B$.

□

For $r > 0$ and $S \subset \mathbb{C}$, let $N_r(S)$ denote the r-neighborhood of S, *i.e.*,

$$N_r(S) = \{z \in \mathbb{C} : |z - p| < r \text{ for some } p \in S\}.$$

Theorem 2.2.8. *Let $S = \{z_1, z_2, \dots\}$ be a countable infinite subset of $\mathbb{C}$. Define an $F \in \mathbb{C}[z][[t]]$ by*

$$F(z,t) = \sum_{n=0}^{\infty} C_n \Big[\prod_{j=1}^{n}(z - z_j)\Big] t^n,$$

where $C_n = (n/\gamma_n)^n$, and

$$\gamma_n = \min(\frac{1}{2} \min_{1 \le i < j \le n+1} |z_i - z_j|, 1/n).$$

Then $\mathrm{Conv}(F) = S$.

Proof. Note that γ_n is positive since z_i are pairwise distinct. Let $L_k = \{z_1, \cdots, z_k\}$, and $U_j = N_{\gamma_j}(L_j)$. We now prove that

$$\bigcap_{j=k}^{\infty} U_j = L_k. \tag{2.6}$$

It is obvious that $L_k \subset \bigcap_{j=k}^{\infty} U_j$. We only need to prove the reversed inclusion, which would follow from the following statement:

$$\bigcap_{s=k}^{\infty} U_s \subset N_{\gamma_j}(L_k), \text{ for } j \ge k. \tag{2.7}$$

We prove this statement by induction on j. It is obvious for $j = k$ since $N_{\gamma_k}(L_k) = U_k$. Suppose the statement is true for $j = N \ge k$. Let $z \in \bigcap_{s=k}^{\infty} U_s$. For $i \ne j$, $1 \le i, j \le N+1$, since $|z_j - z_i| \ge 2\gamma_N$, we see that $N_{\gamma_N}(z_i) \cap N_{\gamma_N}(z_j) = \emptyset$. It follows that

$$\Big(\bigcup_{\ell=k+1}^{N+1} N_{\gamma_N}(z_\ell)\Big) \cap \Big(N_{\gamma_N}(L_k)\Big) = \emptyset. \tag{2.8}$$

By the induction hypothesis $z \in N_{\gamma_N}(L_k)$ and by (2.8), $z \notin \bigcup_{\ell=k+1}^{N+1} N_{\gamma_N}(z_\ell)$. Since $\gamma_{N+1} \le \gamma_N$ we see that $\bigcup_{\ell=k+1}^{N+1} N_{\gamma_{N+1}}(z_\ell) \subset \bigcup_{\ell=k+1}^{N+1} N_{\gamma_N}(z_\ell)$, hence

$$z \notin \bigcup_{\ell=k+1}^{N+1} N_{\gamma_{N+1}}(z_\ell). \tag{2.9}$$

On the other hand, we know that

$$z \in U_{N+1} = \Big(\bigcup_{\ell=k+1}^{N+1} N_{\gamma_{N+1}}(z_\ell)\Big) \cup \Big(N_{\gamma_{N+1}}(L_k)\Big). \tag{2.10}$$

By (2.9) and (2.10), $z \in N_{\gamma_{N+1}}(L_k)$. This completes the induction step, and therefore the statement is proved.

Now let $P_0(z) = 1$ and

$$P_n(z) = (\frac{n}{\gamma_n})^n \prod_{k=1}^{n}(z - z_k), \tag{2.11}$$

and

$$f(z,t) = 1 + P_1(z)t + \cdots + P_n(z)t^n + \cdots .$$

For $n \geq k$, we have $P_n(z_k) = 0$. It follows that $S \subset \text{Conv}(f)$.

Now suppose that $z \notin S$. By (2.6), $z \notin \bigcap_{j=k}^{\infty} U_j$ for each k. So there exists a $j_k \geq k$ such that $z \notin U_{j_k}$. In this way we get a sequence $\{j_k\}_k$ such that $z \notin U_{j_k}, k = 1, 2, \cdots$. Then

$$|P_{j_k}(z)| \geq (\frac{j_k}{\gamma_{j_k}})^{j_k} \prod_{i=1}^{j_k}(\gamma_i) \geq (j_k)^{j_k}, \tag{2.12}$$

since $\gamma_i \geq \gamma_{j_k}$ for $i \leq j_k$. This implies that $z \notin \text{Conv}(f)$. Therefore $S = \text{Conv}(f)$. □

Professor Tejinder Neelon suggested that it might be possible to choose the coefficients C_n so that $\text{Conv}(F) = \{z_1, z_2, \dots\}$.

We have the following corollary.

Corollary 2.2.9. *The set $\mathbb{Q}$ of rational numbers is a convergence set.*

Remark: The set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not a convergence set, since it is not an F_σ set. This can be seen as follows. Suppose that $\mathbb{R} \setminus \mathbb{Q}$ is an F_σ set. Then

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{j=1}^{\infty} C_j, \tag{2.13}$$

where C_j are closed sets. Since $\mathbb{R} \setminus \mathbb{Q}$ has empty interior in the topological space $\mathbb{R}$, each C_j has empty interior. It follows that $\mathbb{R} \setminus \mathbb{Q}$ is of first category. Since $\mathbb{Q}$ is of first category, we see that $\mathbb{R}$ is also of first category, contradicting Baire's category theorem ([13, page 97]).

Theorem 2.2.10. *Let $\Omega \subset \mathbb{C}$, let S be a countable infinite dense subset of Ω, let $\{C_n\}$ be a sequence of positive numbers, and let $a \in \Omega \setminus S$. Then there exist an enumeration $\{z_1, z_2, \dots\}$ of S, such that $a \in \text{Conv}(F)$, where F is defined by*

$$F(z,t) = \sum_{n=0}^{\infty} C_n \Big[\prod_{j=1}^{n}(z - z_j)\Big] t^n.$$

Proof. Let $S = \{s_1, s_2, \dots\}$. We choose $z_1, z_2, \dots$ as follows. Choose $z_2 = s_1$. Since S is dense we can pick $z_1 \in S \setminus \{z_2\}$ close to a so that

$$|C_2(a - z_1)(a - z_2)| \leq 1 \text{ and } |C_1(a - z_1)| \leq 1.$$

Choose $z_4 = s_j$, where $j = \min\{i \in \mathbb{N},\ s_i \in S \setminus \{z_1, z_2\}\}$. We choose $z_3 \in S \setminus \{z_1, z_2, z_4\}$ close to a so that

$$|C_4(a - z_1)(a - z_2)(a - z_3)(a - z_4)| \leq 1 \text{ and } |C_3(a - z_1)(a - z_2)(a - z_3)| \leq 1.$$

We now proceed by induction. Suppose $\{z_1, z_2, \cdots, z_{2k-1}, z_{2k}\}$ are chosen. Here is how we choose the next pair $z_{2k+1},\ z_{2k+2}$. Note that $S \setminus \{z_1, \cdots, z_{2k}\}$ is dense in Ω. First pick $z_{2k+2} = s_j$, where $j = \min\{i \in \mathbb{N} : s_i \in S \setminus \{z_1, \cdots, z_{2k}\}\}$, and pick $z_{2k+1} \in S \setminus \{z_1, \cdots, z_{2k}, z_{2k+2}\}$ close to a so that

$$|C_{2k+1}\big(\prod_{j=1}^{2k}(a - z_j)\big)(a - z_{2k+1})| \leq 1 \tag{2.14}$$

and

$$|C_{2k+2}\big(\prod_{j=1}^{2k}(a - z_j)\big)(a - z_{2k+1})(a - z_{2k+2})| \leq 1. \tag{2.15}$$

Now the series

$$F(a, t) = \sum_{n=0}^{\infty} C_n \prod_{j=1}^{n}(a - z_j)t^n \tag{2.16}$$

is convergent because of the inequalities 2.14, and 2.15. Hence $a \in \mathrm{Conv}(F)$. It is clear from the way the element $\{z_n\}$ are chosen that $\{z_1, z_2, \cdots\} = S$. □

CHAPTER 3

Quasi-Simply-Connected Sets and Convergence Sets

3.1 Quasi-Simply-Connected Sets

Recall that a compact K in $\mathbb{C}^n$ is said to be polynomially convex if for each $\zeta \notin K$ there exists a polynomial $P(z)$ such that $|P(\zeta)| > \max_{z \in K} |P(z)|$. It is a consequence of the maximum modulus principle that if K is polynomially convex then $\mathbb{C} \setminus K$ is connected. It follows from Mergelyan's theorem (see, *e.g.*, [13, page 390]) that if $\mathbb{C} \setminus K$ is connected then K is polynomially convex. Therefore we have the following proposition.

Proposition 3.1.1. *Let $K \subset \mathbb{C}$ be a compact set. Then K is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected.* □

For a compact set K in $\mathbb{C}$, let $\lambda(K)$ be the union of bounded components of $\mathbb{C} \setminus K$. Then, the polynomial hull $\widehat{K} = K \cup \lambda(K)$.

We have stated some properties of convergence sets, and we gave several examples with different kinds of topologies. In this chapter we will try to find necessary conditions of the convergence set. We need to define the central concept of this section.

Definition 3.1.2. An F_σ subset K of $\mathbb{C}$ is said to be *quasi-simply-connected* if it is the union of a countable collection of polynomially convex compact sets.

Proposition 3.1.3. *Let $\{K_j\}$ be a sequence of quasi-simply-connected sets. Then $\cup_{j=1}^{\infty} K_j$ is quasi-simply-connected.*

Proof. Since each K_j is quasi-simply-connected, we see that $K_j = \cup_{\ell=1}^{\infty} F_{j\ell}$, where $F_{j\ell}$ are polynomially convex. It follows that $\cup_{j=1}^{\infty} K_j = \cup_{j,\ell=1}^{\infty} F_{j\ell}$. Since the family $\{F_{j\ell}\}$ is clearly countable, the set $\cup_{j,\ell=1}^{\infty} F_{j\ell}$ is quasi-simply-connected. □

Let $\overline{S}$ denote the closure of a set S.

Example 3.1.4. Every open set in the complex plane is a quasi-simply-connected set.

Proof. Suppose $E \subset \mathbb{C}$ is open. For $w \in E$ there exists $r_w > 0$ such that E contains the open disc $\Delta(w, r_w)$. The cover $\{\Delta(w, r_w/2) : w \in E\}$ of E contains a countable subcover, $\{\Delta(w_j, r_{w_j}/2)\}$. It follows that

$$E = \bigcup_{j=1}^{\infty} \overline{\Delta(w_j, r_{w_j}/2)}. \tag{3.1}$$

Since each $\overline{\Delta(w_j, r_{w_j}/2)}$ is polynomially convex, the set E is quasi-simply-connected. □

Example 3.1.5. *The unit circle is a quasi-simply-connected set.*

Proof. Let

$$\gamma_j = \{e^{i\theta} : \ 0 \le \theta \le \Big(1 - \frac{1}{j}\Big)2\pi\}, \ j = 1, 2, \ldots$$

Then γ_j are polynomially convex and the unit circle is the union of γ_j. □

Example 3.1.6. (Sierpinski triangle): It is constructed as follows. Take an equilateral triangle of side length equal to one, remove the inverted equilateral triangle of half length having the same center, then repeat this process for the remaining triangles infinitely many times. The resulting set is called the Sierpinski triangle. The Sierpinski triangle is a compact set that is not a quasi-simply-connected set, as we show in the following proposition. Thus the Sierpinski triangle is not a convergence set, by Theorem 3.2.1.

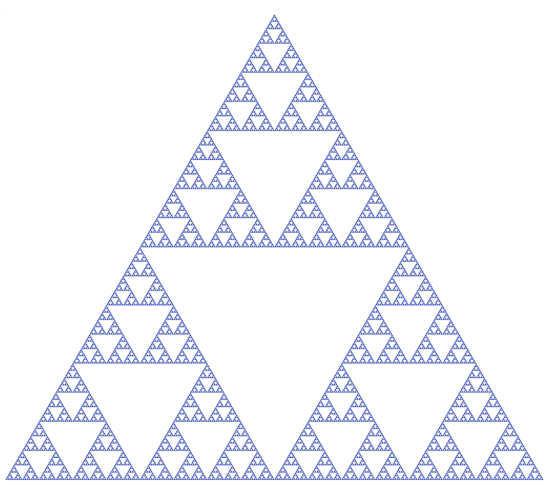

Figure 3.1: Sierpinski Triangle

Definition 3.1.7. *A set in a topological space X is said to be of first category* if it is a countable union of nowhere dense sets. A set that is not of first category is said to be of *second category.*

Baire's category theorem ([13, page 97]) states that a complete metric space as a subset of itself is of second category.

Proposition 3.1.8. *The Sierpinski triangle S is not a quasi-simply-connected set.*

Proof. Suppose that the Sierpinski triangle is a quasi-simply-connected set. Then there exists a sequence of polynomially convex compact sets K_j, $j = 1, 2, \ldots$ such that $S = \cup_{j=1}^{\infty} K_j$. But as the closed set of the complete metric space $\mathbb{C}$, the Sierpinski triangle is itself a complete metric space, and hence a set of second category. So there is at least one K_j and an open subset $U \subset \mathbb{C}$, such that $U \cap S \subset K_j$ and $U \cap S$ is nonempty. By the construction of the Sierpinski triangle, some small triangle Γ belongs to $U \cap K_j$. It follows that the interior of the triangle Γ is a bounded connected component of $\mathbb{C} \setminus K_j$, contradicting the assumption that K_j is polynomially convex. Therefore, the Sierpinski triangle is not quasi-simply-connected. □

Proposition 3.1.9. *Let K_1, K_2 be compact sets in $\mathbb{C}$ with $K_1 \cap K_2 = \emptyset$. Then $(K_1 \cup K_2)\widehat{\ } = \widehat{K}_1 \cup \widehat{K}_2$.*

Proof. It is suffices to show that $(K_1 \cup K_2)\widehat{\ } \subset \widehat{K}_1 \cup \widehat{K}_2$. Let U be a bounded connected component of $\mathbb{C} \setminus (K_1 \cup K_2)$. Then $\partial \widehat{\overline{U}}$ is a connected set that belongs to $K_1 \cup K_2$. Thus $\partial \widehat{\overline{U}} \subset K_1$ or K_2. It follows that $\widehat{\overline{U}} \subset \widehat{K}_1 \cup \widehat{K}_2$. Therefore, $(K_1 \cup K_2)\widehat{\ } \subset \widehat{K}_1 \cup \widehat{K}_2$. □

We now prove the main theorems of this section.

Theorem 3.1.10. *Let E be a quasi-simply-connected set. Then there exist polynomially convex compact sets E_n, $n = 1, 2, \ldots,$ such that $E_n \subset E_{n+1}$ for $n \geq 1$ and $E = \cup_{n=1}^{\infty} E_n$.*

Proof. Since E is a quasi-simply-connected set,

$$E = \bigcup_{j=1}^{\infty} K_j, \tag{3.2}$$

where K_j is polynomially convex compact set for each j. Set

$$F_{n1} = L_{n1} = K_1,\ L_{nj} = K_j \setminus N_{1/n}\Big(\bigcup_{l=1}^{j-1} K_l\Big),\ F_{nj} = \widehat{L}_{nj},\ \text{for } 2 \le j \le n, \tag{3.3}$$

and

$$E_n{}' = \bigcup_{j=1}^{n} L_{nj},\ E_n = \bigcup_{j=1}^{n} F_{nj}. \tag{3.4}$$

Note that for a fixed n, $L_{nj} \cap L_{ni} = \emptyset$, for $i \neq j$. Hence

$$E_n = \bigcup_{j=1}^{n} F_{nj} = \bigcup_{j=1}^{n} \widehat{L}_{nj} = \widehat{E'_n}, \tag{3.5}$$

by Proposition 3.1.9. The set E_n is a polynomially convex compact set, since it is the polynomial hull of $\widehat{E'_n}$. Now $L_{nj} \subset L_{n+1,j}$, hence $E_n{}' \subset E'_{n+1}$, and therefore $E_n \subset E_{n+1}$. We now prove that

$$E = \bigcup_{n=1}^{\infty} E_n. \tag{3.6}$$

Let $z \in E$ then there exist j such that $z \in K_j$. Pick such smallest j then $Z \in L_{nj}$ for a large n. Hence $z \in E_n{}'$, and so $z \in E_n$. We proved that $E \subset \bigcup_{n=1}^{\infty} E_n$. Now let $z \in \bigcup_{n=1}^{\infty} E_n$ then there exist n such that, $z \in E_n = \bigcup_{j=1}^{n} F_{nj} \subset \bigcup_{j=1}^{n} \widehat{K_j} = \bigcup_{j=1}^{n} K_j \subset E$ we proved that $\bigcup_{n=1}^{\infty} E_n \subset E$. Therefore $E = \bigcup_{n=1}^{\infty} E_n$. □

Let $d(.,.)$ denote the euclidean distance.

Lemma 3.1.11. *Let $E, \{K_n\}, \{E_n\}$ be as in the previous theorem. Let $U_n = N_{\frac{1}{3n}}(E_n)$. Then for every integer $m > 0$. we have*

$$\bigcap_{j=m}^{\infty} U_j \subset E. \tag{3.7}$$

Proof. Seeking for a contradiction, suppose that there is an $m > 0$ and

$$z \in (\bigcap_{j=m}^{\infty} U_j) \setminus E. \tag{3.8}$$

We claim that

$$z \in N_{\frac{1}{3n}}\big(\bigcup_{j=1}^{m} F_{nj}\big),\ \text{for } n = m, m+1, \ldots. \tag{3.9}$$

We prove (3.9) by induction on n. When $n = m$, the set in (3.9) is U_m, which contains z by (3.8), so (3.9) holds for $n = m$. Assume that $n > m$ and (3.9) is true for $n - 1$. Put

$$Q = \bigcup_{j=m+1}^{n} F_{nj}, \ R = Q \cap \big(\bigcup_{j=1}^{m} F_{nj}\big), \text{ and } S = \big(\bigcup_{j=1}^{m} F_{nj}\big) \setminus R. \tag{3.10}$$

Then

$$N_{\frac{1}{n}}(R) \subset Q, \ N_{\frac{1}{n}}(S) \cap Q = \emptyset. \tag{3.11}$$

Since (3.9) is assumed to hold for $n-1$, and since the set $N_{\frac{1}{3n-3}}\big(\bigcup_{j=1}^{m} F_{n-1,j}\big)$ is a subset of $N_{\frac{1}{3n-3}}(S \cup R)$, we see that

$$z \in N_{\frac{1}{3n-3}}(S \cup R). \tag{3.12}$$

On the other hand, $z \notin N_{\frac{1}{3n-3}}\big(R\big)$, because

$$N_{\frac{1}{3n-3}}\big(R\big) \subset N_{\frac{1}{n}}\big(R\big) \subset Q \subset E, \tag{3.13}$$

and we assumed that $z \notin E$ in (3.7). It follows that

$$z \in N_{\frac{1}{3n-3}}(S). \tag{3.14}$$

By the triangle inequality,

$$\begin{aligned} d(z, Q) &\geq d(Q, S) - d(z, S) \\ &\geq \frac{1}{n} - \frac{1}{3n-3} \\ &\geq \frac{1}{3n} \ . \end{aligned}$$

Thus

$$z \notin N_{\frac{1}{3n}}(Q) \ . \tag{3.15}$$

By (3.8), $z \in U_n = N_{\frac{1}{3n}}(Q \cup S)$, which, together with (3.15), implies that

$$z \in N_{\frac{1}{3n}}(S) \ . \tag{3.16}$$

This completes the induction step for (3.9). Since $\bigcup_{j=1}^{m} F_{nj} \subset \bigcup_{j=1}^{m} K_j$, we see that

$$\mathrm{d}\big(z, \bigcup_{j=1}^{m} K_j\big) < \frac{1}{3n} \ , \ \forall n \geq m. \tag{3.17}$$

Therefore, $z \in \bigcup_{j=1}^{m} K_j \subset E$, contradicting (3.8). The proof is complete. □

3.2 Quasi-Simply-Connected Sets and Convergence Sets

Theorem 3.2.1. *Let E be a convergence set in $\mathbb{C}$. Then E is a quasi-simply-connected set. Moreover, there exists an ascending sequence $\{E_j\}$ of polynomially convex compact sets such that $E = \cup_{j=1}^{\infty} E_j$.*

Proof. The set E is the convergence set $\mathrm{Conv}(f)$ for some

$$f(z,t) = f_0(z) + f_1(z)t + \cdots + f_n(z)t^n + \cdots , \tag{3.18}$$

where $f_n(z)$ are polynomials. Set

$$E_{jn} = \{z \in \mathbb{C} : |z| \leq j, |f_n(z)| \leq j^n\},\ \forall j, n \in \mathbb{N}, \tag{3.19}$$

and

$$E_j = \bigcap_{n=1}^{\infty} E_{jn},\ j \in \mathbb{N}. \tag{3.20}$$

The sets $\{E_j\}$ have the following property

$$E_j \subset E_{j+1}, \quad \text{for } j \geq 1. \tag{3.21}$$

By the proof of Theorem 2.2.6, $E = \cup_{j=1}^{\infty} E_j$. It follows from the definition of polynomially convex sets that each E_{jn} is polynomially convex. It is also a direct consequence of the definition that the intersection of a family of polynomially convex sets is polynomially convex. Therefore each E_j is polynomially convex.

□

Lemma 3.2.2. *Let K be polynomially convex compact set in $\mathbb{C}$, U an open set containing K, and m a positive integer. Then there exist a finite number of polynomials $\{P_1(z), \ldots, P_\ell(z)\}$, such that*

$$||P_j||_K \leq 1,\ j = 1, \cdots, \ell, \tag{3.22}$$

and

$$\max_j \{|P_1(z)|, \ldots, |P_\ell(z)|\} \geq m\ ,\ \forall z \in \mathbb{C} \setminus U. \tag{3.23}$$

Proof. Find a positive number C such that for $P_1(z) = Cz$,

$$||P_1||_K \leq 1. \tag{3.24}$$

Consider

$$F = \{z \ : \ |z| \leq \frac{m}{C}\} \cap \{\mathbb{C} \setminus U\}. \tag{3.25}$$

Note that F is compact. Due to the polynomial convexity of K, for each $z_0 \in F$ there exists a polynomial Q such that

$$||Q||_K \leq 1 \ \text{ and } \ |Q(z_0)| \geq m. \tag{3.26}$$

Now in some neighborhood $V(z_0)$ of z_0. $|Q(z)| \geq m$ for each $z \in V(z_0)$. Since F is compact there are a finite number of open sets $V(z_1), \ldots, V(z_{\ell-1})$ covering F. The corresponding polynomials are $Q_1(z), \ldots, Q_{\ell-1}(z)$. Denote $P_j(z) = Q_{j-1}(z)$, $\forall j = 2, \ldots, \ell$. □

Now we prove the main theorem of this thesis.

Theorem 3.2.3. *Every quasi-simply-connected set in* $\mathbb{C}$ *is a convergence set.*

Proof. Let E be a quasi-simply-connected set. By Theorem 3.1.10, there exist polynomially convex compact sets E_n such that $E = \bigcup_{n=1}^{\infty} E_n$, and $E_n \subset E_{n+1}$ for $n \geq 1$. We now prove that

$$E = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} U_n. \tag{3.27}$$

By Theorem 3.2.1, $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} U_n \subset E$. If $z \in E$, then $z \in E_k$ for some k, hence $z \in E_n \subset U_n$ for $n \geq k$. It follows that

$$z \in \bigcap_{n=k}^{\infty} U_n \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} U_n, \tag{3.28}$$

which proves (3.27).

Now for each $k \in \mathbb{N}$, let $P_{k1}, \cdots, P_{kn_k}$ be the polynomials for E_k and U_k from Lemma 3.2.2. So

$$|P_{ks}(z)|_{E_k} \leq 1 \ , \ 1 \leq s \leq n_k \tag{3.29}$$

and for every $z \in \mathbb{C} \setminus U_k$ there exists a j, $1 \leq j \leq n_k$, such that

$$|P_{kj}(z)| > k. \tag{3.30}$$

Enumerate the countable set of all polynomials $\{\{P_{kj}\}_{j=1}^{n_k}\}_{k=1}^{\infty} = \{h_\ell\}_{\ell=1}^{\infty}$. Define

$$f_0(z) = 1, \; f_\ell(z) = h_\ell^\ell(z) \; , \; \forall \ell \geq 1. \tag{3.31}$$

For $z \in E$ there exist p, q such that $z \in E_k$ for $k \geq p$, and for all $\ell \geq q$, $h_\ell = P_{kj}$ with $k \geq p$. Therefore $|f_\ell(z)| \leq 1$, for all $\ell \geq q$. To summarize, $z \in E$ implies that $z \in \mathrm{Conv}(f)$. Thus $E \subset \mathrm{Conv}(f)$.

Now consider $z \notin E = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} U_n$. Then for each m, we have $z \notin \bigcap_{n=m}^{\infty} U_n$. Therefore there exists $k \geq m$ such that $z \notin U_k$, hence there exists j with $|P_{kj}(z)| > k$. So there exists ℓ such that $h_\ell(z) = P_{kj}(z)$ and

$$|f_\ell(z)| > k^\ell \geq m^\ell. \tag{3.32}$$

To summarize, for each $z \notin E$, and for each positive integer m, there exists ℓ such that $|f_\ell(z)| > m^\ell$. Consequently, $z \notin E$ implies that $z \notin \mathrm{Conv}(f)$. Hence $\mathrm{Conv}(f) \subset E$. Therefore $E = \mathrm{Conv}(f)$. □

Corollary 3.2.4. *Let $E \subset \mathbb{C}$ be open. Then E is a convergence set.*

We proved that every open set is quasi-simply-connected. So E is a convergence set.

Corollary 3.2.5. *The union of a countable collection of convergence sets is a convergence set.*

Example 3.2.6. *Let the comb set E be defined by*

$$E = \{z \in \mathbb{C} : \Re z \in (\mathbb{Q} \cap [0,1]), 0 \leq \Im z \leq 1\}. \tag{3.33}$$

Then E is a convergence set.

Proof. It is clear that $E = \bigcup_{\alpha \in \mathbb{Q} \cap [0,1]} E_\alpha$, where $E_\alpha = \{z \in \mathbb{C} : \Re z = \alpha, \Im z \in [0,1]\}$ are polynomially convex. □

Corollary 3.2.7. *The intersection of two quasi-simply-connected sets is a quasi-simply-connected set.*

It is obvious that the above corollary can be extended to the case of a finite number of sets. But the intersection of a countable collection of quasi-simply-connected sets is not necessarily a quasi-simply-connected set.

Example 3.2.8. *The Sierpinski triangle is not a convergence set.*

We have proved in Proposition 3.1.8 that Sierpinski triangle is not quasi-simply-connected set. By Theorem 3.2.3 it is not a convergence set too.

3.3 Complex Dynamics

The Mandelbrot set M is defined through a family of complex quadratic polynomials given by

$$P_c(z): \ z \mapsto z^2 + c, \tag{3.34}$$

where c is a complex parameter. For each c, one considers the behavior of the sequence

$$0, \ P_c(0), \ P_c(P_c(0)), \ldots, \tag{3.35}$$

obtained by iterating $P_c(z)$ starting at the critical point $z = 0$, which either escapes to infinity or stays within a disc of some finite radius. Let $f_0(z) = 0$, and $f_n(z) = z + (f_{n-1}(z))^2$ for $n \geq 1$. Then $\deg(f_n) = 2^{n-1}$ for $n \geq 1$. The Mandelbrot set is defined to be

$$M = \{z \in \mathbb{C} \ : \{f_n(z)\} \text{ is bounded}\}. \tag{3.36}$$

Example 3.3.1. *The Mandelbrot set* is a convergence set.

Proof. One can define the Mandelbrot set in the following equivalent way (see [7, page 1]):

$$M = \bigcap_{n=0}^{\infty} Q_n, \text{ where } Q_n = \{z : |f_n(z)| \leq 2\}.$$

Since each Q_n is polynomially convex, we see that M is polynomially convex. Therefore M is a convergence set. □

Note that $M = \text{Conv}(g)$, where

$$g(z,t) = \sum_{n=1}^{\infty} f_n(z) t^n.$$

REFERENCES

LIST OF REFERENCES

[1] S.S. Abhyankar, T.T. Moh, *A reduction theorem for divergent power series*, J. Reine Angew. Math., 241(1970), 27-33.

[2] A.F. Beardon, *Iteration of rational functions*, 3rd ed, Springer, New York, 1965.

[3] J. Bochnak, *Analytic functions in Banach spaces*, Studia Math. 35(1970), 273-292.

[4] B.L. Fridman, D. Ma, *Osgood-Hartogs type properties of power series and smooth functions*, Pacific J. Math., 251(2011), 67-79.

[5] B.L. Fridman, D. Ma, T.S. Neelon, *Nonlinear convergence sets of divergent power series*, preprint.

[6] R.C. Gunning, *Introduction to holomorphic functions of several variables*, 2nd ed, Wadsworth and Brooks/Cole, Belmont, California, 1990.

[7] A. Klebanoff, π *in the Mandelbrot set*, http://home.comcast.net/∼davejanelle/mandel.pdf (2013).

[8] P. Lelong, *On a problem of M.A. Zorn*, Proc. Amer. *Math. Soc.,* 2(1951), 11-19.

[9] N. Levenberg, R.E. Molzon, *Convergence sets of a formal power series*, Math. Z., 197(1988), 411-420.

[10] D. Ma, T.S. Neelon, *On convergence sets of formal power series*, preprint.

[11] T.S. Neelon, *Restrictions of power series and functions to algebraic surfaces*, Analysis (Munich), 29(2009), 1-15.

[12] J. Ribon, *Holomorphic extensions of formal objects*, Ann. Scuola Norm, 3(2004), 657-680.

[13] W. Rudin, *Real and complex analysis*, 3rd ed, McGraw-Hill, New York, 1987.

[14] A. Sathaye, *Convergence sets of divergent power series*, J. Reine Angew. Math., 283(1976), 86-98.

[15] J. Siciak, *Extremal plurisubharmonic functions and capacities in* $\mathbb{C}^n$, Sophia Kokyoroku in Math. 14, Tokyo, 1982.

[16] K. Spallek, P. Tworzewski, T. Winiarski, *Osgood-Hartogs-theorem of mixed type*, Math. Ann, 288(1990), 75-88.