NUMERICAL SIMULATION OF WALL-PRESSURE FLUCTUATIONS DUE TO TURBULENT BOUNDARY LAYER

A Dissertation by

Niloufar Mahmoudnejad

Master of Science, Iran University of Science and Technology, 2003

Bachelor of Science, Amir Kabir University of Technology, 2001

Submitted to the Department of Aerospace Engineering
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

December 2011

© Copyright 2011 by Niloufar Mahmoudnejad

All Rights Reserved

NUMERICAL SIMULATION OF WALL-PRESSURE FLUCTUATIONS DUE TO TURBULENT BOUNDARY LAYER

The following faculty members have examined the final copy of this dissertation for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the Doctor of Philosophy with a major in Aerospace Engineering.

Klaus A. Hoffmann, Committee Chair

Walter Horn, Committee Member

Roy Myose, Committee Member

Hamid Lankarani, Committee Member

Eddie Irani, Committee Member

Accepted for the College of Engineering

Zulma Toro-Ramos, Dean

Accepted for the Graduate School

J. David McDonald, Dean

DEDICATION

To my parents, my husband Ashkan, and my sisters Nastaran and Newsha for their constant love
and support

Men Love to wonder, and that is the seed of science.

Ralph Waldo Emerson

ACKNOWLEDGMENTS

I would like to express my deep and sincere gratitude to my advisor, Dr. Klaus A. Hoffmann. I am thankful for his guidance, instruction, and support. Above all, I sincerely thank Dr. Hoffmann for funding me throughout my doctoral studies.

I extend my gratitude to my committee members, Dr. W. Horn, Dr. H. Lankarani, Dr. R. Myose, and Dr. E. Irani, for their helpful comments, support, and suggestions.

I am grateful to my beloved husband, Ashkan Sahraie Jahromi, for his support, understanding, and help throughout my dissertation. Without his assistance, I would not have been able to finish this study.

I take this opportunity to express my sincere gratitude to my father, mother, and sisters for providing me spiritual support.

I am grateful to all the faculty and staff of the Department of Aerospace Engineering for their constant support and guidance throughout the duration of my studies at Wichita State University.

I am sincerely grateful to Spirit AeroSystems (Judy Gallman, Mark Moeller, Sang Lee), Hawker Beechcraft, Cessna Aircraft Company, Boeing Company, and Bombardier (Terri Miller) for supporting NIS projects and providing valuable suggestions.

I wish to give my sincere thanks to the Ablah Library staff at Wichita State University and the Interlibrary Loan personnel for their hard work providing research recourses.

A very special thanks to Kristie Bixby for her invaluable help with the language editing of my papers and for her kindness.

Special thanks to Dr. Shih-Chieh Lo for his suggestions and assistance in parallelization of my computer code.

Countless other friends have made the past few years absolutely delightful. I take this opportunity to thank all my friends and colleagues for their valuable help and support in successfully completing this study during these years.

ABSTRACT

Pressure fluctuations associated with turbulent boundary layer have been a prominent issue over the past few decades. In order to simulate pressure fluctuations beneath a turbulent boundary layer, a numerical investigation was performed in the current study. Four different turbulence models were employed to calculate the pressure and velocity fluctuations. A new approach of direct numerical simulation (DNS) was developed, as well. The proposed DNS scheme was hybrid of sixth-order weighted compact scheme (WCS) and modified weighted essentially non-oscillatory (WENO) scheme, which is called modified WENO-WCS scheme (MWWS) hereafter. A variety of benchmark problems were investigated to evaluate the accuracy of the proposed numerical scheme. Several empirical/semi-empirical mean square pressure models and single-point wall-pressure spectrum models were investigated to compare mean square wall pressure values. Reynolds-averaged Navier-Stokes based on Spalart-Allmaras (RANS-SA) and Delayed detached-eddy simulation based on Spalart-Allmaras (DDES-SA) turbulence models showed agreement with the Lowson, Lilley and Hodgson, and Goody models. Shear stress transport (RANS-SST) and DDES-SST models showed agreement with the Lowson, Farabee and Casarella, Lilley and Hodgson, and Goody models. The MWWS scheme was in agreement with Lowson and Goody models.

Five single-point wall-pressure spectrum models were investigated and compared with numerical results. In low frequency region, results obtained by DDES-SA model and MWWS scheme were in agreement with the Goody model, while RANS-SA, RANS-SST, and DDES-SST turbulence models showed agreement with the Robertson model. In High frequency region, all investigated numerical methods were in agreement with the Goody and Efimtsov (1) models.

TABLE OF CONTENTS

Chapter		Page
1.	LITERATURE REVIEW	1
1.1	Introduction	1
1.2	Historical Perspective	2
1.2.1	Numerical Schemes	2
1.2.2	Pressure Fluctuations Due to Turbulent Boundary Layer Investigations ...	7
2.	GOVERNING EQUATIONS AND NUMERICAL APPROACH	13
2.1	Motivation	13
2.2	Numerical Procedure	14
2.2.1	ENO Reconstruction Function and Conservative Law	14
2.2.2	WENO Scheme (Jiang and Shu) [10]	15
2.2.3	Weighted Compact Scheme (WCS)	17
2.2.4	Advantages and Drawbacks of WENO and WCS, and the Proposed Approach	19
2.2.5	Time Discretizations and TVD	21
2.2.6	Characteristic Filter	23
2.2.7	Bandwidth Optimization	24
2.2.8	Mapped WENO Scheme	26
2.2.9	Proposed Modified WCS-WENO Scheme (MWWS)	26
2.3	Governing Equations	27
2.3.1	One-Dimensional Euler Equations	27
2.3.2	Two-Dimensional Euler Equations	27
2.3.3	Two-Dimensional Navier-Stokes Equations	32
2.3.4	Three-Dimensional Navier-Stokes Equations	35
2.4	Similarities and Differences of the Proposed Scheme in One-, Two-, and Three-Dimensional Flowfields	43
3.	VALIDATION OF THE PROPOSED NUMERICAL SCHEME	45
3.1	Chapter Overview	45
3.2	One-Dimensional Benchmark Problems	45
3.2.1	Linear Wave Equation with Jump Initial Function	45
3.2.2	Linear First-Order Wave Equation with Different Initial Conditions	46
3.2.3	The Burgers' Equation	52
3.2.4	Linear Advection Equation	55
3.2.5	Buckley Leverett Problem	57
3.2.6	The Shock Tube Problem	58
3.2.7	Shock-Entropy Interaction	59

TABLE OF CONTENTS (continued)

Chapter	Page
3.2.8	Riemann Problem of Lax 66
3.3	Two-Dimensional Benchmark Problems 67
3.3.1	Moving Vortex 67
3.3.2	Shock-Vortex Interaction 71
3.3.3	Double Mach Reflection 74
3.3.4	Shock Wave-Mixing Layer Interaction 79
4.	SIMULATION OF WALL-PRESSURE FLUCTUATIONS WITH EMPIRICAL/SEMI-EMPIRICAL MODELS 84
4.1	Turbulent Boundary Layer Definition 84
4.2	Turbulent Boundary Layer Noise 85
4.3	Empirical/Semi-Empirical Models 86
4.3.1	Single-Point Wall-Pressure Spectrum Empirical Models 87
4.3.2	Mean Square Pressure Empirical Models 94
5.	SIMULATION OF WALL-PRESSURE FLUCTUATIONS WITH NUMERICAL TECHNIQUES 97
5.1	Turbulence Modeling 97
5.1.1	RANS-SA (Spalart-Allmaras) Turbulence Model 98
5.1.2	Delayed Detached-Eddy Simulation based on RANS-SA (DDES-SA) Turbulence Model 100
5.1.3	RANS-SST (Standard Menter Shear Stress Transport) Turbulence Model 102
5.1.4	Delayed Detached-Eddy Simulation based on RANS-SST (DDES-SST) Turbulence Model 103
5.2	Direct Numerical Simulation Technique 104
5.2.1	Inflow Generation Method 104
5.2.2	Three-Dimensional DNS Code 105
6.	RESULTS 106
6.1	Three-Dimensional Grid System and Flow Conditions 106
6.1.1	Grid System for Turbulence Modeling Methods 107
6.1.2	Grid System for Direct Numerical Simulation Scheme 107

TABLE OF CONTENTS (continued)

Chapter	Page
6.2 DDES-SA Turbulence Modeling Results	109
6.3 RANS-SA Turbulence Modeling Results	121
6.4 DDES-SST Turbulence Modeling Results	133
6.5 RANS-SST Turbulence Modeling Results	144
6.6 DNS Results (MWWS Scheme)	155
6.7 Comparison of the Results	165
7. SUMMARY AND CONCLUSIONS	170
REFERENCES	173

LIST OF TABLES

Table	Page
3-1. Shock-Shear Layer Interaction, Flow Properties in Different Regions of the Flow, Comparison of Numerical Results with the Standard Gas-Dynamic Solution (Bold and Underlined values)	83
4-1. Summary of Frequency Range Slope Predictions [116].....	94
4-2. Scaling Variables	94
5-1. Cobalt Software Turbulence Models	98
6-1. Computation Time of Numerical Schemes.....	166
6-2. Mean Square Wall-Pressure Values.....	166
6-3. Comparison of the Mean Square Wall-Pressure Values by Different Models	167
6-4. Comparison of Power Spectral Density by Different Methods	168

LIST OF FIGURES

Figure		Page
2-1.	Schematics of WENO scheme stencils for $\hat{f}_{j+1/2}$ ($r=3$).....	15
2-2.	WCS stencil candidates.....	18
2-3.	Candidate stencils for the WENO-SYM numerical flux $\hat{f}_{j+1/2}$ ($r=3$).....	25
3-1.	Linear wave equation with jump initial function (IM = 81, CFL = 0.8)	46
3-2.	Comparison of the solutions for case one by different schemes (IM = 51, CFL = 0.8).....	48
3-3.	Solutions of case one at different time levels (CFL = 0.7)	48
3-4.	Comparison of the results for case two by different schemes (IM = 101, CFL = 0.7)	50
3-5.	Comparison of the results of the different schemes for case three (IM = 101, CFL = 0.8)	51
3-6.	High-frequency sine wave (IM = 121, CFL = 0.8).....	52
3-7.	The Burgers' equation with nonlinear convex flux, numerical results by different schemes (IM = 81, CFL = 0.8)	54
3-8.	The Burgers' equation solution, comparison of numerical error distribution by different schemes	55
3-9.	Linear advection equation (IM = 101, CFL = 0.2)	56
3-10.	Linear advection equation, comparison of numerical error distributions for different schemes	57
3-11.	The Buckley Leverett problem with non-convex flux (IM = 81, CFL = 0.8)	58
3-12.	Velocity distribution for shock tube problem (IM = 101, CFL = 0.6).....	59
3-13.	Shock-entropy interaction, initial and final reference solution.....	61
3-14.	Density distribution of shock-entropy problem by different schemes (IM = 301, CFL = 0.6)	62
3-15.	Results obtained by different schemes (IM = 301, CFL = 0.6).....	63
3-16.	Pressure distribution of shock-entropy problem (IM = 301, CFL = 0.6)	64
3-17.	MWWS results with different κ values	65

LIST OF FIGURES (continued)

Figure	Page
3-18.	Results obtained by MWWS scheme with different β values..... 66
3-19.	Solution of Riemann problem of Lax with different schemes (IM = 101, CFL = 0.6). 67
3-20.	Two-dimensional moving vortex, twenty density contours from 0.5 to 0.95 (81×81, CFL = 0.6) 69
3-21.	Two-dimensional moving vortex, density and numerical distributions at different time levels 70
3-22.	Two-dimensional shock-vortex interaction, thirty pressure contours from 0.8 to 1.3 (251×101, CFL = 0.6) 73
3-23.	Two-dimensional shock-vortex interaction, comparison of pressure at the central lines downstream the shock ($t = 0.6$)..... 73
3-24.	Two-dimensional shock-vortex interaction, Ninety pressure contours from 1.19 to 1.37 ($t = 0.6$) 74
3-25.	Schematic illustration of two-dimensional double Mach reflection configuration 75
3-26.	Two-dimensional double Mach reflection, general view 75
3-27.	Two-dimensional double Mach reflection, Euler solution, thirty density contours from 1.5 to 20.5 (241×61 , CFL = 0.6)..... 76
3-28.	Two-dimensional double Mach reflection, Euler solution, thirty density contours from 1.5 to 20.5 (481×121 , CFL = 0.6)..... 77
3-29.	Two-dimensional double Mach reflection, Euler solution, thirty density contours from 1.5 to 20.5 (961×241 , CFL = 0.6)..... 77
3-30.	Two-dimensional double Mach reflection, Euler solution, blow-up region 77
3-31.	Two-dimensional double Mach reflection, Navier-Stokes solution, thirty density contours from 1.5 to 20.5 (481×121 , CFL = 0.6) 79
3-32.	Schematic of the shock-mixing layer interaction [108] 80
3-33.	Shock-shear layer interaction ($t = 120$)..... 82
6-1.	General configuration of flat plate 106

LIST OF FIGURES (continued)

Figure	Page
6-2. Three-dimensional grid system of flat plate for turbulence modeling methods	107
6-3. Three-dimensional grid system of flat plate for DNS technique	108
6-4. Sensor points locations of flat plate along the center line	109
6-5. DDES-SA model simulation of flat plate, boundary layer development.....	110
6-6. DDES-SA model simulation of flat plate, pressure contours	110
6-7. Locations of sensor points for numerical simulations	111
6-8. DDES-SA model simulation, velocity fluctuations at different locations	112
6-9. DDES-SA model simulation, pressure fluctuations and power spectral density plots at (0.25,0,0)	115
6-10. DDES-SA model simulation, power spectral density at (0.25,0,0) normalized by inner boundary layer variables	116
6-11. DDES-SA model simulation, power spectral density at (0.25,0,0) normalized by outer boundary layer variables	117
6-12. DDES-SA model simulation, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables	117
6-13. DDES-SA model simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)	118
6-14. RMS wall-pressure fluctuations (numerical model: DDES-SA)	119
6-15. Mean square wall-pressure values along the center line of flat plate (numerical model: DDES-SA)	120
6-16. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: DDES-SA)	121
6-17. RANS-SA model simulation of flat plate, boundary layer development	122
6-18. RANS-SA model simulation of flat plate, pressure contours	122
6-19. RANS-SA model simulation, velocity fluctuations at different locations.....	123

LIST OF FIGURES (continued)

Figure	Page
6-20. RANS-SA model simulation, pressure fluctuations and power spectral density plots at (0.25,0,0)	127
6-21. RANS-SA model simulation, power spectral density at (0.25,0,0) normalized by inner boundary layer variables	128
6-22. RANS-SA model simulation, power spectral density at (0.25,0,0) normalized by outer boundary layer variables	128
6-23. RANS-SA model simulation, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables	129
6-24. RANS-SA model simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)	130
6-25. RMS wall-pressure fluctuations (numerical model: RANS-SA).....	131
6-26. Mean square wall-pressure values along the center line of flat plate (numerical model: RANS-SA)	132
6-27. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: RANS-SA)	133
6-28. DDES-SST model simulation of flat plate, boundary layer development.....	134
6-29. DDES-SST model simulation of flat plate, pressure contours	134
6-30. DDES-SST model simulation, velocity fluctuations at different locations	135
6-31. DDES-SST model simulation, pressure fluctuations and power spectral density plots at (0.25,0,0)	138
6-32. DDES-SST model simulation, power spectral density at (0.25,0,0) normalized by inner boundary layer variables	139
6-33. DDES-SST model simulation, power spectral density at (0.25,0,0) normalized by outer boundary layer variables	140

LIST OF FIGURES (continued)

Figure	Page
6-34. DDES-SST model simulation, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables	140
6-35. DDES-SST model simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)	141
6-36. RMS wall-pressure fluctuations (numerical model: DDES-SST)	142
6-37. Mean square wall-pressure values along the center line of flat plate (numerical model: DDES-SST).....	143
6-38. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: DDES-SST).....	144
6-39. RANS-SST model simulation of flat plate, boundary layer development.....	145
6-40. RANS-SST model simulation of flat plate, pressure contours	145
6-41. RANS-SST model simulation, velocity fluctuations at different locations	146
6-42. RANS-SST model simulation, pressure fluctuations and power spectral density plots at (0.25,0,0).....	149
6-43. RANS-SST model simulation, power spectral density at (0.25,0,0) normalized by inner boundary layer variables	150
6-44. RANS-SST model simulation, power spectral density at (0.25,0,0) normalized by outer boundary layer variables	151
6-45. RANS-SST model simulation, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables	151
6-46. RANS-SST model simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)	152
6-47. RMS wall-pressure fluctuations (numerical model: RANS-SST).....	153
6-48. Mean square wall-pressure values along the center line of flat plate (numerical model: RANS-SST)	154

LIST OF FIGURES (continued)

Figure	Page
6-49. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: RANS-SST)	155
6-50. MWWS scheme, velocity fluctuations at different locations	156
6-51. MWWS scheme, pressure fluctuations and power spectral density plots at (0.25,0,0)	159
6-52. MWWS scheme, power spectral density at (0.25,0,0) normalized by inner boundary layer variables	160
6-53. MWWS scheme, power spectral density at (0.25,0,0) normalized by outer boundary layer variables	160
6-54. MWWS scheme, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables	161
6-55. MWWS scheme simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)	162
6-56. RMS wall-pressure fluctuations (numerical model: MWWS scheme).....	163
6-57. Mean square wall-pressure values along the center line of flat plate (numerical model: MWWS scheme).....	164
6-58. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: MWWS scheme).....	165

LIST OF NOMENCLATURES

a	Speed of sound
C_f	Skin friction coefficient
C_p	Specific heat
C_k^r	Optimal weight allocated to each stencil
e	Energy
$\hat{f}$	Numerical flux function
H	Primitive function of $\hat{f}$
IM	Number of grid points in X direction
IS_k	Smoothness indicator
J	Jacobian of transformation
JM	Number of grid points in Y direction
k	Thermal conductivity
L	Length
M	Mach number
M_c	Convective Mach number
p	Pressure/Power in numerical schemes
$P(p)$	Probability density function of the pressure fluctuations
$\overline{p'^2}$	Mean square pressure

LIST OF NOMENCLATURES (continued)

Pr	Prandtl number
q	Dynamic pressure
q_k^r	Candidate flux on k^{th} stencil
R	Specific gas constant
Re	Reynolds number
R_T	Ratio of the outer-layer to inner-layer timescale
S	Magnitude of the vorticity/ Entropy
Sh	Strouhal number ($\omega\delta/U_\tau$)
t	Time
T	Temperature
u	X-component of the velocity
U_∞	Free stream velocity
U_τ	Friction (dynamic) velocity ($(\tau_w / \rho_w)^{1/2}$)
v	Y-component of the velocity
w	Z-component of the velocity
$W(f)$	Power spectrum level
α	Local numerical viscosity
γ	Ratio of specific heats

LIST OF NOMENCLATURES (continued)

δ	Boundary layer thickness
δ^*	Displacement thickness
δ_s	Laminar sublayer thickness
Δx	Size of spatial mesh in X direction
Δt	Time step
$\theta_{j+1/2}^l$	Harten switch
κ	Characteristic filter parameter
λ	Wavelength
μ	Dynamic viscosity
ν	Kinematic viscosity
ν_t	Turbulent eddy viscosity
ρ	Density
τ_w	Wall shear stress
$\Phi_{pp}(\omega)$	Single-point wall-pressure spectrum
$\Phi_{j+1/2}^*$	Nonlinear dissipation function of characteristic filters
$\phi_{j+1/2}^{l*}$	Elements of $\Phi_{j+1/2}^*$ computed by shock capturing schemes
ω_k	Allocated weight to each stencil

LIST OF ABBREVIATIONS

ACM	Artificial compression method
AUSM	Advection upstream splitting method
CFD	Computational fluid dynamics
CFL	Courant–Friedrichs–Lewy number
DDES	Delayed detached-eddy simulation
DES	Detached-eddy simulation
DFT	Discrete Fourier transform
DNS	Direct numerical simulation
ENO	Essentially non-oscillatory
ERR	Error function
FFT	Fast Fourier transform
LES	Large eddy simulation
LF	Lax-Friedrichs
LLF	Local Lax-Friedrichs
MUSCL	Monotone upstream-centered schemes for conservation laws
MWWS	Modified WENO-WCS scheme
PSD	Power spectral density
RANS	Reynolds-averaged Navier-Stokes
RANS-SM	Reynolds-averaged Navier-Stokes statistical model
RK4	Fourth-order Runge-Kutta
RMS	Root mean square

LIST OF ABBREVIATIONS (continued)

SA	Spalart-Allmaras turbulence model
SARC	SA model with rotation/curvature correction
SGS	Subgrid-scale
SST	Shear-stress transport turbulence model
TBL	Turbulent boundary layer
TVD	Total variation diminishing
WCS	Weighted compact scheme
WENO	Weighted essentially non-oscillatory

CHAPTER 1

LITERATURE REVIEW

1.1 Introduction

The numerical simulation of turbulent flows has been a challenging issue during the past decades, and attempts to find better schemes have been limited by the capacity and speed of digital computers. With the recent progress in high-speed computers, more numerical schemes have been developed by computational fluid dynamics (CFD) scientists. There are generally three different approaches for solving the equations of fluid dynamics involving turbulent flows [1]:

- ✓ Reynolds-averaged Navier-Stokes (RANS)
- ✓ Large eddy simulation (LES)
- ✓ Direct numerical simulation (DNS)

RANS approach is a set of time-averaged equations of motion, which can be used over a wide range of geometries and flow conditions. This approach is popular because of its simplification to the Navier-Stokes equations.

LES is another numerical technique, which was formulated in the late 1960s. Literatures show that LES approach is more accurate than RANS for acoustic problems and the flows with separation.

DNS approach implies an exact solution to the systems of equations. However, the problem is limited to the computational resources. Therefore, researchers have been attempting to develop higher order accurate schemes, which require less number of grid points.

Better understanding of the existing DNS schemes can guide the development of more accurate schemes with improved resolution characteristics. The main goal of current research is

developing a higher order DNS scheme in order to calculate the pressure fluctuations due to turbulent boundary layers. Therefore, literature survey is performed in two segments. In the first segment, the background of the numerical schemes is presented. The previous researches performed by several investigators in areas of noise generation and pressure fluctuations measurement/calculation due to turbulent boundary layers are explained and discussed in the second segment.

1.2 Historical Perspective

1.2.1 Numerical Schemes

One of the complicated issues in fluid dynamics is the existence of shocks or other discontinuities in certain regions of the flowfield, and the interaction of shocks and the turbulent boundary layer. Many CFD investigators have developed shock-capturing schemes based on upwind or upwind-biased schemes. Some of the shock capturing schemes, which had great successes are Godunov [2], monotone upstream-centered schemes for conservation laws (MUSCL) [3], Roe [4], total variation diminishing (TVD) [5], essentially non-oscillatory (ENO) [6-8] and weighted essentially non-oscillatory (WENO) [9, 10]. Most of these schemes are CPU-intensive and require modification and treatment of the boundary points [11]. Another disadvantage of these schemes is misrepresenting the high-frequency sound waves as shocks [12]. The solutions may also show undesirable amplitude and/or errors in phases in the vicinity of shocks [11]. Among these schemes, ENO and WENO have been more successful in solving problems that contain both shocks and smooth regions. In 1987, Harten et al. [6] developed ENO scheme that was successful in direct numerical simulation. The basic idea in ENO is avoiding the stencil, which contains shock. In order to calculate the derivatives, ENO selects the smoothest stencil among several candidates [13]. Based on the same concept of ENO, Jiang and Shu [10]

developed the WENO scheme, whereby smoothness functions are introduced and derivatives are calculated based on the allocated weight of each stencil. These schemes are based on upwind or upwind-biased methods, which are suitable for hyperbolic system [13]. In 1992, Lele [14] proposed a high-order compact scheme, which was more suitable for turbulence simulation with the Navier-Stokes equations. Unlike the traditional finite difference schemes, which use the Lagrange interpolation, Lele used Hermitian interpolation. In compact formulation, both derivatives and function values are used without any increase in the width of the stencils. Therefore, the result is a tri-diagonal or penta-diagonal system. The standard compact scheme has high order of accuracy and good spectral resolution, which are the main concerns in shock-capturing schemes. The advantages of this scheme are noticeable. However, there are some challenging issues in implementation. In problems that contain discontinuities or shock waves, visible wiggles around the discontinuities are observed due to grid-to-grid oscillations. Several methods are used to reduce the oscillations, such as adding a damping term, using TVD schemes, or applying filters to the scheme. However, in some applications, none of the added dissipation approaches will result in a noticeable improved solution. In 1996, Kim and Lee [15] optimized the compact finite difference schemes in order to achieve maximum resolution characteristics. Based on their analytical optimization, the overall error characteristics of the schemes are dependent on the resolution, truncation error, and their multidiagonality. They showed that the results of the fourth-order Runge-Kutta time advancing are better than the higher order ones for the optimized schemes. They also found that the optimized sixth-order tridiagonal scheme is more efficient and economic.

Deng and Maekawa [16] developed a uniform fourth-order compact scheme in order to capture the discontinuities. They proposed a compact adaptive interpolation of variables at cell

edges, which jumps automatically to local one as discontinuities are encountered. This approach resulted in oscillation free discontinuities. They compared their scheme with ENO scheme and showed that both schemes perform the same in capturing the discontinuities. However, their scheme was simpler and more efficient because of the compact property.

In 1997, Deng and Mao [17] introduced the weighted technique in the compact high order nonlinear schemes. Based on the method, they could obtain the three-, fourth-, and fifth-order weighted compact nonlinear schemes. They also investigated the dispersive and dissipative features of their scheme by Fourier analysis. They showed that the interpolations of variables at cell edges were the main factor in accuracy of the scheme.

Shu [18] carried out a valuable investigation on ENO and WENO schemes for hyperbolic conservation laws, which became a source for further investigations. A common procedure of elimination or reduction of the oscillations near the discontinuities has been addition of an artificial viscosity or application of limiters. These methods are helpful; however, they have some disadvantages. First, most of them are problem dependent and differ from case to case. Second, these methods require logical lines be added to the code, which results in more computational effort and time.

Hixon [19] developed a new class of compact scheme based on factorization. The scheme reduced the tridiagonal matrix of the standard sixth-order compact into two independent bidiagonal matrices, which were solved in parallel. By proposing simpler boundary stencil specification, the computational effort was reduced.

Based on the efforts, which were carried out by several investigators, it was shown that compact schemes had better resolution with higher order of accuracy. In 2001, a new class of compact scheme, called the weighted compact scheme (WCS), was developed by Jiang et

al. [20] for problems containing shocks and discontinuities. This scheme has the same idea as the WENO scheme in assigning the weight for each stencil. They used a compact stencil, which improved resolution and accuracy. This scheme was also successful in capturing the shock waves and discontinuities without oscillation. The weights were defined in such a way that the stencils had less influence on the scheme and resulted in reduced grid-to-grid oscillation. The scheme was applied to several one-dimensional problems with shock waves, discontinuities, and shock fluctuation interaction and compared them with the standard compact scheme. Standard compact scheme failed because of the oscillations generated near the discontinuities. However, weighted compact scheme showed better resolution with no numerical oscillation in the regions of discontinuities. The details of this scheme are presented in the next chapter.

In 2002, Deng [21] introduced the fifth-order dissipative weighted compact nonlinear scheme and investigated the dissipative and dispersive characteristics of the scheme. The analysis showed that the accuracy of the scheme was dominated by the interpolation of variables at cell edges. Based on his numerical tests, he found that weighted compact nonlinear scheme worked well in one- and two-dimensional flowfields.

Solving Navier-Stokes equations by traditional schemes in problems containing shocks and discontinuities may not produce satisfactory results. Sengupta et al. [22] proposed a new high spectral accuracy compact difference scheme by constrained optimization of error in spectral space. In a similar approach, Mawlood et al. [23] developed a fourth-order compact space discretization method based on the advection upstream splitting method (AUSM). They discretized the diffusive flux term of the Navier-Stokes equations by a central fourth-order compact method.

Due to weaknesses in the original weighted compact scheme proposed by Jiang et al. [20], a new scheme was introduced [24]. The scheme included a shock detector and a smart filter and weight function. It was applied for shock tube and shock entropy interaction problems. They found that the smart filter can remove the oscillations near the shocks while it does not smear the physical solution. The new weight function was able to improve the shock capturing ability of the scheme. Subsequently, Xie et al. [25] applied the improved scheme to the shock-boundary layer interaction in two dimensions. They showed that this scheme was feasible and suitable for the problems containing both shock waves and complex vortex structures. Xie and Liu [13] introduced a new type of weighted compact and non-compact scheme based on the feature of discrete data sets instead of the physics. Data set is divided into three regions of smooth, oscillatory, and non-differentiable or shock. In the new scheme, central weighted compact scheme is used in smooth and oscillatory regions, while in shock regions it switches to the non-compact scheme. Their numerical results showed that the scheme was capable of capturing the one-dimensional shock sharply and obtained much higher resolution for one-dimensional shock entropy interaction than the 5th order WENO scheme.

Shen and Yang [26] proposed a hybrid finite compact with WENO scheme for shock calculations. This hybrid scheme combines the advantages of finite compact and WENO schemes. Their numerical results showed that the new scheme has better resolution of high wave numbers than any of those two schemes. They also compared their results with artificial compression method (ACM) filter schemes of Yee et al. [11] and the results showed that the ACM filter schemes have higher efficiencies than their scheme. However, these types of filter schemes are problem dependent. Another effort was carried out by Xin et al. [27], which investigated the performance of weighted compact fifth order nonlinear schemes. They showed

that weighted compact nonlinear schemes have better characteristics than explicit upwind biased fifth-order schemes and Pade.

Su et al. [28] analyzed the accuracy and numerical dissipation and dispersion errors of the weighted compact scheme proposed by Jiang et al. [20]. Their analysis showed that the WCS and WENO schemes could be combined to provide robust results with better shock capturing capability. Nonomura et al. [29] investigated the coefficients and resolution of higher order weighted compact nonlinear scheme. They calculated the coefficients of seventh- and ninth-order of weighted compact nonlinear scheme. They investigated the order of accuracy of scheme analytically and numerically and showed that the seventh-order weighted compact nonlinear scheme solves the two-dimensional problems with better resolution than the ninth-order.

Recently, investigators have focused on finding an effective shock/discontinuity detector, which reduces the computation time. Oliveria et al. [12] developed a method with shock detector for shock boundary layer interaction. The goal was to develop a shock detector, which could distinguish between the high-frequency waves and critical points as shocks. They compared their results with two other shock detectors of Harten's and WENO and showed that their new detector was superior to both ones.

1.2.2 Pressure Fluctuations Due to Turbulent Boundary Layer Investigations

Investigation of the pressure fluctuations in association with turbulent boundary layer has been an interesting issue over the past few decades. The main concerns about turbulent boundary layers and pressure fluctuations are structural vibration and noise generation. Understanding the behavior of the velocity and pressure fluctuations due to the turbulent boundary layers is essential to understanding the source of noise generation. Therefore, many researchers have performed extensive experimental and computational investigations in this field.

In 1962, Willmarth and Wooldridge [30] carried out an experimental study in order to measure the wall-pressure fluctuations. Their results showed that in a fully turbulent tube flow, low-frequency pressure fluctuations had the highest convection speed. They also found that the transverse and longitudinal scales of small- and large-scale wall-pressure fluctuations were approximately the same. Williams and Lyon [31] derived an equation for the resonant vibration response of a flat plate for a low speed turbulent boundary layer. Several investigations were performed by Corcos [32, 33] who is one of the pioneers in this field. In 1963, he studied the effect of transducer size in measurements and found some undesirable large errors. Corcos also proposed a model for the frequency cross spectrum, which introduced undesirable features; these were later addressed by other investigators. He continued his investigations and in another study discussed the measurements of the statistical properties of the pressure field at the wall of the turbulent attached shear flows [33]. The effect of a finite size circular transducer on measurements of the wall-pressure beneath a turbulent boundary layer was investigated by Willmarth and Roos [34]. In 1964, Williams [35] proposed a theory describing surface pressure fluctuations of the boundary layer on a rigid surface, which illustrated the effect of compressibility. Chase [36] measured the area averaged turbulent boundary layer pressure spectrum by transducers of streamwise size with arbitrary spatial responses in the high-frequency domain. He also proposed a specific approximate form of wave number frequency spectrum of pressure on wall for the turbulent boundary layer. He appears to be the first to propose the elliptic model [37]. In 1982, Efimtsov [38] performed an experimental study to measure the pressure fluctuations on the surface of aircraft. The tests were carried out at large Reynolds numbers and Mach number range of 0.41 to 2.1. Another experimental investigation in a wind tunnel was performed by Schewe [39], who measured wall-pressure fluctuations with several

pressure transducers of different sizes. He pointed out that the ratio between the pressure transducer size and the smallest important length-scale of the flow should be as small as possible. Chandiramani [40] modified the theory of transmission of sound across an isotropic flat plate to apply at high, convective wavenumbers. Kim [41] performed a numerical investigation on pressure fluctuations in a turbulent channel flow. The database was obtained from a direct numerical simulation (DNS). It was shown that the slow pressure fluctuations were larger than the rapid pressure fluctuations throughout the channel except very near the wall, where they had the same magnitude.

Modeling turbulence is another concern in aeroacoustics as well as in underwater sound. Mellen [42] carried out an investigation on convective turbulence modeling and found that the Corcos model predictions were too high below the convective peak. He proposed the elliptic model to modify the frequency cross spectrum of Corcos model. A numerical investigation was carried out by Choi and Moin [43] where the three-dimensional frequency wavenumber spectrum of wall-pressure fluctuations were computed by DNS. Chase [44] studied the properties of the wave-vector-frequency spectrum of fluctuating pressure on a smooth plane in turbulent boundary layer flow at low Mach numbers. Farabee and Casarella [45] performed an experimental investigation on spectral features of wall-pressure fluctuations beneath turbulent boundary layers. Their results showed the sensitivity of the wall-pressure field to enriched organized structures in the outer flow. Keith et al. [46] compared the turbulent boundary layer wall-pressure spectra from several investigations with each other. Mellen [37] again studied the analytical model proposed by Corcos and extended the comparison of the Corcos model and elliptic model. The comparison included further illustration of the implications for wave vector filter experiments. Lueptow [47] studied the effect of pressure transducer size on the accuracy of

the wall-pressure beneath a turbulent boundary layer. Manoha [48] performed an experimental investigation in a hydrodynamic tunnel and validated the available semi-empirical models. The comparison between simulated and measured estimators indicated that the Chase's model was more successful than the other models in low wavenumber and convective domains. Bull [49], Sawada et al. [50], and Abraham and Keith [51] are other investigators who performed investigations in the field of wall-pressure fluctuations of turbulent boundary layers. Efimtsov et al. performed an extensive investigation on forward [52] and backward [53] steps and proposed pressure fluctuation spectra for each of them. Smol'yakov [54] developed a method for calculating the wall-pressure spectra in turbulent boundary layers. The method was based on the modeling of the wavenumber spectrum of the sources caused by interaction of the turbulence mean shear type. He also compared the calculated spectra and root mean square (RMS) values of pressure fluctuations with the experimental data, which showed a good agreement. Tkachenko [55] studied the conditions for the existence of the similarity of the nondimensional cross spectra for frequency and spacing. It was shown that the absence of the field energy in some region of the wave spaces leads to the violation of similarity of the cross spectra at the corresponding frequencies. Wang [56] developed a numerical procedure using a combination of large eddy simulation (LES) with wall modeling for simulating wall bounded flows. This cost-effective method predicted low-order velocity statistics in good agreement with LES results. Smol'yakov [57] continued his investigations on the noise of turbulent boundary layer over plates and showed that the logarithmic region of the velocity profile is responsible for the zone of the quadrupole noise spectrum with a hyperbolic frequency dependence. It was also shown that the radiation levels of a boundary layer flow in a certain frequency range are increased by surface roughness. Efimtsov et al. [58] analyzed the flight test data of pressure fluctuations (on

Tu-144LL) in front of forward-facing and behind backward-facing steps. The Mach number range was from 0.57 to 1.97. A direct numerical simulation was performed by Kim et al. [59] to examine the relationship between wall-pressure fluctuations and near-wall streamwise vortices in a turbulent boundary layer. It was found that wall-pressure fluctuations were linked with upstream vortices in the buffer zone. Other studies in measuring wall-pressure fluctuations at different conditions were performed by Leclercq and Bohineust [60], Efimtsov et al. [61, 62], and Smol'yakov and Tkachenko [63]. In 2004, Goody [64] presented an empirical model of the surface pressure spectrum. The predicted results by the model were in good agreement with experimental data.

The literature survey in this field shows that most of the related investigations were performed analytically or experimentally. However, with increased efficiency and accuracy of recently developed numerical schemes, accurate predictions of pressure fluctuations are now possible. Lee et al. [65] performed an investigation using Reynolds-averaged Navier–Stokes (RANS) simulation. They validated both equilibrium (zero pressure gradient) and non-equilibrium flows over a backward-facing step. The predicted spectrum was in good agreement with measurements in the redeveloped boundary layer region downstream of the reattachment zone. However, in the middle and high frequencies, it was lower than the measured values in the reattachment region. Efimtsov et al. [66] investigated the effect of supersonic flow parameters on the spectral density of wall-pressure fluctuations in the interaction region of shock and boundary layer, and showed that the influence of the Reynolds number was weak. Efimtsov and Lazarev [67] analyzed the effects of increasing the sound transmission loss of panels with the help of resonant systems on the basis of equivalent representations. Smol'yakov [68] developed a model for the statistical characteristics of the turbulent pressure field in a boundary layer. The

model was characterized by convertibility, which allows changes from cross spectra to wavenumber frequency spectra and back. The results were in good agreement with the experimental results. Peltier and Hambric [69] developed a stochastic model for the space-time turbulence boundary layer wall-pressure spectrum, which uses data from the RANS solution. Goody et al. [70] compared the pressures measured on the hull of model 5488 with the predictions performed using two methods. The first method was the empirical model proposed by Goody, and the second method was a RANS statistical model (RANS-SM). Recently, more numerical investigations have been carried out by researchers. Dietiker and Hoffmann [71] performed a numerical investigation to predict the wall-pressure fluctuations over a backward-facing step. They used detached eddy simulation (DES) with Menter's shear stress transport turbulence model. They compared their results with the results obtained by the empirical relations proposed by Efimtsov et al. [53] and available experimental results. The frequency content matched the experimental results, and the power spectral density was in good agreement with the empirical model. Lee et al. [72] presented a methodology to predict the frequency spectrum of fluctuating wall-pressure based on RANS solution. Recently, Pirozzoli and Bernardini [73] studied the structure of wall-pressure fluctuations of a transonic shock/boundary layer interaction through DNS.

It should be noted that experimental pressure fluctuation measurement within the boundary layer is not a simple task. By placing a pressure probe within the boundary layer, the flowfield is highly affected, and measured data is corrupted. On the other hand, numerical simulation does not disturb the flow and provides insight into the phenomenon.

CHAPTER 2

GOVERNING EQUATIONS AND NUMERICAL APPROACH

2.1 Motivation

The compact schemes have been commonly used during the last decades. The most notable advantage of compact schemes in comparison with the standard finite difference schemes is providing higher order of accuracy without any increase in stencil width [20]. The resolution, which is the largest captured wave number [20], is an important issue in comparing the different schemes with each other. Lele [14] proposed a new scheme (centered compact), which had spectral-like resolution. Unlike the standard finite difference schemes, which use the Lagrange interpolation, the new scheme used the Hermitian formulation. In this scheme, both function values and derivatives are used. Therefore, the result is a tri-diagonal or penta-diagonal system. The advantages of this scheme are noticeable. However, there are some difficulties in employing it. For instance, in problems which contain discontinuities or shock waves, the scheme encounters difficulties. There are visible wiggles around the discontinuities because of the grid-to-grid oscillation. There are several methods to reduce the oscillation such as adding a damping term, using TVD schemes, or applying filters to the scheme. However, in some cases none of the mentioned approaches can improve the scheme. Therefore, despite the advantages of the compact schemes, the existing problems in shock capturing schemes directed the investigators to switch to the schemes such as ENO and WENO which avoid the shock containing stencil. In order to calculate the derivatives, ENO selects the smoothest stencil. However, in WENO scheme the stencils are controlled based on the allocated weights. The weighted compact scheme (WCS) proposed by Jiang et al. [20], was another successful scheme based on the concept of WENO scheme applied to the standard compact scheme. Similar to the WENO scheme, WCS

employs three stencils with allocated weights. The only difference between the WENO and WCS is the building block for each candidate. WENO is developed based on the Lagrange polynomials and WCS is made of Hermitian formulation.

The new scheme, which is discussed in the following sections, is a combination of modified WENO and WCS. Therefore, both schemes are explained and discussed in this chapter.

2.2 Numerical Procedure

2.2.1 ENO Reconstruction Function and Conservative Law

The one-dimensional conservation law is given by

$$u_t(x, t) + f_x(u(x, t)) = 0. \quad (2.1)$$

By applying a conservative approximation to the spatial derivative, equation (2.1) becomes

$$\frac{du_j}{dt} = -\frac{1}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}) \quad (2.2)$$

where Δx is the grid size of the uniform grid and $\hat{f}_{j+1/2}$ and $\hat{f}_{j-1/2}$ are the numerical flux functions. Shu and Osher [8] proved that the primitive function of $\hat{f}$ at the cell interfaces can be exactly calculated by the given point values f_j . Let H be the primitive function of $\hat{f}$, then

$$H(x_{j+1/2}) = \Delta x \sum_{i=-\infty}^j f_i. \quad (2.3)$$

$\hat{f}$ at cell interfaces is the derivative of its primitive function H . Then

$$\hat{f}_{j+1/2} = H'_{j+1/2} \quad (2.4)$$

The primitive function H is a discrete set of data or function, which will require numerical methods to obtain its derivatives. The procedure of finding f'_x from f with

calculation of H and $\hat{f}$ is called reconstruction. The numerical schemes should be applied to the primitive function. Thus, the conservation property is attained.

2.2.2 WENO Scheme (Jiang and Shu) [10]

Recently, WENO scheme has become one of the more popular approaches to solve the hyperbolic conservation equations. The advantage of WENO scheme over ENO is the ability to obtain higher order of accuracy for the same stencil size. In order to illustrate the WENO scheme, one should consider the scalar, one-dimensional advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0. \quad (2.5)$$

By using ENO reconstruction, equation (2.5) can be written as semi-discretized form of equation (2.2). In the WENO scheme of Jiang and Shu, there are r candidate stencils. Each stencil contains r number of points, denoted by S_k for which $k=0, \dots, r-1$. Figure 2-1 provides the schematic of these stencils for $r=3$.

Figure 2-1. Schematics of WENO scheme stencils for $\hat{f}_{j+1/2}$ ($r=3$)

In its most general form, $\hat{f}_{j+1/2}$ can be written as [74]

$$\hat{f}_{j+1/2} = \sum_{k=0}^{r-1} \omega_k q_k^r \quad (2.6)$$

where q_k^r is the candidate flux on each k^{th} stencil and, ω_k is the allocated bias weight of the same stencil. The weight can be calculated by

$$\omega_k = \frac{\gamma_k}{\sum_{k=0}^{r-1} \gamma_k} \quad (2.7)$$

which γ_k is defined as

$$\gamma_k = \frac{C_k^r}{(\varepsilon + IS_k)^p}. \quad (2.8)$$

IS_k is smoothness indicator on each stencil, and ε is a very small number to prevent division by zero. In the proposed scheme, ε is set to 10^{-40} , and C_k^r is the optimal weight that satisfies the requirement $\sum_{k=0}^{r-1} C_k^r = 1$. The power p in equation (2.8) plays an important role in the sensitivity of the weights to the smoothness measurements. Usually this power is set to 2 for WENO schemes.

Jiang and Shu [10] defined the IS_k by

$$IS_k = \sum_{m=1}^{r-1} \int_{x_{j-1/2}}^{x_{j+1/2}} (\Delta x)^{2m-1} \left(\frac{\partial^m q_k^r}{\partial x^m} \right)^2 dx \quad (2.9)$$

where the candidate stencil flux is calculated by

$$q_k^r \Big|_{j+1/2} = \sum_{l=1}^{r-1} a_{k,l}^r f(u_{j-r+1+k+l}). \quad (2.10)$$

If $r \leq 4$, the smoothness measurement is computed by

$$IS_k = \sum_{m=1}^{r-1} \left(\sum_{l=1}^{r-1} d_{l,m,k} f(u_{j-r+1+k+l}) \right)^2. \quad (2.11)$$

The values of $a_{k,l}^r$ and $d_{l,m,k}$ are listed in Ref. [74] for $r=3$ and $r=4$. The weights depend on the local smoothness of the data. Candidate stencils containing shock will have zero weight. By setting $r=3$, the fifth-order accuracy of WENO is achieved. The optimal weights are set as

$$C_0^3 = \frac{1}{10}, \quad C_1^3 = \frac{6}{10}, \quad \text{and} \quad C_2^3 = \frac{3}{10}.$$

It should be noted that in smooth regions, ω_k reverts to the optimal weights [75]. The smoothness indicators are

$$\begin{aligned} IS_0 &= \frac{13}{12} (f_{j-2} - 2f_{j-1} + f_j)^2 + \frac{1}{4} (f_{j-2} - 4f_{j-1} + 3f_j)^2 \\ IS_1 &= \frac{13}{12} (f_{j-1} - 2f_j + f_{j+1})^2 + \frac{1}{4} (f_{j-1} - f_{j+1})^2 \\ IS_2 &= \frac{13}{12} (f_j - 2f_{j+1} + f_{j+2})^2 + \frac{1}{4} (f_{j+2} - 4f_{j+1} + 3f_j)^2. \end{aligned} \quad (2.12)$$

The first term on the right-hand side measures the curvature, and the second term is related to the slope [20].

The WENO scheme can provide an efficient and robust numerical technique, which has great successes in capturing shocks and discontinuities. However, the WENO scheme proposed by Jiang and Shu [10], which is called WENO-JS hereafter, is too dissipative for turbulent flows. There have been several efforts to overcome the deficiencies.

2.2.3 Weighted Compact Scheme (WCS)

The successes of WENO and ENO schemes have encouraged investigators to apply the concept to compact schemes as well.

The finite difference approximation of f'_j can be written based on the Hermite interpolation as [14]

$$\beta_- f'_{j-2} + \alpha_- f'_{j-1} + f'_j + \alpha_+ f'_{j+1} + \beta_+ f'_{j+2} = \frac{1}{h} (b_- f_{j-2} + a_- f_{j-1} + c f_j + a_+ f_{j+1} + b_+ f_{j+2}). \quad (2.13)$$

By setting $\beta_- = \beta_+ = 0$, a symmetric tri-diagonal system is achieved. According to Lele's discussion, one can obtain a one-parameter family of compact scheme

$$\alpha f'_{j-1} + f'_j + \alpha f'_{j+1} = \left[-\frac{1}{12}(4\alpha-1)f_{j-2} - \frac{1}{3}(\alpha+2)f_{j-1} + \frac{1}{3}(\alpha+2)f_{j+1} + \frac{1}{12}(4\alpha-1)f_{j+2} \right] / h \quad (2.14)$$

Similar to the WENO scheme, WCS scheme also employs three candidate stencils. The stencil candidates are shown in Figure 2-2.

Figure 2-2. WCS stencil candidates

By setting $\alpha = \frac{1}{3}$, equation (2.14) becomes a standard sixth-order compact scheme as follows:

$$\frac{1}{3}f'_{j-1} + f'_j + \frac{1}{3}f'_{j+1} \approx \left[-\frac{1}{36}f_{j-2} - \frac{7}{9}f_{j-1} + \frac{7}{9}f_{j+1} + \frac{1}{36}f_{j+2} \right] / h. \quad (2.15)$$

The procedure for deriving weighted compact formulation is identical to the WENO scheme. The

only difference is the values of the optimal weights, which are $C_0 = \frac{1}{18}$, $C_1 = \frac{16}{18}$, and $C_2 = \frac{1}{18}$.

In this scheme, the weights are very sensitive to the power p . Liu et al. [24] employed a linear function for p . However, similar to WENO scheme, this power has been set to 2 in the majority of cases. When $p = 0$, the sixth-order standard compact scheme is recovered.

By employing equations (2.7), (2.8), and (2.12) one can derive

$$\begin{aligned}
\hat{f}_{j+1/2} &\approx \omega_{0,j+1/2} \left(\frac{1}{2} f_{j-1} + \frac{5}{2} f_j + 2\Delta x \cdot f'_j \right) / 3 + \omega_{1,j+1/2} \left[\frac{3}{4} (f_{j+1} + f_j) + \frac{1}{4} \Delta x f'_j - \frac{1}{4} \Delta x f'_{j+1} \right] / 1.5 \\
&+ \omega_{2,j+1/2} \left(\frac{1}{2} f_{j+2} + \frac{5}{2} f_{j+1} - 2\Delta x \cdot f'_{j+1} \right) / 3 \\
\hat{f}_{j-1/2} &\approx \omega_{0,j-1/2} \left(\frac{1}{2} f_{j-2} + \frac{5}{2} f_{j-1} + 2\Delta x \cdot f'_{j-1} \right) / 3 + \omega_{1,j-1/2} \left[\frac{3}{4} (f_j + f_{j-1}) + \frac{1}{4} \Delta x f'_{j-1} - \frac{1}{4} \Delta x f'_j \right] / 1.5 \\
&+ \omega_{2,j-1/2} \left(\frac{1}{2} f_{j+1} + \frac{5}{2} f_j - 2\Delta x \cdot f'_j \right) / 3
\end{aligned} \tag{2.16}$$

and finally f'_j can be computed by

$$f'_j = \frac{\hat{f}_{j+1/2} - \hat{f}_{j-1/2}}{\Delta x} \tag{2.17}$$

The final equation can be written as

$$\begin{aligned}
&\left(\frac{1}{6} \omega_{1,j-1/2} + \frac{2}{3} \omega_{0,j-1/2} \right) f'_{j-1} + \left[1 - \left(\frac{1}{6} \omega_{1,j+1/2} + \frac{2}{3} \omega_{0,j+1/2} + \frac{1}{6} \omega_{1,j-1/2} + \frac{2}{3} \omega_{2,j-1/2} \right) \right] f_j \\
&+ \left(\frac{2}{3} \omega_{2,j+1/2} + \frac{1}{6} \omega_{1,j+1/2} \right) f'_{j+1} \approx \frac{1}{\Delta x} \left[\frac{1}{6} \omega_{2,j+1/2} f_{j+2} + \left(\frac{5}{6} \omega_{2,j+1/2} + \frac{1}{2} \omega_{1,j+1/2} - \frac{1}{6} \omega_{2,j-1/2} \right) f_{j+1} \right. \\
&+ \left(\frac{1}{2} \omega_{1,j+1/2} + \frac{5}{6} \omega_{0,j+1/2} - \frac{1}{2} \omega_{1,j-1/2} - \frac{5}{6} \omega_{2,j-1/2} \right) f_j \\
&\left. + \left(\frac{1}{6} \omega_{0,j+1/2} - \frac{1}{2} \omega_{1,j-1/2} - \frac{5}{6} \omega_{0,j-1/2} \right) f_{j-1} - \frac{1}{6} \omega_{0,j-1/2} f_{j-2} \right]
\end{aligned} \tag{2.18}$$

In equation (2.18), both function values and derivatives are involved and should be solved by a tri-diagonal algorithm.

2.2.4 Advantages and Drawbacks of WENO and WCS, and the Proposed Approach

A higher order of accuracy and better resolution are the advantages of both the WCS and WENO schemes. However, there are some deficiencies which could cause the schemes to fail in some cases. As discussed previously, WENO-JS is too dissipative [75]. A preferred scheme for compressible turbulent flows should have minimal dispersion and dissipation errors, which can properly capture the shocks and discontinuities. To overcome these deficiencies, a new hybrid scheme was developed by Adams and Shariff [76], which was a combination of ENO and

compact upwind schemes. In this scheme, ENO was automatically activated near the discontinuities. In a similar approach, Pirozzoli [77] coupled a conservative compact scheme with WENO which had better results. Ren et al. [78], Hill and Pullin [79], Kim and Kwon [80], and several other investigators employed similar concepts to introduce better procedures for solving problems containing shocks and discontinuities. WCS has sixth order of accuracy, which is superior to WENO with the same stencil sizes. However, this scheme does not perform well for Euler equations with shocks. Near the shocks and discontinuities, visible oscillations are observed. Several investigators developed filters for removing the oscillation. However, majority of these filters were not successful in removing the oscillations near the shocks and discontinuities.

In order to improve and utilize the advantages of both schemes, a new approach is proposed. This new scheme couples WCS and WENO schemes with modifications. Here a modified fourth-order Runge-Kutta with TVD is employed for time discretization. Several of the TVD and ENO dissipations with modifications appear to be good candidates as filters [11]. In the present effort, Harten and Yee upwind TVD is used with a filter designed by Yee et al. [11]. The filter is a generalization of the ACM filter of Harten. This filter minimizes the numerical dissipations for the scheme. A symmetric WENO, which was proposed by Martin et al. [75], is used for one-dimensional simulations to maximize the order of accuracy and bandwidth and minimizing the dissipation errors. However, this modification does not perform well for two-dimensional simulations, which will be discussed later.

In order to ensure that WENO scheme has fifth-order of accuracy in all areas including the shock regions, another modification is applied. Henrick et al. [81] used a new equation for

weights which provides the sufficient condition for WENO scheme to maintain the fifth-order of accuracy in all regions.

The new scheme is hybrid of sixth-order WCS and modified WENO-JS which shows better results in comparison with traditional schemes. The modifications are introduced and discussed in the following sections.

2.2.5 Time Discretizations and TVD

In the proposed scheme, time integration is performed using fourth-order TVD Runge-Kutta method. It helps to reduce the oscillation introduced by the schemes. This method, which is called RK4 hereafter, for one-dimensional flow is formulated as [82]

$$\begin{aligned}
Q_i^{(1)} &= Q_i^n \\
Q_i^{(2)} &= Q_i^n - \frac{\Delta t}{4} \left(\frac{\partial E}{\partial x} \right)_i^{(1)} \\
Q_i^{(3)} &= Q_i^n - \frac{\Delta t}{3} \left(\frac{\partial E}{\partial x} \right)_i^{(2)} \\
Q_i^{(4)} &= Q_i^n - \frac{\Delta t}{2} \left(\frac{\partial E}{\partial x} \right)_i^{(3)} \\
Q_i^{n+1} &= Q_i^n - \Delta t \left(\frac{\partial E}{\partial x} \right)_i^{(4)}
\end{aligned} \tag{2.19}$$

which is augmented by one additional step to reduce the dispersion error as follows

$$Q_i^{n+1} = Q_i^{n+1} - \frac{\Delta t}{2\Delta x} (\phi_{i+1/2}^n - \phi_{i-1/2}^n) \tag{2.20}$$

Several TVD approaches can be amended to RK4. In equation (2.20), each of the flux limiters (ϕ) can be used. In the current research, Harten-Yee upwind TVD is employed. In this method, the flux vector limiters are computed by [1]

$$\phi_{i+1/2} = \sigma(\alpha_{i+1/2})(G_{i+1} + G_i) - \psi(\alpha_{i+1/2} + \beta_{i+1/2})\delta_{i+1/2} \tag{2.21}$$

$$\phi_{i-1/2} = \sigma(\alpha_{i-1/2})(G_i + G_{i-1}) - \psi(\alpha_{i-1/2} + \beta_{i-1/2})\delta_{i-1/2}. \tag{2.22}$$

The terms in equations (2.21) and (2.22) are calculated by

$$(\delta_{i+1/2})_1 = (X_{i+1/2}^{-1})_{\text{first row}} (Q_{i+1} - Q_i) \quad (2.23)$$

$$\sigma(\alpha_{i+1/2}) = \frac{1}{2} \psi(\alpha_{i+1/2}) + \frac{\Delta t}{\Delta x} (\alpha_{i+1/2})^2 \quad (2.24)$$

$$\beta_{i+1/2} = \sigma(\alpha_{i+1/2}) \begin{cases} \frac{G_{i+1} - G_i}{\delta_{i+1/2}} & \text{for } \delta_{i+1/2} \neq 0 \\ 0 & \text{for } \delta_{i+1/2} = 0 \end{cases} \quad (2.25)$$

$$\psi(y) = \begin{cases} |y| & \text{for } |y| \geq \varepsilon \\ \frac{y^2 + \varepsilon^2}{2\varepsilon} & \text{for } |y| < \varepsilon \end{cases} \quad (2.26)$$

while $0 \leq \varepsilon \leq 0.125$, X is the eigenvector matrix, and α denotes $S\lambda$. All the above equations are the terms for calculating $\phi_{i+1/2}$, and subsequently $\phi_{i-1/2}$ can be computed in a similar technique. Different limiters have been introduced for calculating G which are

$$G_i = \min \text{mod}(\delta_{i-1/2}, \delta_{i+1/2}) \quad (2.27)$$

$$G_i = \frac{\delta_{i+1/2} \delta_{i-1/2} + |\delta_{i+1/2} \delta_{i-1/2}|}{\delta_{i+1/2} + \delta_{i-1/2}} \quad (2.28)$$

$$G_i = \frac{\delta_{i-1/2} [(\delta_{i+1/2})^2 + \omega] + \delta_{i+1/2} [(\delta_{i-1/2})^2 + \omega]}{(\delta_{i+1/2})^2 + (\delta_{i-1/2})^2 + 2\omega} \quad (2.29)$$

with $10^{-7} \leq \omega \leq 10^{-5}$.

$$G_i = \min \text{mod} \left[2\delta_{i-1/2}, 2\delta_{i+1/2}, \frac{1}{2}(\delta_{i+1/2} + \delta_{i-1/2}) \right] \quad (2.30)$$

$$G_i = S \times \max \left[0, \min(2|\delta_{i+1/2}|, S \times \delta_{i-1/2}), \min(|\delta_{i+1/2}|, 2S \times \delta_{i-1/2}) \right] \quad (2.31)$$

where $S = \text{Sgn}(\delta_{i+1/2}) = \frac{|\delta_{i+1/2}|}{\delta_{i+1/2}}$, and

$$\min \text{mod}(a, b, c, \dots, n) = S \times \max \left[0, \min(|a|, S \times b, S \times c, \dots, S \times n) \right] \quad (2.32)$$

with $S = \frac{|a|}{a}$. For the proposed scheme, equation (2.31) is used as TVD limiter.

It should be noted that Δt in equations (2.19) and (2.20) is calculated by

$$\Delta t = \frac{\text{CFL}}{\frac{\max_{1 \leq i \leq \text{IM}} (|u| + a)}{\Delta x}} \quad (2.33)$$

where a is the speed of sound.

2.2.6 Characteristic Filter

Yee et al. [11] proposed characteristic type filters to reduce the numerical instabilities. The procedure is similar to Harten's ACM filter [83]. The Harten switch has a self-adjusting capability. Yee employed characteristic-based filters in order to better control the dissipation for complex shock waves, shear, and turbulence interactions. It should be noted that traditional shock-capturing schemes cannot eliminate high-frequency errors. Adding a characteristic filter can remedy this problem. The filter operator is introduced by

$$L_{\text{TVD}}(F, G)_{j,k} = \frac{1}{\Delta x} [\tilde{F}_{j+1/2,k} - \tilde{F}_{j-1/2,k}] + \frac{1}{\Delta y} [\tilde{G}_{j,k+1/2} - \tilde{G}_{j,k-1/2}] \quad (2.34)$$

where $\tilde{F}$ and $\tilde{G}$ are the x and y numerical fluxes. The numerical flux $\tilde{F}_{j+1/2,k}$ can be cast into the following form:

$$\tilde{F}_{j+1/2,k} = \frac{1}{2} \left[F_{j+1,k} + F_{j,k} + X_{j+\frac{1}{2}} \Phi_{j+\frac{1}{2}} \right]. \quad (2.35)$$

The last term in equation (2.35) is the nonlinear dissipation. The filter numerical flux, which is the modified form of the nonlinear dissipation term, is calculated by

$$\tilde{F}_{j+1/2,k}^* = \frac{1}{2} X_{j+1/2} \Phi_{j+1/2}^* \quad (2.36)$$

which should not be confused with $\tilde{F}_{j+1/2,k}$. The elements of $\Phi_{j+1/2}^*$ denoted by $\phi_{j+1/2}^{l*}$ are

$$\phi_{j+1/2}^{l*} = \kappa \theta_{j+1/2}^l \phi_{j+1/2}^l. \quad (2.37)$$

In equation (2.37), the function $\kappa \theta_{j+1/2}^l$ plays an important role in achieving high-frequency of the fine-scale flow structures. The parameter κ is case-dependent, with the range of $0.03 \leq \kappa \leq 2$. The function $\theta_{j+1/2}^l$ is the recommended Harten switch, which is calculated by

$$\theta_{j+1/2}^l = \max(\theta_{j-m+1}^l, \dots, \theta_{j+m}^l) \quad (2.38)$$

where

$$\theta_j^l = \left| \frac{|\delta_{j+1/2}^l| - |\delta_{j-1/2}^l|}{|\delta_{j+1/2}^l| + |\delta_{j-1/2}^l|} \right|^p. \quad (2.39)$$

In above equation, $\delta_{j+1/2}^l$ can be calculated by equation (2.23). Yee employed a simplified version of equation (2.38), which is

$$\theta_{j+1/2}^l = \max(\theta_j^l, \theta_{j+1}^l) \quad (2.40)$$

and used $p = 1$ in equation (2.39). In the proposed scheme, equation (2.40) and $p = 1$ are used for all the numerical examples. The function $\phi_{j+1/2}^l$ can be found in any of the TVD schemes, which is calculated by Harten-Yee upwind TVD in the new scheme. The numerical examples show that the scheme is sensitive to the value of the κ . By using the filter, numerical instabilities are suppressed, and there is a significant change in shock-capturing capability of the scheme.

2.2.7 Bandwidth Optimization

As stated previously, in the proposed scheme the modified WENO-JS scheme is coupled with WCS. One of the other modifications applied to WENO scheme is employing symmetric optimal stencils, which reduces the dissipation errors. Numerical experiments carried out by

Weirs and Candler [74] show that the bandwidth optimized stencils can improve the performance of WENO with greater resolution efficiency. Based on the investigations carried out by Weirs and Candler, there are two sources of the dissipation for WENO scheme. The first is the optimal stencil and the second is the smoothness measurement. In this approach, instead of using r stencils, $r + 1$ is used. Therefore, equation (2.6) becomes

$$\hat{f}_{j+1/2} = \sum_{k=0}^r \omega_k q_k^r \quad (2.41)$$

and the weights are calculated by

$$\omega_k = \frac{\gamma_k}{\sum_{k=0}^r \gamma_k}. \quad (2.42)$$

The candidate stencils for this technique are shown in Figure 2-3. The definitions for γ_k and IS_k apply for $k = r$.

Figure 2-3. Candidate stencils for the WENO-SYM numerical flux $\hat{f}_{j+1/2}$ ($r = 3$)

In order to prevent the selection of the fully downwinded candidate near the shocks, IS_r is modified by

$$IS_r = \max_{0 \leq k \leq r} (IS_k). \quad (2.43)$$

The coefficients for calculating the additional terms can be found in the work of Weirs and Candler [74]. For the $r=3$ case, the optimal weights are $\tilde{C}_0^3 = 0.09464755$, $\tilde{C}_1^3 = 0.42807421$, $\tilde{C}_2^3 = 0.40828933$, and $\tilde{C}_3^3 = 0.06898891$.

2.2.8 Mapped WENO Scheme

Jiang and Shu [84] showed that the WENO-JS had fifth-order convergence properties. However, Henrick et al. [81] carried out an investigation, which showed its rate of convergence as being less than fifth order at critical points. They derived the necessary and sufficient conditions on the weights to achieve fifth order of accuracy in all regions. Therefore, new nonlinear weights with the same smoothness measurement indicators of the WENO-JS scheme were derived. By applying this modification, the scheme at the critical points, where the first derivative of the function is zero but the second derivative is not zero, has fifth order of accuracy. Due to allocating large nonlinear weights to the stencils containing discontinuities, the numerical dissipation errors are reduced.

The new weights can be computed from

$$\omega_k^M = \frac{\gamma_k^*}{\sum_{l=0}^r \gamma_l^*}. \quad (2.44)$$

It should be noted that in equation (2.44), the bandwidth modification has been implemented.

The function γ_k^* can be calculated from

$$\gamma_k^* = \frac{\omega_k (C_k^r + (C_k^r)^2 - 3C_k^r \omega_k + \omega_k^2)}{(C_k^r)^2 + (1 - 2C_k^r) \omega_k}. \quad (2.45)$$

2.2.9 Proposed Modified WCS-WENO Scheme (MWWS)

Subsequent to modifications of the WENO scheme, the new technique couples WCS and WENO schemes. The proposed scheme is calculated by

$$Q_{\text{MWWS}} = \beta Q_{\text{WCS}} + (1 - \beta) Q_{\text{Modified WENO}}. \quad (2.46)$$

This combination of schemes allows for accurate wave propagation, which is important in calculating the power spectral density and pressure fluctuations. In order to show the efficiency of the new scheme, several benchmark problems are presented. The best value of β for all of the benchmark problems is 0.65, which is used for three-dimensional case, as well.

2.3 Governing Equations

In this section, the governing equations of motion for one- and two-dimensional inviscid and viscous flow are presented. The results of the one- and two-dimensional flow for different benchmark problems are shown in next chapter.

2.3.1 One-Dimensional Euler Equations

The governing equation for the one-dimensional inviscid flow can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0 \quad (2.47)$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho e_t \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho e_t + p)u \end{bmatrix}. \quad (2.48)$$

In equation (2.48), ρ is the density, u is the fluid velocity, p is pressure, and $e_t = e + \frac{1}{2}u^2$.

2.3.2 Two-Dimensional Euler Equations

The governing equation of motion for the two-dimensional inviscid flow is

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \quad (2.49)$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_t \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (\rho e_t + p)u \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (\rho e_t + p)v \end{bmatrix}. \quad (2.50)$$

For all two-dimensional benchmark problems, the equations of motion are transformed from physical space to computational space. Therefore, the equation (2.49) becomes

$$\frac{\partial \bar{Q}}{\partial \tau} + \frac{\partial \bar{E}}{\partial \xi} + \frac{\partial \bar{F}}{\partial \eta} = 0 \quad (2.51)$$

where

$$\begin{aligned} \bar{Q} &= Q / J \\ \bar{E} &= (\xi_t Q + \xi_x E + \xi_y F) / J \\ \bar{F} &= (\eta_t Q + \eta_x E + \eta_y F) / J. \end{aligned} \quad (2.52)$$

In equation (2.52), J is defined as the Jacobian of transformation. The flux Jacobian matrices (A and B) are calculated from the following equations

$$A = \frac{\partial \bar{E}}{\partial \bar{Q}} = \begin{bmatrix} \xi_t & \xi_x & \xi_y & 0 \\ -u(u\xi_x + v\xi_y) & \xi_t + \xi_x(3-\gamma)u & (1-\gamma)\xi_x v & (\gamma-1)\xi_x \\ +\xi_x[0.5(\gamma-1)(u^2 + v^2)] & +\xi_y v & +\xi_y u & \\ -v(u\xi_x + v\xi_y) & \xi_x v - (\gamma-1)\xi_y u & \xi_t + \xi_x u & (\gamma-1)\xi_y \\ +\xi_y[0.5(\gamma-1)(u^2 + v^2)] & +\xi_y(3-\gamma)v & +\xi_y(3-\gamma)v & \\ (\xi_x u + \xi_y v) & \xi_x[\frac{a^2}{\gamma-1} + 0.5(u^2 + v^2)] & \xi_y[\frac{a^2}{\gamma-1} + 0.5(u^2 + v^2)] & \xi_t + \\ [\frac{a^2}{1-\gamma} + (0.5\gamma-1)(u^2 + v^2)] & -(\gamma-1)(\xi_x u + \xi_y v)u & -(\gamma-1)(\xi_x u + \xi_y v)v & \gamma(\xi_x u + \xi_y v) \end{bmatrix} \quad (2.53)$$

and

$$B = \frac{\partial \bar{F}}{\partial \bar{Q}} = \begin{bmatrix} \eta_t & \eta_x & \eta_y & 0 \\ -u(\eta_x + v\eta_y) & \eta_t + \eta_x(3-\gamma)u & (1-\gamma)\eta_x v & (\gamma-1)\eta_x \\ +\eta_x[0.5(\gamma-1)(u^2+v^2)] & +\eta_y v & +\eta_y u & \\ -v(u\eta_x + v\eta_y) & \eta_x v - (\gamma-1)\eta_y u & \eta_t + \eta_x u & (\gamma-1)\eta_y \\ +\eta_y[0.5(\gamma-1)(u^2+v^2)] & & +\eta_y(3-\gamma)v & \\ (\eta_x u + \eta_y v) & \eta_x[\frac{a^2}{\gamma-1} + 0.5(u^2+v^2)] & \eta_y[\frac{a^2}{\gamma-1} + 0.5(u^2+v^2)] & \eta_t + \\ [\frac{a^2}{1-\gamma} + (0.5\gamma-1)(u^2+v^2)] & -(\gamma-1)(\eta_x u + \eta_y v)u & -(\gamma-1)(\eta_x u + \eta_y v)v & \gamma(\eta_x u + \eta_y v) \end{bmatrix} \quad (2.54)$$

where a is the speed of sound. The eigenvalues of these matrices are determined as

$$\begin{aligned} \lambda_\xi(1) &= \xi_t + \xi_x u + \xi_y v \\ \lambda_\xi(2) &= \lambda_\xi(1) \\ \lambda_\xi(3) &= \lambda_\xi(1) + a\sqrt{\xi_x^2 + \xi_y^2} \\ \lambda_\xi(4) &= \lambda_\xi(1) - a\sqrt{\xi_x^2 + \xi_y^2} \\ \lambda_\eta(1) &= \eta_t + \eta_x u + \eta_y v \\ \lambda_\eta(2) &= \lambda_\xi(1) \\ \lambda_\eta(3) &= \lambda_\xi(1) + a\sqrt{\xi_x^2 + \xi_y^2} \\ \lambda_\eta(4) &= \lambda_\xi(1) - a\sqrt{\xi_x^2 + \xi_y^2}. \end{aligned} \quad (2.55)$$

The eigenvector matrices and their inverse are

$$X_A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -\frac{\xi_y}{\xi_x} & \frac{u\xi_x + v\xi_y}{\xi_x} & u + \frac{a\xi_x}{\sqrt{\xi_x^2 + \xi_y^2}} & u - \frac{a\xi_x}{\sqrt{\xi_x^2 + \xi_y^2}} \\ 1 & 0 & v + \frac{a\xi_y}{\sqrt{\xi_x^2 + \xi_y^2}} & v - \frac{a\xi_y}{\sqrt{\xi_x^2 + \xi_y^2}} \\ \frac{v\xi_x - u\xi_y}{\xi_x} & \frac{2uv\xi_y + (u^2 - v^2)\xi_x}{2\xi_x} & \frac{u^2 + v^2}{2} + \frac{a^2}{\gamma-1} + \frac{a(u\xi_x + v\xi_y)}{\sqrt{\xi_x^2 + \xi_y^2}} & \frac{u^2 + v^2}{2} + \frac{a^2}{\gamma-1} - \frac{a(u\xi_x + v\xi_y)}{\sqrt{\xi_x^2 + \xi_y^2}} \end{bmatrix} \quad (2.56)$$

and

$$X_B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ \frac{\eta_y}{\eta_x} & \frac{u\eta_x + v\eta_y}{\eta_x} & u + \frac{a\eta_x}{\sqrt{\eta_x^2 + \eta_y^2}} & u - \frac{a\eta_x}{\sqrt{\eta_x^2 + \eta_y^2}} \\ 1 & 0 & v + \frac{a\eta_y}{\sqrt{\eta_x^2 + \eta_y^2}} & v - \frac{a\eta_y}{\sqrt{\eta_x^2 + \eta_y^2}} \\ \frac{v\eta_x - u\eta_y}{\eta_x} & \frac{2uv\eta_y + (u^2 - v^2)\eta_x}{2\eta_x} & \frac{u^2 + v^2}{2} + \frac{a^2}{\gamma - 1} + \frac{a(u\eta_x + v\eta_y)}{\sqrt{\eta_x^2 + \eta_y^2}} & \frac{u^2 + v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{a(u\eta_x + v\eta_y)}{\sqrt{\eta_x^2 + \eta_y^2}} \end{bmatrix} \quad (2.57)$$

$$X_A^{-1} = \begin{bmatrix} \frac{(1-\gamma)v(u^2 + v^2)}{2a^2} + \frac{\xi_y(u\xi_x + v\xi_y)}{\xi_x^2 + \xi_y^2} & \frac{(\gamma-1)uv}{a^2} - \frac{\xi_x\xi_y}{\xi_x^2 + \xi_y^2} & \frac{(\gamma-1)v^2}{a^2} + \frac{\xi_x^2}{\xi_x^2 + \xi_y^2} & \frac{(1-\gamma)v}{a^2} \\ 1 + \frac{(1-\gamma)(u^2 + v^2)}{2a^2} & \frac{(\gamma-1)u}{a^2} & \frac{(\gamma-1)v}{a^2} & \frac{(1-\gamma)}{a^2} \\ \frac{(\gamma-1)(u^2 + v^2)}{4a^2} - \frac{u\xi_x + v\xi_y}{2a\sqrt{\xi_x^2 + \xi_y^2}} & \frac{(1-\gamma)u}{2a^2} + \frac{\xi_x}{2a\sqrt{\xi_x^2 + \xi_y^2}} & \frac{(1-\gamma)v}{2a^2} + \frac{\xi_y}{2a\sqrt{\xi_x^2 + \xi_y^2}} & \frac{(\gamma-1)}{2a^2} \\ \frac{(\gamma-1)(u^2 + v^2)}{4a^2} + \frac{u\xi_x + v\xi_y}{2a\sqrt{\xi_x^2 + \xi_y^2}} & \frac{(1-\gamma)u}{2a^2} - \frac{\xi_x}{2a\sqrt{\xi_x^2 + \xi_y^2}} & \frac{(1-\gamma)v}{2a^2} - \frac{\xi_y}{2a\sqrt{\xi_x^2 + \xi_y^2}} & \frac{(\gamma-1)}{2a^2} \end{bmatrix} \quad (2.58)$$

and

$$X_B^{-1} = \begin{bmatrix} \frac{(1-\gamma)v(u^2+v^2)}{2a^2} + \frac{\eta_y(u\eta_x+v\eta_y)}{\eta_x^2+\eta_y^2} & \frac{(\gamma-1)uv}{a^2} - \frac{\eta_x\eta_y}{\eta_x^2+\eta_y^2} & \frac{(\gamma-1)v^2}{a^2} + \frac{\eta_x^2}{\eta_x^2+\eta_y^2} & \frac{(1-\gamma)v}{a^2} \\ 1 + \frac{(1-\gamma)(u^2+v^2)}{2a^2} & \frac{(\gamma-1)u}{a^2} & \frac{(\gamma-1)v}{a^2} & \frac{(1-\gamma)}{a^2} \\ \frac{(\gamma-1)(u^2+v^2)}{4a^2} - \frac{u\eta_x+v\eta_y}{2a\sqrt{\eta_x^2+\eta_y^2}} & \frac{(1-\gamma)u}{2a^2} + \frac{\eta_x}{2a\sqrt{\eta_x^2+\eta_y^2}} & \frac{(1-\gamma)v}{2a^2} + \frac{\eta_y}{2a\sqrt{\eta_x^2+\eta_y^2}} & \frac{(\gamma-1)}{2a^2} \\ \frac{(\gamma-1)(u^2+v^2)}{4a^2} + \frac{u\eta_x+v\eta_y}{2a\sqrt{\eta_x^2+\eta_y^2}} & \frac{(1-\gamma)u}{2a^2} - \frac{\eta_x}{2a\sqrt{\eta_x^2+\eta_y^2}} & \frac{(1-\gamma)v}{2a^2} - \frac{\eta_y}{2a\sqrt{\eta_x^2+\eta_y^2}} & \frac{(\gamma-1)}{2a^2} \end{bmatrix}. \quad (2.59)$$

In some cases, ξ_x or η_x may become zero. In this situation the eigenvalues are calculated

from

$$\begin{aligned} \lambda_{\xi_0}(1) &= \xi_t + \xi_y v \\ \lambda_{\xi_0}(2) &= \lambda_{\xi_0}(1) \\ \lambda_{\xi_0}(3) &= \xi_t + (v+a)\xi_y \\ \lambda_{\xi_0}(4) &= \xi_t + (v-a)\xi_y \end{aligned} \quad (2.60)$$

and eigenvector matrix and inverse of it can be written as

$$X_{A0} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & u & u \\ 0 & v & v+a & v-a \\ u & \frac{v^2-u^2}{2} & \frac{a^2}{\gamma-1} + \frac{(u^2+v^2+2av)}{2} & \frac{a^2}{\gamma-1} + \frac{(u^2+v^2-2av)}{2} \end{bmatrix} \quad (2.61)$$

and

$$X_{A0}^{-1} = \begin{bmatrix} -\frac{(\gamma-1)(u^3+uv^2)}{2a^2} & 1+\frac{(\gamma-1)u^2}{a^2} & \frac{(\gamma-1)uv}{a^2} & \frac{(1-\gamma)u}{a^2} \\ 1-\frac{(\gamma-1)(u^2+v^2)}{2a^2} & \frac{(\gamma-1)u}{a^2} & \frac{(\gamma-1)v}{a^2} & \frac{(1-\gamma)}{a^2} \\ \frac{(\gamma-1)(u^2+v^2)-2av}{4a^2} & \frac{(1-\gamma)u}{2a^2} & \frac{a-(\gamma-1)v}{2a^2} & \frac{(\gamma-1)}{2a^2} \\ \frac{(\gamma-1)(u^2+v^2)+2av}{4a^2} & \frac{(1-\gamma)u}{2a^2} & -\frac{(a+(\gamma-1)v)}{2a^2} & \frac{(\gamma-1)}{2a^2} \end{bmatrix}. \quad (2.62)$$

For $\eta_x = 0$, the procedure is exactly the same. The only difference is employing η_y instead of ξ_y in all of the equations.

The equation (2.19), becomes

$$\begin{aligned} \bar{Q}_{i,j}^{(1)} &= \bar{Q}_{i,j}^n \\ \bar{Q}_{i,j}^{(2)} &= \bar{Q}_{i,j}^n - \frac{\Delta\tau}{4} \left[\left(\frac{\partial E}{\partial \xi} \right)_{i,j}^{(1)} + \left(\frac{\partial F}{\partial \eta} \right)_{i,j}^{(1)} \right] \\ \bar{Q}_{i,j}^{(3)} &= \bar{Q}_{i,j}^n - \frac{\Delta\tau}{3} \left[\left(\frac{\partial E}{\partial \xi} \right)_{i,j}^{(2)} + \left(\frac{\partial F}{\partial \eta} \right)_{i,j}^{(2)} \right] \\ \bar{Q}_{i,j}^{(4)} &= \bar{Q}_{i,j}^n - \frac{\Delta\tau}{2} \left[\left(\frac{\partial E}{\partial \xi} \right)_{i,j}^{(3)} + \left(\frac{\partial F}{\partial \eta} \right)_{i,j}^{(3)} \right] \\ \bar{Q}_{i,j}^{n+1} &= \bar{Q}_{i,j}^n - \Delta\tau \left[\left(\frac{\partial E}{\partial \xi} \right)_{i,j}^{(4)} + \left(\frac{\partial F}{\partial \eta} \right)_{i,j}^{(4)} \right]. \end{aligned} \quad (2.63)$$

The time step $\Delta\tau$ in equation (2.63) is defined by

$$\Delta\tau = \frac{\text{CFL}}{\frac{\max_{1 \leq i \leq \text{IM}, 1 \leq j \leq \text{JM}} (|u|+a)_{i,j}}{\Delta\xi} + \frac{\max_{1 \leq i \leq \text{IM}, 1 \leq j \leq \text{JM}} (|v|+a)_{i,j}}{\Delta\eta}}. \quad (2.64)$$

2.3.3 Two-Dimensional Navier-Stokes Equations

The governing equation in the physical space is given by

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = \frac{\partial E_v}{\partial x} + \frac{\partial F_v}{\partial y}. \quad (2.65)$$

It should be noted that coordinate transformation from physical to computational space is required for Navier-Stokes equations. Therefore, equation (2.65) becomes

$$\frac{\partial \bar{Q}}{\partial \tau} + \frac{\partial \bar{E}}{\partial \xi} + \frac{\partial \bar{F}}{\partial \eta} = \frac{\partial \bar{E}_v}{\partial \xi} + \frac{\partial \bar{F}_v}{\partial \eta} \quad (2.66)$$

where

$$\begin{aligned} \bar{E}_v &= (\xi_x E_v + \xi_y F_v) / J \\ \bar{F}_v &= (\eta_x E_v + \eta_y F_v) / J \end{aligned} \quad (2.67)$$

or by employing viscous shear stresses, one can obtain

$$\begin{aligned} \bar{E}_v &= \frac{\mu}{\text{Re}_\infty J} \left[\begin{array}{c} 0 \\ \cdots \cdots \cdots \\ a_1 u_\xi + a_3 v_\xi + c_1 u_\eta + c_3 v_\eta \\ \cdots \cdots \cdots \\ a_3 u_\xi + a_2 v_\xi + c_4 u_\eta + c_2 v_\eta \\ \cdots \cdots \cdots \\ \frac{1}{2} a_1 (u^2)_\xi + \frac{1}{2} a_2 (v^2)_\xi + a_3 (uv)_\xi + \frac{1}{\text{Pr}(\gamma-1)M_\infty^2} a_4 T_\xi \\ + \frac{1}{2} c_1 (u^2)_\eta + \frac{1}{2} c_2 (v^2)_\eta + c_3 uv_\eta + c_4 vu_\eta + \frac{1}{\text{Pr}(\gamma-1)M_\infty^2} c_5 T_\eta \end{array} \right] \\ \bar{F}_v &= \frac{\mu}{\text{Re}_\infty J} \left[\begin{array}{c} 0 \\ \cdots \cdots \cdots \\ c_1 u_\xi + c_4 v_\xi + b_1 u_\eta + b_3 v_\eta \\ \cdots \cdots \cdots \\ c_3 u_\xi + c_2 v_\xi + b_3 u_\eta + b_2 v_\eta \\ \cdots \cdots \cdots \\ \frac{1}{2} c_1 (u^2)_\xi + \frac{1}{2} c_2 (v^2)_\xi + c_3 uv_\xi + c_4 vu_\xi + \frac{1}{\text{Pr}(\gamma-1)M_\infty^2} c_5 T_\xi \\ + \frac{1}{2} b_1 (u^2)_\eta + \frac{1}{2} b_2 (v^2)_\eta + b_3 (uv)_\eta + \frac{1}{\text{Pr}(\gamma-1)M_\infty^2} b_4 T_\eta \end{array} \right] \end{aligned} \quad (2.68)$$

where

$$\begin{aligned}
a_1 &= \frac{4}{3}\xi_x^2 + \xi_y^2 & a_2 &= \xi_x^2 + \frac{4}{3}\xi_y^2 \\
a_3 &= \frac{1}{3}\xi_x\xi_y & a_4 &= \xi_x^2 + \xi_y^2 \\
b_1 &= \frac{4}{3}\eta_x^2 + \eta_y^2 & b_2 &= \eta_x^2 + \frac{4}{3}\eta_y^2 \\
b_3 &= \frac{1}{3}\eta_x\eta_y & b_4 &= \eta_x^2 + \eta_y^2 \\
c_1 &= \frac{4}{3}\eta_x\xi_x + \eta_y\xi_y & c_2 &= \xi_x\eta_x + \frac{4}{3}\xi_y\eta_y \\
c_3 &= \eta_x\xi_y - \frac{2}{3}\xi_x\eta_y & c_4 &= \xi_x\eta_y - \frac{2}{3}\xi_y\eta_x \\
c_5 &= \xi_x\eta_x + \xi_y\eta_y.
\end{aligned} \tag{2.69}$$

In equation (2.66), the convective terms are approximated by MWWS and the diffusion terms are calculated by a fourth-order central difference scheme.

More details about calculating the viscous terms can be found in Ref. [85]. The RK4 for the two-dimensional Navier-Stokes equation becomes

$$\begin{aligned}
\bar{Q}_{i,j}^{(1)} &= \bar{Q}_{i,j}^n \\
\bar{Q}_{i,j}^{(2)} &= \bar{Q}_{i,j}^n - \frac{\Delta\tau}{4} \left[\left(\frac{\partial E}{\partial \xi} \right)_{i,j}^{(1)} + \left(\frac{\partial F}{\partial \eta} \right)_{i,j}^{(1)} - \left(\frac{\partial E_v}{\partial \xi} \right)_{i,j}^{(1)} - \left(\frac{\partial F_v}{\partial \eta} \right)_{i,j}^{(1)} \right] \\
\bar{Q}_{i,j}^{(3)} &= \bar{Q}_{i,j}^n - \frac{\Delta\tau}{3} \left[\left(\frac{\partial E}{\partial \xi} \right)_{i,j}^{(2)} + \left(\frac{\partial F}{\partial \eta} \right)_{i,j}^{(2)} - \left(\frac{\partial E_v}{\partial \xi} \right)_{i,j}^{(2)} - \left(\frac{\partial F_v}{\partial \eta} \right)_{i,j}^{(2)} \right] \\
\bar{Q}_{i,j}^{(4)} &= \bar{Q}_{i,j}^n - \frac{\Delta\tau}{2} \left[\left(\frac{\partial E}{\partial \xi} \right)_{i,j}^{(3)} + \left(\frac{\partial F}{\partial \eta} \right)_{i,j}^{(3)} - \left(\frac{\partial E_v}{\partial \xi} \right)_{i,j}^{(3)} - \left(\frac{\partial F_v}{\partial \eta} \right)_{i,j}^{(3)} \right] \\
\bar{Q}_{i,j}^{n+1} &= \bar{Q}_{i,j}^n - \Delta\tau \left[\left(\frac{\partial E}{\partial \xi} \right)_{i,j}^{(4)} + \left(\frac{\partial F}{\partial \eta} \right)_{i,j}^{(4)} - \left(\frac{\partial E_v}{\partial \xi} \right)_{i,j}^{(4)} - \left(\frac{\partial F_v}{\partial \eta} \right)_{i,j}^{(4)} \right].
\end{aligned} \tag{2.70}$$

The time step calculation is similar to the previous section and can be calculated by equation (2.64).

2.3.4 Three-Dimensional Navier-Stokes Equations

The governing equation in the physical space is given by

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = \frac{\partial E_v}{\partial x} + \frac{\partial F_v}{\partial y} + \frac{\partial G_v}{\partial z}. \quad (2.71)$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_t \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (\rho e_t + p)u \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ (\rho e_t + p)v \end{bmatrix}, \quad G = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ (\rho e_t + p)w \end{bmatrix} \quad (2.72)$$

By transformation from physical to computational space the governing equation becomes

$$\frac{\partial \bar{Q}}{\partial \tau} + \frac{\partial \bar{E}}{\partial \xi} + \frac{\partial \bar{F}}{\partial \eta} + \frac{\partial \bar{G}}{\partial \zeta} = \frac{\partial \bar{E}_v}{\partial \xi} + \frac{\partial \bar{F}_v}{\partial \eta} + \frac{\partial \bar{G}_v}{\partial \zeta} \quad (2.73)$$

where

$$\begin{aligned} \bar{Q} &= Q / J \\ \bar{E} &= (\xi_t Q + \xi_x E + \xi_y F + \xi_z G) / J \\ \bar{F} &= (\eta_t Q + \eta_x E + \eta_y F + \eta_z G) / J \\ \bar{G} &= (\zeta_t Q + \zeta_x E + \zeta_y F + \zeta_z G) / J \\ \bar{E}_v &= (\xi_x E_v + \xi_y F_v + \xi_z G_v) / J \\ \bar{F}_v &= (\eta_x E_v + \eta_y F_v + \eta_z G_v) / J \\ \bar{G}_v &= (\zeta_x E_v + \zeta_y F_v + \zeta_z G_v) / J. \end{aligned} \quad (2.74)$$

Similar to two-dimensional Navier-Stokes equations, the flux Jacobian matrices (A, B, and C) can be calculated from the following equations:

$$A = \frac{\partial \bar{E}}{\partial \bar{Q}} = \begin{bmatrix} \xi_t & \xi_x & \xi_y & \xi_z & 0 \\ -u(u\xi_x + v\xi_y + w\xi_z) + \xi_x[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \xi_t + \xi_x(2-\gamma)u + (\xi_x u + \xi_y v + \xi_z w) & \xi_y u - (\gamma-1)\xi_x v & \xi_z u - (\gamma-1)\xi_x w & (\gamma-1)\xi_x \\ -v(u\xi_x + v\xi_y + w\xi_z) + \xi_y[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \xi_x v - (\gamma-1)\xi_y u + (\xi_x u + \xi_y v + \xi_z w) & \xi_t + \xi_y(2-\gamma)v & \xi_z v - (\gamma-1)\xi_y w & (\gamma-1)\xi_y \\ -w(u\xi_x + v\xi_y + w\xi_z) + \xi_z[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \xi_x w - (\gamma-1)\xi_z u + (\xi_x u + \xi_y v + \xi_z w) & \xi_y w - (\gamma-1)\xi_z v & \xi_t + \xi_z(2-\gamma)w + (\xi_x u + \xi_y v + \xi_z w) & (\gamma-1)\xi_z \\ (\xi_x u + \xi_y v + w\xi_z)[- \gamma e_t + (\gamma-1)(u^2 + v^2 + w^2)] & \xi_x[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\xi_x u + \xi_y v + \xi_z w)u & \xi_y[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\xi_x u + \xi_y v + \xi_z w)v & \xi_z[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\xi_x u + \xi_y v + \xi_z w)w & \xi_t + \gamma + (\xi_x u + \xi_y v + \xi_z w) \end{bmatrix} \quad (2.75)$$

$$B = \frac{\partial \bar{F}}{\partial \bar{Q}} = \begin{bmatrix} \eta_t & \eta_x & \eta_y & \eta_z & 0 \\ -u(u\eta_x + v\eta_y + w\eta_z) + \eta_x[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \eta_t + \eta_x(2-\gamma)u + (\eta_x u + \eta_y v + \eta_z w) & \eta_y u - (\gamma-1)\eta_x v & \eta_z u - (\gamma-1)\eta_x w & (\gamma-1)\eta_x \\ -v(u\eta_x + v\eta_y + w\eta_z) + \eta_y[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \eta_x v - (\gamma-1)\eta_y u + (\eta_x u + \eta_y v + \eta_z w) & \eta_t + \eta_y(2-\gamma)v & \eta_z v - (\gamma-1)\eta_y w & (\gamma-1)\eta_y \\ -w(u\eta_x + v\eta_y + w\eta_z) + \eta_z[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \eta_x w - (\gamma-1)\eta_z u + (\eta_x u + \eta_y v + \eta_z w) & \eta_y w - (\gamma-1)\eta_z v & \eta_t + \eta_z(2-\gamma)w + (\eta_x u + \eta_y v + \eta_z w) & (\gamma-1)\eta_z \\ (\eta_x u + \eta_y v + w\eta_z)[- \gamma e_t + (\gamma-1)(u^2 + v^2 + w^2)] & \eta_x[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\eta_x u + \eta_y v + \eta_z w)u & \eta_y[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\eta_x u + \eta_y v + \eta_z w)v & \eta_z[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\eta_x u + \eta_y v + \eta_z w)w & \eta_t + \gamma + (\eta_x u + \eta_y v + \eta_z w) \end{bmatrix} \quad (2.76)$$

$$C = \frac{\partial \bar{G}}{\partial Q} = \begin{bmatrix} \zeta_t & \zeta_x & \zeta_y & \zeta_z & 0 \\ -u(u\zeta_x + v\zeta_y + w\zeta_z) + \zeta_x[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \zeta_t + \zeta_x(2-\gamma)u + (\zeta_x u + \zeta_y v + \zeta_z w) & \zeta_y u - (\gamma-1)\zeta_x v & \zeta_z u - (\gamma-1)\zeta_x w & (\gamma-1)\zeta_x \\ -v(u\zeta_x + v\zeta_y + w\zeta_z) + \zeta_y[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \zeta_x v - (\gamma-1)\zeta_y u & \zeta_t + \zeta_y(2-\gamma)v + (\zeta_x u + \zeta_y v + \zeta_z w) & \zeta_z v - (\gamma-1)\zeta_y w & (\gamma-1)\zeta_y \\ -w(u\zeta_x + v\zeta_y + w\zeta_z) + \zeta_z[0.5(\gamma-1)(u^2 + v^2 + w^2)] & \zeta_x w - (\gamma-1)\zeta_z u & \zeta_y w - (\gamma-1)\zeta_z v & \zeta_t + \zeta_z(2-\gamma)w + (\zeta_x u + \zeta_y v + \zeta_z w) & (\gamma-1)\zeta_z \\ (\zeta_x u + \zeta_y v + w\zeta_z) \cdot [-\gamma e_t + (\gamma-1)(u^2 + v^2 + w^2)] & \zeta_x[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\zeta_x u + \zeta_y v \zeta_z w)u & \zeta_y[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\zeta_x u + \zeta_y v \zeta_z w)v & \zeta_z[\gamma e_t - 0.5(\gamma-1)(u^2 + v^2 + w^2)] - (\gamma-1)(\zeta_x u + \zeta_y v \zeta_z w)w & \zeta_t + \gamma + (\zeta_x u + \zeta_y v + \zeta_z w) \end{bmatrix} \quad (2.77)$$

The eigenvalues of matrix A can be determined as

$$\begin{aligned}
\lambda_{\xi}(1) &= \lambda_{\xi}(2) = \lambda_{\xi}(3) = \xi_t + U \\
\lambda_{\xi}(4) &= \xi_t + U + a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2} \\
\lambda_{\xi}(5) &= \xi_t + U - a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2} \\
\lambda_{\eta}(1) &= \lambda_{\eta}(2) = \lambda_{\eta}(3) = \eta_t + V \\
\lambda_{\eta}(4) &= \eta_t + V + a\sqrt{\eta_x^2 + \eta_y^2 + \eta_z^2} \\
\lambda_{\eta}(5) &= \eta_t + V - a\sqrt{\eta_x^2 + \eta_y^2 + \eta_z^2} \\
\lambda_{\zeta}(1) &= \lambda_{\zeta}(2) = \lambda_{\zeta}(3) = \zeta_t + W \\
\lambda_{\zeta}(4) &= \zeta_t + W + a\sqrt{\zeta_x^2 + \zeta_y^2 + \zeta_z^2} \\
\lambda_{\zeta}(5) &= \zeta_t + W - a\sqrt{\zeta_x^2 + \zeta_y^2 + \zeta_z^2}
\end{aligned} \tag{2.78}$$

where

$$\begin{aligned}
U &= \xi_t + \xi_x u + \xi_y v + \xi_z w \\
V &= \eta_t + \eta_x u + \eta_y v + \eta_z w \\
W &= \zeta_t + \zeta_x u + \zeta_y v + \zeta_z w
\end{aligned} \tag{2.79}$$

The eigenvector matrix and its inverse for matrix A are

$$X_A = \begin{bmatrix}
0 & 1 & 0 & 1 & 1 \\
-\frac{\xi_y}{\xi_x} & \frac{U}{\xi_x} & -\frac{\xi_z}{\xi_x} & u + \frac{\xi_x a}{\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} & u - \frac{\xi_x a}{\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} \\
1 & 0 & 0 & v + \frac{\xi_y a}{\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} & v - \frac{\xi_y a}{\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} \\
0 & 0 & 1 & w + \frac{\xi_z a}{\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} & w - \frac{\xi_z a}{\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} \\
\frac{-\xi_y u + \xi_x v}{\xi_x} & \frac{u(\xi_y v + \xi_z w)}{\xi_x} & \frac{\xi_x w - \xi_z u}{\xi_x} & \gamma e + \frac{0.5(u^2 + v^2 + w^2)}{+ \frac{aU}{\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}}} & \gamma e + \frac{0.5(u^2 + v^2 + w^2)}{- \frac{aU}{\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}}}
\end{bmatrix} \tag{2.80}$$

and

$$X_A^{-1} = \begin{bmatrix} \frac{\xi_y U}{\xi_x^2 + \xi_y^2 + \xi_z^2} - \frac{v}{2\gamma e} (u^2 + v^2 + w^2) & \frac{uv}{\gamma e} - \frac{\xi_x \xi_y}{\xi_x^2 + \xi_y^2 + \xi_z^2} & \frac{v^2}{\gamma e} + \frac{\xi_x^2 + \xi_y^2}{\xi_x^2 + \xi_y^2 + \xi_z^2} & \frac{vw}{\gamma e} - \frac{\xi_y \xi_z}{\xi_x^2 + \xi_y^2 + \xi_z^2} & \frac{-v}{\gamma e} \\ 1 - \frac{(u^2 + v^2 + w^2)}{2\gamma e} & \frac{u}{\gamma e} & \frac{v}{\gamma e} & \frac{w}{\gamma e} & \frac{-1}{\gamma e} \\ \frac{\xi_z U}{\xi_x^2 + \xi_y^2 + \xi_z^2} - \frac{w}{2\gamma e} (u^2 + v^2 + w^2) & \frac{uw}{\gamma e} - \frac{\xi_x \xi_z}{\xi_x^2 + \xi_y^2 + \xi_z^2} & \frac{vw}{\gamma e} - \frac{\xi_y \xi_z}{\xi_x^2 + \xi_y^2 + \xi_z^2} & \frac{w^2}{\gamma e} + \frac{\xi_x^2 + \xi_y^2}{\xi_x^2 + \xi_y^2 + \xi_z^2} & \frac{-w}{\gamma e} \\ \frac{-U}{2a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} + \frac{(u^2 + v^2 + w^2)}{4\gamma e} & \frac{\xi_x}{2a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} - \frac{u}{2\gamma e} & \frac{\xi_y}{2a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} - \frac{v}{2\gamma e} & \frac{\xi_z}{2a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} - \frac{w}{2\gamma e} & \frac{1}{2\gamma e} \\ \frac{U}{2a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} + \frac{(u^2 + v^2 + w^2)}{4\gamma e} & \frac{-\xi_x}{2a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} - \frac{u}{2\gamma e} & \frac{-\xi_y}{2a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} - \frac{v}{2\gamma e} & \frac{-\xi_z}{2a\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}} - \frac{w}{2\gamma e} & \frac{1}{2\gamma e} \end{bmatrix} \quad (2.81)$$

The eigenvector matrices and their inverse for matrices B and C have the same form and can be derived easily.

The viscous flux vectors can be calculated from the following equations:

$$\bar{E}_v = \frac{\mu}{\text{Re}_\infty J} \left[\begin{array}{c} 0 \\ \hline a_1 u_\xi + a_5 v_\xi + a_7 w_\xi + d_1 u_\eta + d_7 v_\eta + d_9 w_\eta + e_1 u_\zeta + e_7 v_\zeta + e_9 w_\zeta \\ \hline a_5 u_\xi + a_2 v_\xi + a_6 w_\xi + d_5 u_\eta + d_2 v_\eta + d_{10} w_\eta + e_5 u_\zeta + e_2 v_\zeta + e_{10} w_\zeta \\ \hline a_7 u_\xi + a_6 v_\xi + a_3 w_\xi + d_6 u_\eta + d_8 v_\eta + d_3 w_\eta + e_6 u_\zeta + e_8 v_\zeta + e_3 w_\zeta \\ \hline 0.5a_1(u^2)_\xi + 0.5a_2(v^2)_\xi + 0.5a_3(w^2)_\xi + a_5(uv)_\xi + a_6(vw)_\xi + a_7(uw)_\xi \\ \hline + \frac{1}{\text{Pr}(\gamma-1)\text{M}_\infty^2} a_4 T_\xi + 0.5d_1(u^2)_\eta + 0.5d_2(v^2)_\eta + 0.5d_3(w^2)_\eta + d_5 v u_\eta \\ \hline + d_6 w u_\eta + d_7 u v_\eta + d_8 w v_\eta + d_9 u w_\eta + d_{10} v w_\eta + \frac{1}{\text{Pr}(\gamma-1)\text{M}_\infty^2} d_4 T_\eta \\ \hline + 0.5e_1(u^2)_\zeta + 0.5e_2(v^2)_\zeta + 0.5e_3(w^2)_\zeta + e_5 v u_\zeta + e_6 w u_\zeta + e_7 u v_\zeta \\ \hline + e_8 w v_\zeta + e_9 u w_\zeta + e_{10} v w_\zeta + \frac{1}{\text{Pr}(\gamma-1)\text{M}_\infty^2} e_4 T_\zeta \end{array} \right] \quad (2.82)$$

where, for a perfect gas,

$$\frac{1}{\text{Pr}(\gamma-1)\text{M}_\infty^2} T = \frac{\gamma}{\text{Pr}} [e_t - 0.5(u^2 + v^2 + w^2)] \quad (2.83)$$

$$\bar{F}_v = \frac{\mu}{\text{Re}_\infty J} \left[\begin{array}{c} 0 \\ \hline d_1 u_\xi + d_5 v_\xi + d_6 w_\xi + b_1 u_\eta + b_5 v_\eta + b_7 w_\eta + f_1 u_\zeta + f_7 v_\zeta + f_9 w_\zeta \\ \hline d_7 u_\xi + d_2 v_\xi + d_8 w_\xi + b_5 u_\eta + b_2 v_\eta + b_6 w_\eta + f_5 u_\zeta + f_2 v_\zeta + f_{10} w_\zeta \\ \hline d_9 u_\xi + d_{10} v_\xi + d_3 w_\xi + b_7 u_\eta + b_6 v_\eta + b_3 w_\eta + f_6 u_\zeta + f_8 v_\zeta + f_3 w_\zeta \\ \hline 0.5d_1(u^2)_\xi + 0.5d_2(v^2)_\xi + 0.5d_3(w^2)_\xi + d_5 u v_\xi + d_6 u w_\xi + d_7 v u_\xi \\ \hline + d_8 v w_\xi + d_9 w u_\xi + d_{10} w v_\xi + \frac{1}{\text{Pr}(\gamma-1)\text{M}_\infty^2} d_4 T_\xi \\ \hline + 0.5b_1(u^2)_\eta + 0.5b_2(v^2)_\eta + 0.5b_3(w^2)_\eta + b_5(uv)_\eta + b_6(vw)_\eta + b_7(uw)_\eta \\ \hline + \frac{1}{\text{Pr}(\gamma-1)\text{M}_\infty^2} b_4 T_\eta + 0.5f_1(u^2)_\zeta + 0.5f_2(v^2)_\zeta + 0.5f_3(w^2)_\zeta \\ \hline + f_5 v u_\zeta + f_6 w u_\zeta + f_7 u v_\zeta + f_8 w v_\zeta + f_9 u w_\zeta + f_{10} v w_\zeta \\ \hline + \frac{1}{\text{Pr}(\gamma-1)\text{M}_\infty^2} f_4 T_\zeta \end{array} \right] \quad (2.84)$$

$$\bar{G}_v = \frac{\mu}{\text{Re}_\infty J} \left[\begin{array}{c} 0 \\ \hline e_1 u_\xi + e_5 v_\xi + e_6 w_\xi + f_1 u_\eta + f_5 v_\eta + f_6 w_\eta + c_1 u_\zeta + c_5 v_\zeta + c_7 w_\zeta \\ \hline e_7 u_\xi + e_2 v_\xi + e_8 w_\xi + f_7 u_\eta + f_2 v_\eta + f_8 w_\eta + c_5 u_\zeta + c_2 v_\zeta + c_6 w_\zeta \\ \hline e_9 u_\xi + e_{10} v_\xi + e_3 w_\xi + f_9 u_\eta + f_{10} v_\eta + f_3 w_\eta + c_7 u_\zeta + c_6 v_\zeta + c_3 w_\zeta \\ \hline 0.5e_1(u^2)_\xi + 0.5e_2(v^2)_\xi + 0.5e_3(w^2)_\xi + e_5 uv_\xi + e_6 uw_\xi + e_7 vu_\xi \\ + e_8 vw_\xi + e_9 wu_\xi + e_{10} wv_\xi + \frac{1}{\text{Pr}(\gamma-1)M_\infty^2} e_4 T_\xi + 0.5f_1(u^2)_\eta \\ + 0.5f_2(v^2)_\eta + 0.5f_3(w^2)_\eta + f_3 uv_\eta + f_6 uw_\eta + f_7 vu_\eta + f_8 vw_\eta \\ + f_9 wu_\eta + f_{10} wv_\eta + \frac{1}{\text{Pr}(\gamma-1)M_\infty^2} f_4 T_\eta + 0.5c_1(u^2)_\zeta + 0.5c_2(v^2)_\zeta \\ + 0.5c_3(w^2)_\zeta + c_5(uv)_\zeta + c_6(vw)_\zeta + c_7(uw)_\zeta + \frac{1}{\text{Pr}(\gamma-1)M_\infty^2} c_4 T_\zeta \end{array} \right] \quad (2.85)$$

where

$$\begin{aligned} a_1 &= \frac{4}{3} \xi_x^2 + \xi_y^2 + \xi_z^2 & a_2 &= \xi_x^2 + \frac{4}{3} \xi_y^2 + \xi_z^2 \\ a_3 &= \xi_x^2 + \xi_y^2 + \frac{4}{3} \xi_z^2 & a_4 &= \xi_x^2 + \xi_y^2 + \xi_z^2 \\ a_5 &= \frac{1}{3} \xi_x \xi_y & a_6 &= \frac{1}{3} \xi_y \xi_z \\ a_7 &= \frac{1}{3} \xi_x \xi_z \end{aligned} \quad (2.86)$$

$$\begin{aligned} b_1 &= \frac{4}{3} \eta_x^2 + \eta_y^2 + \eta_z^2 & b_2 &= \eta_x^2 + \frac{4}{3} \eta_y^2 + \eta_z^2 \\ b_3 &= \eta_x^2 + \eta_y^2 + \frac{4}{3} \eta_z^2 & b_4 &= \eta_x^2 + \eta_y^2 + \eta_z^2 \\ b_5 &= \frac{1}{3} \eta_x \eta_y & b_6 &= \frac{1}{3} \eta_y \eta_z \\ b_7 &= \frac{1}{3} \eta_x \eta_z \end{aligned} \quad (2.87)$$

$$\begin{aligned}
c_1 &= \frac{4}{3}\zeta_x^2 + \zeta_y^2 + \zeta_z^2 \\
c_3 &= \zeta_x^2 + \zeta_y^2 + \frac{4}{3}\zeta_z^2 \\
c_5 &= \frac{1}{3}\zeta_x\zeta_y \\
c_7 &= \frac{1}{3}\zeta_x\zeta_z \\
c_2 &= \zeta_x^2 + \frac{4}{3}\zeta_y^2 + \zeta_z^2 \\
c_4 &= \zeta_x^2 + \zeta_y^2 + \zeta_z^2 \\
c_6 &= \frac{1}{3}\zeta_y\zeta_z
\end{aligned} \tag{2.88}$$

$$\begin{aligned}
d_1 &= \frac{4}{3}\xi_x\eta_x + \xi_y\eta_y + \xi_z\eta_z \\
d_3 &= \xi_x\eta_x + \xi_y\eta_y + \frac{4}{3}\xi_z\eta_z \\
d_5 &= \xi_x\eta_y - \frac{2}{3}\xi_y\eta_x \\
d_7 &= \xi_y\eta_x - \frac{2}{3}\xi_x\eta_y \\
d_9 &= \xi_z\eta_x - \frac{2}{3}\xi_x\eta_z \\
d_2 &= \xi_x\eta_x + \frac{4}{3}\xi_y\eta_y + \xi_z\eta_z \\
d_4 &= \xi_x\eta_x + \xi_y\eta_y + \xi_z\eta_z \\
d_6 &= \xi_x\eta_z - \frac{2}{3}\xi_z\eta_x \\
d_8 &= \xi_y\eta_z - \frac{2}{3}\xi_z\eta_y \\
d_{10} &= \xi_z\eta_y - \frac{2}{3}\xi_y\eta_z
\end{aligned} \tag{2.89}$$

$$\begin{aligned}
e_1 &= \frac{4}{3}\xi_x\zeta_x + \xi_y\zeta_y + \xi_z\zeta_z \\
e_3 &= \xi_x\zeta_x + \xi_y\zeta_y + \frac{4}{3}\xi_z\zeta_z \\
e_5 &= \xi_x\zeta_y - \frac{2}{3}\xi_y\zeta_x \\
e_7 &= \xi_y\zeta_x - \frac{2}{3}\xi_x\zeta_y \\
e_9 &= \xi_z\zeta_x - \frac{2}{3}\xi_x\zeta_z \\
e_2 &= \xi_x\zeta_x + \frac{4}{3}\xi_y\zeta_y + \xi_z\zeta_z \\
e_4 &= \xi_x\zeta_x + \xi_y\zeta_y + \xi_z\zeta_z \\
e_6 &= \xi_x\zeta_z - \frac{2}{3}\xi_z\zeta_x \\
e_8 &= \xi_y\zeta_z - \frac{2}{3}\xi_z\zeta_y \\
e_{10} &= \xi_z\zeta_y - \frac{2}{3}\xi_y\zeta_z
\end{aligned} \tag{2.90}$$

$$\begin{aligned}
f_1 &= \frac{4}{3}\eta_x\zeta_x + \eta_y\zeta_y + \eta_z\zeta_z \\
f_3 &= \eta_x\zeta_x + \eta_y\zeta_y + \frac{4}{3}\eta_z\zeta_z \\
f_5 &= \eta_x\zeta_y - \frac{2}{3}\eta_y\zeta_x \\
f_7 &= \eta_y\zeta_x - \frac{2}{3}\eta_x\zeta_y \\
f_9 &= \eta_z\zeta_x - \frac{2}{3}\eta_x\zeta_z \\
f_2 &= \eta_x\zeta_x + \frac{4}{3}\eta_y\zeta_y + \eta_z\zeta_z \\
f_4 &= \eta_x\zeta_x + \eta_y\zeta_y + \eta_z\zeta_z \\
f_6 &= \eta_x\zeta_z - \frac{2}{3}\eta_z\zeta_x \\
f_8 &= \eta_y\zeta_z - \frac{2}{3}\eta_z\zeta_y \\
f_{10} &= \eta_z\zeta_y - \frac{2}{3}\eta_y\zeta_z.
\end{aligned} \tag{2.91}$$

2.4 Similarities and Differences of the Proposed Scheme in One-, Two-, and Three-Dimensional Flowfields

In order to examine the accuracy and resolution of the proposed scheme, several benchmark problems were considered in one-dimensional and two-dimensional flowfields, which are presented in chapter 3. In all one-dimensional cases, the proposed scheme performs very well and is superior to the other traditional schemes. However, the proposed scheme requires some additional modifications for two-dimensional cases. The fact is that the bandwidth optimization, which was explained in section 2.2.7, fails for two-dimensional cases. It cannot reduce the oscillations near the shock regions. Even increasing the grid size or decreasing the time increment is not helpful. In order to overcome this problem, in all two-dimensional cases the bandwidth optimization was replaced by flux splitting. There are many options for flux splitting. In the proposed scheme, Lax-Friedrichs (LF) flux splitting method is used. This method was proposed in 1954 [86]. It can be performed globally or locally. In all the two- and three-dimensional cases, Local Lax-Friedrichs (LLF) is employed. The simplest form of LF splitting is

$$f^{\pm}(u) = \frac{1}{2}(f(u) \pm \alpha u). \quad (2.92)$$

In equation (2.92) α , which is the local numerical viscosity, is defined as $\alpha = \max_u |f'(u)|$.

The proposed scheme in three-dimensional flowfield is an extension of two-dimensional scheme. It should be noted that all the equations are first nondimensionalized and subsequently solved. The nondimensionalized variables are

$$\begin{aligned}
(x^*, y^*, z^*) &= \frac{(x, y, z)}{L_{ref}}, \quad t^* = \frac{U_{ref} t}{L_{ref}}, \quad \rho^* = \frac{\rho}{\rho_{ref}} \\
(u^*, v^*, w^*) &= \frac{(u, v, w)}{U_{ref}}, \quad p^* = \frac{p}{\rho_{ref} U_{ref}^2}, \quad T^* = \frac{T}{T_{ref}} \\
e_t^* &= \frac{e_t}{U_{ref}^2}.
\end{aligned} \tag{2.93}$$

The resulting nondimensional reference parameters are as follows

$$\text{Re}_{ref} = \frac{\rho_{ref} U_{ref} L_{ref}}{\mu_{ref}}, \quad \text{Pr}_{ref} = \frac{c_{p_{ref}} \mu_{ref}}{k_{ref}} \tag{2.94}$$

The subscript *ref* denotes reference variables and the asterisk shows dimensionless variables.

CHAPTER 3

VALIDATION OF THE PROPOSED NUMERICAL SCHEME

3.1 Chapter Overview

In this chapter, numerical results of the traditional schemes and the proposed scheme (MWWS) are presented and compared with each other. The results are presented in two different sections. In section one, one-dimensional cases are discussed and section two presents the two-dimensional benchmark problems in two categories of Euler and Navier-Stokes equations.

3.2 One-Dimensional Benchmark Problems

The one-dimensional benchmark problems are presented in two categories. First category is solving the scalar conservation law in one dimensional space. The second category includes the system of Euler equations in one dimension. For all of the following examples, the time integration is carried out employing RK4 method.

3.2.1 Linear Wave Equation with Jump Initial Function

In this problem, the one-dimensional convection equation which is shown by

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad (3.1)$$

is solved with the following initial condition

$$u(x,0) = \begin{cases} 1.0 & 0.1 \leq x \leq 0.4 \\ 0.5 & \text{otherwise.} \end{cases} \quad (3.2)$$

The calculation proceeds until $t = 0.3$. As discussed previously, the value of κ in equation (2.37) can be chosen in the range of $0.03 \leq \kappa \leq 2$ which is set to be 0.05 in this benchmark problem. The solutions are illustrated in Figure 3-1. The results indicate that both WCS and WENO schemes work well but first order upwind is not suitable for shocks. This

problem clearly shows the reason for selecting WENO and WCS as the foundation of the proposed scheme.

Figure 3-1. Linear wave equation with jump initial function (IM = 81, CFL = 0.8)

In this case, the WCS scheme shows better result than WENO. However, the solutions by both schemes are in good agreement with the exact solution. Among all the schemes, MWWS result shows the best agreement with the analytical solution.

3.2.2 Linear First-Order Wave Equation with Different Initial Conditions

First-order wave equation is given by

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} \quad (3.3)$$

while a is set to 40π for the first two cases and 1.0 for the remaining. Four different initial conditions are used to compare the results of the different schemes. The domain of the solution for the first two cases is set to $0 \leq x \leq 10\pi$.

3.2.2.1 Case One

This test case is adapted from Ref. [1]. The initial condition is specified as

$$u(x,0) = \begin{cases} 0 & 0 \leq x \leq 2\pi \\ 0.5\{\sin[(3x-7\pi)/2]+1\} & 2\pi \leq x \leq 8\pi/3 \\ \sin[3(3\pi-x)/2] & 8\pi/3 \leq x \leq 10\pi/3 \\ 0.5\{\sin[(3x-11\pi)/2]-1\} & 10\pi/3 \leq x \leq 4\pi \\ 0 & 4\pi \leq x \leq 10\pi. \end{cases} \quad (3.4)$$

The spatial grid is calculated from $\Delta x = 10\pi / (\text{IM}-1)$ and IM is set to 51. The solution proceeds until $t = 0.1$. The value of κ in equation (2.37) is set to be 0.06 in this case. Results obtained from different schemes are shown in Figure 3-2. It is observed that the solutions by central differencing and standard compact schemes produce inaccuracies compared to the analytical solution. Central difference approximation of the convection term does not provide good results and large oscillations are observed in the vicinity of large gradients. Standard compact scheme results are also containing some oscillations. The WENO scheme is more dissipative than the WCS scheme. However, no oscillation is observed in any of these schemes. MWWS scheme shows a fairly good result in comparison with the analytical solution and other schemes. It can be observed that the MWWS scheme preserves the general shape of the wave. Figure 3-3 illustrates the solutions of WCS and WENO schemes at different time levels. As it is shown, WCS is less dissipative than WENO scheme. In both schemes decrease in amplitude and dissipation of the wave to the neighboring points are observed.

Figure 3-2. Comparison of the solutions for case one by different schemes (IM = 51, CFL = 0.8)

Figure 3-3. Solutions of case one at different time levels (CFL = 0.7)

3.2.2.2 Case Two

In this case initial condition is defined as

$$u(x,0) = \begin{cases} 1 + \cos(x) & 2\pi \leq x \leq 4\pi \\ 0 & \text{otherwise.} \end{cases} \quad (3.5)$$

This is a challenging test for evaluating the behavior of the different schemes. The solution proceeds up to $t=0.1$. The value of κ in equation (2.37) is set to be 0.08 in this problem. As it is shown in Figure 3-4, the first order upwind scheme does not perform well for this case. The WCS and WENO schemes show similar results, which do not compare well with the analytical solution. The proposed scheme shows better results in comparison with the other schemes. It is observed that the MWWS scheme develops the accurate solution in smooth regions while capturing the discontinuities in a non-oscillatory manner.

Figure 3-4. Comparison of the results for case two by different schemes (IM = 101, CFL = 0.7)

3.2.2.3 Case Three

This benchmark problem is adapted from Ref. [16]. In this case the initial condition is set to be

$$u(x,0) = \begin{cases} \left[1 - \left(\frac{3}{10}x \right)^2 \right]^{1/2} & |x| < \frac{3}{10} \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

The domain of the solution is set to be $-1 \leq x \leq 1$ with 101 grid points. The solution proceeds until $t = 2$. The value of κ in equation (2.37) is set to be 0.03 in this case. Results of different schemes are shown in Figure 3-5. Standard compact scheme shows large oscillations in all regions. WCS and WENO schemes show better results but some wiggles can be observed

near the large gradients. MWWS result is in good agreement with the exact solution. It follows the shape of the wave without any oscillations. It should be noted that in this case the WCS was very much sensitive to the power p . In this case, p was set to 1.

Figure 3-5. Comparison of the results of the different schemes for case three (IM = 101, CFL = 0.8)

3.2.2.4 Sinuous Wave

In this benchmark problem the linear advection equation is solved with the following initial condition

$$u(x,0) = \sin(20\pi x) \quad (3.7)$$

which is a high-frequency sinuous function. The domain is set to $0 \leq x \leq 1$ and IM is 121. The solution proceeds up to $t=1.0$. The value of κ in equation (2.37) is set to be 0.03 in this

problem. Figure 3-6 illustrates the results by different schemes. MWWS scheme has the best resolution among the schemes. WCS and WENO schemes have more dissipative error than the proposed scheme.

Figure 3-6. High-frequency sine wave (IM = 121, CFL = 0.8)

3.2.3 The Burgers' Equation

In this case equation (2.5), which is the one-dimensional advection equation, is solved with the following nonlinear convex flux

$$f(u) = 1/2u^2 \quad (3.8)$$

and initial condition as follows

$$u(x,0) = 1/2 + \sin(\pi x). \quad (3.9)$$

The solution proceeds up to $t=1/\pi$ and 81 grid points are used to evaluate the performance of each scheme. The domain is set to $-1 \leq x \leq 1$. The value of κ in equation (2.37) is set to be 0.04 in this case. Figure 3-7 illustrates the results by different schemes. WCS and WENO show the same results, while MWWS shows slightly improved result. It is interesting to compare the numerical error distributions for different schemes. The numerical error is calculated by

$$\text{ERR} = \frac{\text{Result}_{\text{Numerical}} - \text{Result}_{\text{Exact/Reference}}}{\text{Result}_{\text{Exact/Reference}}}. \quad (3.10)$$

The error distributions of different schemes are illustrated in Figure 3-8. It can be observed that MWWS scheme provides better results with less amount of error than WENO and WCS schemes.

Figure 3-7. The Burgers' equation with nonlinear convex flux, numerical results by different schemes (IM = 81, CFL = 0.8)

Figure 3-8. The Burgers' equation solution, comparison of numerical error distribution by different schemes

3.2.4 Linear Advection Equation

In this problem, the one-dimensional linear advection equation is solved with following initial condition:

$$u(x+0.5,0) = \begin{cases} -x \sin(3/2x^2) & -1 \leq x < -1/3 \\ |\sin(2\pi x)| & |x| \leq 1/3 \\ 2x-1 - \sin(3\pi x)/6 & 1/3 < x \leq 1. \end{cases} \quad (3.11)$$

The value of κ in equation (2.37) is set to be 0.03 in this problem. Solution at $t=8.0$ and in domain of $-1 \leq x \leq 1$ with 101 grid points is illustrated in Figure 3-9. As it is shown, the exact solution consists of high discontinuities in the derivatives. The WCS and WENO schemes show poor results in comparison with the exact solution. The MWWS scheme shows improved results with less oscillations and higher amplitudes in all regions.

Figure 3-9. Linear advection equation ($IM = 101$, $CFL = 0.2$)

The numerical error distributions are illustrated in Figure 3-10. It is observed that the proposed scheme is superior to other schemes.

Figure 3-10. Linear advection equation, comparison of numerical error distributions for different schemes

3.2.5 Buckley Leverett Problem

Buckley Leverett problem is solved with a nonlinear non-convex flux of

$$f(u) = 4u^2 / [4u^2 + (1-u)^2] \quad (3.12)$$

and initial condition as follows

$$u(x,0) = \begin{cases} 1 & -0.5 \leq x < 0 \\ 0 & \text{otherwise.} \end{cases} \quad (3.13)$$

In this problem 81 grid points are used in a domain of $-1 \leq x \leq 1$. The solution proceeds up to $t = 0.4$. The value of κ in equation (2.37) is set to be 0.9 in this benchmark problem. The results of different schemes are illustrated in Figure 3-11. The WCS scheme shows a poor solution containing oscillations near the large gradients. WENO scheme provides a better result, which is in good agreement with the exact solution. In this problem, the results of the MWWS scheme

with and without filter are compared. It is observed that the filter plays an important role in resolution of the scheme and shock-capturing capability. The MWWS with filter shows a good result in comparison with the exact solution.

Figure 3-11. The Buckley Leverett problem with non-convex flux (IM = 81, CFL = 0.8)

3.2.6 The Shock Tube Problem

In this section shock capturing capability of the new scheme is examined with shock tube problem. The flow is assumed to be one dimensional and inviscid, with the governing equations being the Euler equations (equation (2.47)). Initial condition for this case is set as

$$(\rho, u, p)_0 = \begin{cases} (1, 0, 1) & x \leq 5 \\ (0.125, 0, 0.1) & x > 5 \end{cases} \quad (3.14)$$

At first stage a diaphragm in the middle of the tube separates the gas with different initial conditions. At $t = 0$, the diaphragm is broken and a normal shock wave starts to propagate from

high pressure region into the region with lower pressure. From low pressure region, some expansion waves propagate into the high pressure region.

To solve the Euler equations for this case, Van Leer flux vector splitting is used to calculate the derivatives of the splitting flux. The solution proceeds until $t = 2.0$. The value of κ in equation (2.37) is set to be 0.03 in this case. Figure 3-12 shows the velocity distributions by analytical solution and several other schemes. Details about analytical solution can be found in Ref. [85]. The new scheme performed superior compared to other schemes.

Figure 3-12. Velocity distribution for shock tube problem (IM = 101, CFL = 0.6)

3.2.7 Shock-Entropy Interaction

In this test case the initial condition is set by a moving shock with Mach number of 3 interacting with a perturbed density field as follows

$$(\rho, u, p) = \begin{cases} (3.857143, 2.269369, 10.333333) & x < -4 \\ (1 + 0.2\sin(5x), 0, 1) & x \geq -4 \end{cases} \quad (3.15)$$

and zero order extrapolation boundary condition, which is applied at $x = \pm 5$. The solution proceeded up to $t = 1.8$. Results obtained by MWWS, WENO, and WCS schemes on 7,001 grid points are shown in Figure 3-13. As shown in this figure, all schemes with fine grid system provide the same result, which is employed as the reference solution. The initial and final reference solutions are illustrated in this figure. As can be seen, the location of the moving shock at $t = 1.8$ is $x \approx 2.4$.

In this case, 301 grid points are employed for different schemes and the results are compared to the reference solution. Figure 3-14 presents the solutions by WCS with different TVD schemes and the MWWS scheme. It is clearly shown that Harten-Yee TVD works better than Roe-Sweby and Davis-Yee TVD methods and MWWS scheme. However, none of the TVD schemes is able to capture the shocks sharply. The result of the proposed scheme is in very good agreement with the reference solution. The MWWS scheme can capture the shocks accurately and shows a good resolution.

a) Density distribution

b) Pressure distribution

Figure 3-13. Shock-entropy interaction, initial and final reference solution

Figure 3-14. Density distribution of shock-entropy problem by different schemes ($IM = 301$, $CFL = 0.6$)

Figure 3-15 illustrates the results by WENO, WCS, and MWWS schemes. It is clearly seen that the proposed scheme is superior to both WCS and WENO schemes. Figure 3-15(b) shows a closer view of the results near the shock. WCS result shows better resolution and accuracy in comparison with the WENO scheme. However, the result obtained by the WCS scheme is in poor agreement with the reference solution.

a) Overall view of the results

b) Closer view of the results near the shock

Figure 3-15. Results obtained by different schemes ($IM = 301$, $CFL = 0.6$)

Figure 3-16 also illustrates the pressure distribution obtained by the MWWS scheme. It shows good result in comparison with the reference solution.

Figure 3-16. Pressure distribution of shock-entropy problem (IM = 301, CFL = 0.6)

As discussed previously, the value of κ in equation (2.37) can be chosen in the range of $0.03 \leq \kappa \leq 2$ which is set to be 0.08 in this case. Figure 3-17 shows the effect of changing the value of κ . As can be seen in this figure, by increasing the value, the shock-capturing capability of the scheme is reduced. It is also observed that lower end value of κ overestimates the values of density.

Figure 3-18 illustrates the effect of changing β in equation (2.46). Low values of β result in overestimated results while high values fail to provide accurate results. By employing $\beta = 0.95$, the scheme fails and it is necessary to change the value of κ . However, changing κ value affects the shock-capturing capability of the scheme and provided result is not in good

agreement with the reference solution. It was mentioned that the best value of β is 0.65, which is valid for all of the benchmark problems.

Figure 3-17. MWWS results with different κ values

Figure 3-18. Results obtained by MWWS scheme with different β values

3.2.8 Riemann Problem of Lax

The last benchmark problem in one-dimensional cases is Riemann problem of Lax with the following initial condition

$$(\rho, u, p) = \begin{cases} (0.445, 0.698, 3.528) & x \leq 0 \\ (0.5, 0, 0.571) & x > 0. \end{cases} \quad (3.16)$$

At the boundaries, zero gradient boundary conditions were applied. The solution was stopped at $t = 0.13$. The value of κ in equation (2.37) is set to be 1.0 in this case. The results are shown in Figure 3-19 by three different schemes. In this figure, the WCS and modified WENO schemes are compared to the new scheme. The WCS scheme shows some oscillations near the discontinuities while, the MWWS scheme shows better result with improved resolution.

Figure 3-19. Solution of Riemann problem of Lax with different schemes (IM = 101, CFL = 0.6)

3.3 Two-Dimensional Benchmark Problems

3.3.1 Moving Vortex

The first problem considered in this section is the moving vortex, which was used by several investigators as benchmark problem [11, 18, 87, 88]. The problem is investigated over a large time interval to investigate the dispersion and dissipation properties. The domain is taken as $[0,10] \times [0,10]$ with 81 grid points in each direction. In order to simulate the moving vortex over a long time, all the boundaries are specified as periodic boundaries. Shu [18] showed that higher-order methods are superior to other schemes for long time simulations. The governing equations are two-dimensional Euler equations, which were presented in section 2.3.2. At $t = 0$,

an isentropic vortex located at $(x_v, y_v) = (5, 5)$ is added to the mean flow with the following initial condition

$$\bar{\rho} = 1, \quad \bar{u} = 1, \quad \bar{v} = 0, \quad \bar{p} = 1. \quad (3.17)$$

The vortex perturbation to the mean flow is introduced as

$$\begin{aligned} (\delta u, \delta v) &= \frac{\varepsilon}{2\pi} e^{0.5(1-r^2)} (y_v - y, x - x_v) \\ \delta T &= -\frac{(\gamma-1)\varepsilon^2}{8\gamma\pi^2} e^{(1-r^2)}; \quad T = \frac{p}{\rho} \\ \delta S &= 0; \quad S = \ln \frac{p}{\rho^\gamma} \quad (\text{No perturbation in entropy}) \end{aligned} \quad (3.18)$$

where $r = \sqrt{(x-x_v)^2 + (y-y_v)^2}$, and the vortex strength (ε) is set to 5. The entire flowfield is set to be isentropic, and $\gamma = 1.4$. From the isentropic relations and ideal gas law, the resulting state for the variables is

$$\begin{aligned} \rho &= (\bar{T} + \delta T)^{1/(\gamma-1)} \\ u &= \bar{u} + \delta u \\ v &= \bar{v} + \delta v \\ p &= (\bar{T} + \delta T)^{\gamma/(\gamma-1)}. \end{aligned} \quad (3.19)$$

The CFL is set to 0.6 and uniform grid system is used for this case. The value of κ in equation (2.37) is set to be 0.2 in this case. The results are presented in Figure 3-20.

a) $t = 50$

b) $t = 100$

c) $t = 500$

d) $t = 1000$

Figure 3-20. Two-dimensional moving vortex, twenty density contours from 0.5 to 0.95 (81×81 , $CFL = 0.6$)

It is clearly shown that the solution remains accurate even at $t = 1000$. The smearing of the vortex happens between $t = 500$ and $t = 1000$. Available results of this test case carried out by other researchers show that smearing of the vortex happens long time before $t = 500$. In most of the cases it happens at or before $t = 100$. In comparison with the other researchers' results, the proposed scheme results show better resolution.

a) Density distributions

b) Numerical error distribution

Figure 3-21. Two-dimensional moving vortex, density and numerical distributions at different time levels

3.3.2 Shock-Vortex Interaction

This problem is one of the best benchmark tests to evaluate the accuracy and resolution of the high-order schemes. This case is adapted from Ref. [18] and used by several other researchers [26, 27, 84, 89, 90] to assess the accuracy of their schemes.

The domain is taken to be $[0, 2] \times [0, 1]$. The stationary Mach 1.1 shock is located at $x = 0.5$ and normal to the x axis. The left state of the shock is set to be

$$\rho_1 = 1, \quad u_1 = \sqrt{\gamma} M_1, \quad v_1 = 0, \quad p_1 = 1. \quad (3.20)$$

A small vortex centered at $(x_c, y_c) = (0.25, 0.5)$ is located ahead of the shock. The vortex perturbation to the uniform flow is introduced by

$$\begin{aligned} \delta u &= \varepsilon \tau e^{\alpha(1-\tau^2)} \sin \theta \\ \delta v &= -\varepsilon \tau e^{\alpha(1-\tau^2)} \cos \theta \\ \delta T &= -\frac{(\gamma-1)\varepsilon^2 e^{2\alpha(1-\tau^2)}}{4\alpha\gamma}; \quad T = \frac{p}{\rho} \\ \delta S &= 0; \quad S = \ln \frac{p}{\rho^\gamma} \text{ (No perturbation in entropy)} \end{aligned} \quad (3.21)$$

where $\tau = r / r_c$, $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$, and r_c is the critical radius, which is set to 0.05. The parameter ε shows the strength of the vortex, and its value is set to be 0.3. The factor α is the controlling factor for the vortex decay with value of 0.204. The governing equations are two-dimensional Euler equations. A 251×101 grid system that is uniform in both directions is used for this test case. Solid wall boundary conditions are implemented at the top and bottom boundaries, and zero-order extrapolation boundary conditions are used for right and left boundaries. The value of κ in equation (2.37) is set to be 0.05 in this problem. The results are shown in Figure 3-22. At $t = 0$, the vortex is located on the left side of the shock wave. After that, it starts moving toward the shock wave and passing through it. Subsequently, the shock

wave begins to deform and bifurcation is being created, while series of compression and rarefaction waves form behind the shock. The scheme maintains the circular shape of the vortex when the vortex passes through the shock wave. Several sound waves are generated and propagated in the radial direction. Several researchers [91-94] investigated the mechanism of the sound generation, experimentally and numerically.

In most of the previous studies of this test case, the grid system was refined near the shock wave along the x-direction. However, some small oscillations were observed near the shock wave. In Figure 3-22, although the grid system is uniform, the interaction is nicely captured without any oscillation, and some more details can be observed in flow structure.

The result obtained by the proposed scheme is compared with reference solution adapted from the work of Shen and Zha [95]. The reference solution is calculated by the fifth-order WENO scheme with a refined mesh of 1001×401 . Pressure values at the central line ($y = 0.5$) downstream of the shock ($0.49 < x < 0.84$), are shown in Figure 3-23. As can be observed, the values obtained by the proposed scheme are in good agreement with the reference solution. Another comparison is performed using the results of Jiang and Shu [10] for this case. Figure 3-24 illustrates ninety pressure contours at $t = 0.6$. Jiang and Shu used 251×101 grid system with uniform grid spacing in Y and refined grid spacing in X direction around the shock. As can be seen, result obtained by MWWS scheme with uniform grid system is in good agreement with Jiang and Shu's result, which is obtained by fifth-order WENO scheme with the global Lax-Friedrichs flux splitting.

Figure 3-22. Two-dimensional shock-vortex interaction, thirty pressure contours from 0.8 to 1.3 (251×101, CFL = 0.6)

Figure 3-23. Two-dimensional shock-vortex interaction, comparison of pressure at the central lines downstream the shock ($t = 0.6$)

Figure 3-24. Two-dimensional shock-vortex interaction, Ninety pressure contours from 1.19 to 1.37 ($t = 0.6$)

3.3.3 Double Mach Reflection

This benchmark problem was initially investigated by Woodward and Colella [96] and employed as a benchmark problem by several researchers [10, 18, 97-102]. Double Mach reflection shock structure is generated when the incident shock Mach number is greater than about 2.5 [102]. In particular, the type of reflection pattern is a function of the wedge angle, the incident shock-wave Mach number, and the gas ratio of specific heats. More details about different types of double Mach reflection pattern can be found in Refs. [103, 104]. This classical test example examines the robustness of the numerical algorithms.

In order to simulate the double Mach reflection phenomenon, a shock wave is diagonally sent into a reflecting wall. It can be supposed that a shock is moving horizontally and encounters a wedge, which is inclined by some angle. In the current benchmark problem, as can be seen in Figure 3-25, a right-moving Mach 10 shock located at $(\frac{1}{6}, 0)$ makes a 60 degree angle with a reflecting wall. A detailed view of the flowfield in double Mach reflection region is shown in

Figure 3-26. The incident shock (I), the reflected shock (R'), and the main Mach stem (M₁) meet at the main triple point (TP) and generate a slip line (SL).

Figure 3-25. Schematic illustration of two-dimensional double Mach reflection configuration

Figure 3-26. Two-dimensional double Mach reflection, general view

The computational domain for this case is $[0, 4] \times [0, 1]$. For the bottom boundary, from $x = 0$ to $x = \frac{1}{6}$, the post-shock condition is imposed, and for the rest, reflecting wall boundary condition is used. At the top boundary, exact motion of the Mach 10 shock is simulated. For the left and right boundaries, inflow and outflow boundary conditions are imposed. The computation is carried out until $t = 0.2$. Initial condition for the gas ahead of the shock is

$$\begin{aligned} p_1 &= 1 \\ \rho_1 &= 1.4 \end{aligned} \quad (3.22)$$

The initial post-shock condition is calculated by Rankine-Hugoniot relations

$$\begin{aligned} v_s &= M_s \sqrt{\gamma p_1 / \rho_1} = 10; \quad \gamma = 1.4 \\ p_2 / p_1 &= (2\gamma M_s^2 - (\gamma - 1)) / (\gamma + 1) \\ \rho_2 / \rho_1 &= (\gamma + 1) M_s^2 / ((\gamma - 1) M_s^2 + 2) \\ \rho_1 v_s &= \rho_2 (v_s - v_2). \end{aligned} \quad (3.23)$$

The first study of this case is carried out with a 241×61 grid system corresponding to $\Delta x = \Delta y = 1/60$. The governing equations are two-dimensional Euler equations, which were discussed in section 2.3.2. CFL value is set to be 0.6 and the value of κ in equation (2.37) is set to be 0.55 in this problem. The result is shown in Figure 3-27. It can be clearly seen that as the shock reflects from the lower wall, a jet forms and grows, which is consists of a denser gas. The result agrees well with the results of Woodward and Colella [96].

Figure 3-27. Two-dimensional double Mach reflection, Euler solution, thirty density contours from 1.5 to 20.5 (241×61 , CFL = 0.6)

The solution is repeated for $\Delta x = \Delta y = 1/120$ and $\Delta x = \Delta y = 1/240$, which are illustrated in Figure 3-28 and Figure 3-29, respectively. The blow-up region is illustrated in Figure 3-30 for different grid sizes.

Figure 3-28. Two-dimensional double Mach reflection, Euler solution, thirty density contours from 1.5 to 20.5 (481×121 , CFL = 0.6)

Figure 3-29. Two-dimensional double Mach reflection, Euler solution, thirty density contours from 1.5 to 20.5 (961×241 , CFL = 0.6)

Figure 3-30. Two-dimensional double Mach reflection, Euler solution, blow-up region

Based on the study performed by Shi et al. [99] some of the details of the complicated structures that arise in Euler calculations may be non-physical and only Navier-Stokes simulation can provide the exact solution. Subsequently, the solution is repeated by employing Navier-Stokes equations, which is shown in Figure 3-31. The overall solutions of Euler and Navier-Stokes are similar. However, more details can be seen in Navier-Stokes solution. In the majority of previously performed results, the Mach stem appears to be straight or almost straight. However, several cases can be found where the Mach stem shape is concave or convex. Concave shapes are due to weak incident shock waves, while convex shapes are the result of strong incident shock waves [105]. As can be seen in Figure 3-31, the curled slipstream catches up with the Mach stem, and subsequently, the Mach stem is pushed forward, thus developing a non-straight shape. Based on the research performed by Li and Ben-Dor [105], if the downstream disturbances and front of the curled slipstream do not reach the Mach stem, the Mach stem remains straight and perpendicular to the wedge surface. Otherwise, the Mach stem orientation is changed due to the downstream disturbances.

This phenomenon is observed in Euler solution as well. However, Navier-Stokes solution provides more comprehensive result than the Euler solution.

Figure 3-31. Two-dimensional double Mach reflection, Navier-Stokes solution, thirty density contours from 1.5 to 20.5 (481×121, CFL = 0.6)

3.3.4 Shock Wave-Mixing Layer Interaction

This test case was developed by Sandham and Yee [106] in order to evaluate the accuracy and resolution of numerical schemes for interaction of shock waves with shear layers. The shear layer is a turbulent region of transition between two parallel streams (1 and 2), each with a uniform velocity assumed in the x-direction [107]. It should be noted that shear layers can be developed in time and space. The main difference between these two types of shear layers is that in spatially evolving shear layers, coalescence of vortices induces a change in velocity everywhere in the flow, even upstream, while for the time-developing shear layers, such an event will not affect the previous development of the flow [107].

In this problem, on a domain of 200×40 , a spatially developing shear layer with convective Mach number of 0.6 ($M_c = \frac{u_1 - u_2}{a_1 + a_2}$) and an oblique shock originating from the top left corner are simulated. The schematic of the physical problem is shown in Figure 3-32. The oblique shock interacts with the vortices of the mixing layer and is deflected. The impact occurs at around $x = 88$. It also reflects from the lower boundary, which is introduced as slip wall.

Therefore, the vortices pass through the second shock, and an expansion fan forms above the mixing layer.

Figure 3-32. Schematic of the shock-mixing layer interaction [108]

The inflow is set as a hyperbolic tangent profile as

$$u = 2.5 + 0.5 \tanh(2y). \quad (3.24)$$

With this profile, the upper stream velocity is set to $u_1 = 3$, and the lower stream velocity is $u_2 = 2$. For both streams, equal pressures and stagnation enthalpies are assumed. Other flow properties of the two streams are

$$\begin{aligned} v_1 &= v_2 = 0 \\ p_1 &= p_2 = 0.3327 \\ \rho_1 &= 1.6374 \\ \rho_2 &= 0.3626. \end{aligned} \quad (3.25)$$

The Prandtl number is 0.72, and the Reynolds number is set to be 500. Fluctuations are added to the inflow as

$$v' = \sum_{k=1}^2 a_k \cos(2\pi k t / T + \phi_k) e^{(-y^2/b)} \quad (3.26)$$

with period $T = \lambda / u_c$, convective velocity $u_c = 2.68$, wavelength $\lambda = 30$, $b = 10$, $a_1 = a_2 = 0.05$, $\phi_1 = 0$, and $\phi_2 = \pi / 2$. No perturbations are added to the x-component of the velocity. For the outflow boundary, zero-order extrapolation is used, and the flow properties at

upper boundary are set to be as the flow properties behind an oblique shock with angle $\beta = 12^\circ$. A uniform grid system in both directions with 401×101 grid points is used for this case. A constant time step of $\Delta t = 0.12$ is used, and the solution proceeds up to $t = 120$. The value of κ in equation (2.37) is set to be 0.2 in this benchmark problem. The density and pressure contours at the final time are illustrated in Figure 3-33. The shock waves are distorted by the passage of the vortices. Pressure contours are compared with the reference solution adapted from Ref. [11]. Grid system of 641×161 , which is much finer than the present grid system, is used in the reference solution. Unlike the reference result, no oscillations are observed near the upper boundary of the MWWS solution. As can be seen, the expansion fan and shock waves are clearly captured.

a) Density contours from 0.314432 to 2.8392, grid system of 401×101

b) Pressure contours from 0.218441 to 0.718442, grid system of 401×101

c) Pressure contours from 0.218441 to 0.718442, grid system of 641×161 [11]

Figure 3-33. Shock-shear layer interaction ($t = 120$)

Flow properties of different regions shown in Figure 3-32 are listed in Table 3-1. These values are obtained from the numerical scheme and are in good agreement with the standard gas-dynamic solution adapted from the research of Yee et al. [11]. It should be noted that the values were obtained by averaging the values of several locations within the specified regions. In

column 5, some of the values do not exactly match with the analytical results, which may be due to the shear layer's finite thickness or averaging the values within the region.

TABLE 3-1

SHOCK-SHEAR LAYER INTERACTION, FLOW PROPERTIES IN DIFFERENT REGIONS OF THE FLOW, COMPARISON OF NUMERICAL RESULTS WITH THE STANDARD GAS-DYNAMIC SOLUTION (BOLD AND UNDERLINED VALUES)

Property	1	2	3	4	5
x-Velocity (u)	<u>3.0000</u> 3.0000	<u>2.0000</u> 2.0000	<u>2.9709</u> 2.9709	<u>2.9792</u> 2.9879	<u>1.9001</u> 1.8775
y-Velocity (v)	<u>0.0000</u> 0.0000	<u>0.0000</u> 0.0000	<u>-0.1367</u> -0.1368	<u>-0.1996</u> -0.1998	<u>-0.1273</u> -0.1653
Density (ρ)	<u>1.6374</u> 1.6372	<u>0.3626</u> 0.3626	<u>2.1101</u> 2.1095	<u>1.8823</u> 1.8814	<u>0.4173</u> 0.4118
Pressure (p)	<u>0.3327</u> 0.3326	<u>0.3327</u> 0.3327	<u>0.4754</u> 0.4753	<u>0.4051</u> 0.4049	<u>0.4051</u> 0.3976
Mach No. (M)	<u>5.6250</u> 5.6249	<u>1.7647</u> 1.7646	<u>5.2956</u> 5.2953	<u>5.4396</u> 5.4555	<u>1.6335</u> 1.6262

Note: (1) Inflow (upper stream), (2) Inflow (lower stream), (3) upper stream after oblique shock, (4) upper stream after expansion fan, (5) lower stream after shock wave

CHAPTER 4

SIMULATION OF WALL-PRESSURE FLUCTUATIONS WITH EMPIRICAL/SEMI-EMPIRICAL MODELS

4.1 Turbulent Boundary Layer Definition

Predicting flow behavior over flat plates has been one of the most interesting areas of research since the 1950s. However, it requires more attention when the developed boundary layer is turbulent. The basic idea of a boundary layer was developed by Prandtl. He found out that for high Reynolds number flows, viscous effects are confined to a thin layer near the solid surface [109]. Based on the mentioned concept, this small region cannot be neglected, since all heat, momentum, and mass transfer must take place through it.

A turbulent boundary layer is divided into several regions with specific turbulence behaviors [1]. There is a very thin layer near the surface, which is called viscous or laminar sublayer. In viscous or laminar sublayer, flow velocity decreases towards the surface. In this region turbulent fluctuations are damped and there is no room for large eddies. The thickness of the viscous sublayer depends on the velocity of undisturbed flow. The ratio of the laminar sublayer thickness δ_s to the total boundary layer thickness can be found from [110]

$$\frac{\delta_s}{\delta} = 680 \frac{\ln^2 \text{Re}_x}{\text{Re}_x}. \quad (4.1)$$

The outer region of the boundary layer, which contains turbulent flow, is referred to as fully turbulent zone. These two zones are connected with another region, which is called buffer zone. These distinct regions can be categorized by a non-dimensional velocity $u^+ = \frac{u}{u_\tau}$ and

normal to the surface spatial coordinate $y^+ = y \frac{u_\tau}{\nu}$. Hence, the introduced regions can be identified as follows [1]

$$\text{Viscous sublayer: } y^+ < 2 \sim 8 \quad (4.2)$$

$$\text{Buffer zone: } 2 \sim 8 < y^+ < 2 \sim 50 \quad (4.3)$$

$$\text{Fully turbulent zone: } y^+ > \sim 50. \quad (4.4)$$

There is another classification of a turbulent boundary layer, which divides it to two inner and outer regions. The inner region includes the laminar sublayer, buffer zone, and a part of the fully turbulent zone. Subsequently, the other part of the turbulent boundary layer is considered as outer region. Similar to the previous categorization, these two regions can be identified as [1]

$$\text{Inner region: } y^+ < 100 \sim 400 \quad (4.5)$$

$$\text{Outer region: } y^+ > 100 \sim 400 \quad (4.6)$$

4.2 Turbulent Boundary Layer Noise

The mechanism of noise generation in turbulent boundary layer has remained as challenging issue over the past decades. The interest in this field is noticeable since the majority of engineering applications is turbulent. Recent developments in numerical and experimental efforts have opened up new possibilities to improve the knowledge of noise generation and propagation mechanisms. One of the primary sources of aircraft noise is due to aerodynamic noise or turbulent boundary layer noise. Wall-pressure fluctuations beneath a turbulent boundary layer are called boundary layer noise, or pseudonoise [111]. It has been recognized that the boundary layer noise can be a significant noise problem at speeds greater than 200 (mph) [112] and a significant contributor to the mid- and high-frequency cabin sound-pressure levels [113]. The generated noise has a small amplitude and therefore, the numerical and experimental

simulations become challenging [114]. Several theoretical studies have been performed to predict the features of the pressure field radiated by a turbulent boundary layer. Lighthill's acoustic analogy [115] in the early 1950s is an example and is widely employed in the study of noise generated by a turbulent boundary layer. Lighthill's aero-acoustic analogy is obtained by the rearranged governing Navier-Stokes equations. The farfield sound pressure is calculated by volume integration over the domain containing the sound source.

Noise generated by airflow over an airplane's surface is important for all classes of airplanes. For small airplanes, boundary layer noise is only important at high-frequencies, while high-speed flows in larger airplanes generate significant levels of turbulent boundary layer noise, which is the most important source of cabin noise during cruising [116].

In some cases, severe sound pressure levels can cause intolerable noise, which may affect passenger comfort and cause many difficulties such as crew fatigue, malfunction of equipments, and problems in communications. Controlling sound pressure levels may have some consequences such as reduced performance, reduced cabin volume, added structural weight, or increased cost [117]. Consequently, understanding the sources of noise generation is necessary to find better methods to control it without penalties.

4.3 Empirical/Semi-Empirical Models

There are several methods to analyze acoustics and calculate the pressure fluctuations of turbulent boundary layers. Single-point wall-pressure spectrum models, normalized wavenumber-frequency spectrum, and mean square pressure models are the most remarkable empirical/semi-empirical models in this field. In the current research, several single-point models and mean square pressure models will be reviewed and discussed.

In the present study, power spectral density (PSD), which is the frequency response of a random or periodic signal, is estimated using the Welch method of spectral estimation. It employs fast Fourier transform (FFT) function, which uses the recorded pressures over time at each location. This is a faster version of the discrete Fourier transform (DFT) function that uses the same algorithms but much less time in execution. Matlab has an effective tool for computing the Fourier transform of a signal, which is employed in this study.

4.3.1 Single-Point Wall-Pressure Spectrum Empirical Models

The single-point wall-pressure spectrum models correspond to the distribution of the mean square pressure fluctuations with frequency. In this section, selected empirical models are presented and reviewed. In order to compare the results from different models and the numerical solutions, a theoretical spectral density model is required. Newland [118] introduced the mean value of x as

$$E[x] = \int_{-\infty}^{\infty} xP(x)dx \quad (4.7)$$

Therefore, the mean value of the pressure fluctuations can be written as

$$E[p] = \int_{-\infty}^{\infty} pP(p)dp \quad (4.8)$$

where $P(p)$ is the probability density function of the pressure fluctuations. In this equation, E represents “the statistical expectation” [118]. The mean square value of p , $E[p^2]$, is defined as the average value of p^2 , which is given by

$$E[p^2] = \int_{-\infty}^{\infty} p^2 P(p)dp = p_{rms}^2. \quad (4.9)$$

The main objective of this section is to calculate the pressure fluctuations at a single-point at different times. The auto-correlation function for $P(t)$, which is the expected value of the pressure at time t and again at time $t + \tau$ can be explained as

$$R_{pp}(\tau) = E[p(t)p(t + \tau)]. \quad (4.10)$$

The Fourier transform of $R_{pp}(\tau)$ and its inverse are given by

$$S_{pp}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R_{pp}(\tau) e^{-i\omega\tau} d\tau \quad (4.11)$$

and

$$R_{pp}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_{pp}(\omega) e^{i\omega\tau} d\omega \quad (4.12)$$

where $S_{pp}(\omega)$ is called the spectral density of the pressure process, which is a function of angular frequency. The fundamental definition of the auto-correlation function provides

$$E[p^2] = \int_{-\infty}^{\infty} S_{pp}(\omega) d\omega \quad (4.13)$$

which can be written as

$$E[p^2] = \int_0^{\infty} W_{pp}(f) df \quad (4.14)$$

where f is the frequency (Hz), and $W_{pp}(f)$ is the equivalent one-sided spectral density function.

The single-sided spectrum is related to the double sided spectrum by

$$W_{pp}(f) = 4\pi S_{pp}(\omega) \quad (4.15)$$

and if the single-point wall-pressure spectrum is defined by $\Phi_{pp}(\omega)$, then one can obtain

$$\Phi_{pp}(\omega) = 2S_{pp}(\omega) \quad (4.16)$$

Therefore, the power spectrum level can be calculated by

$$W_{pp}(f) = 2\pi\Phi_{pp}(\omega) \quad (4.17)$$

More details can be found in Refs. [116, 118].

4.3.1.1 Robertson Model

In 1971, Robertson [119] proposed an empirical model based on the previous model developed by Lowson [120]. He noticed that Lowson's proposed model underestimates the spectral levels at low Strouhal numbers and provides too much roll-off at high Strouhal numbers. In his equation, δ^* (displacement thickness) and U_∞ (free stream velocity) were used as normalizing parameters. The proposed equation is presented as

$$\frac{\Phi(\omega)U_\infty}{q_\infty^2\delta^*} = \frac{\overline{p^2}/q_\infty^2}{\frac{\omega_0\delta^*}{U_\infty}\left(1 + (\omega/\omega_0)^{0.9}\right)^2} \quad (4.18)$$

where $\omega_0 = 0.5\frac{U_\infty}{\delta^*}$ and $\overline{p^2}/q_\infty^2 = (0.006)^2/(1 + 0.14M^2)^2$.

The proposed formula was developed based on the experimental findings throughout the Mach number range. Replacing ω by $2\pi f$ and including the 2π factor, the spectral density can be calculated by

$$W(f) = \frac{2\pi\overline{P^2}}{\omega_0\left(1 + (2\pi f/\omega_0)^{0.9}\right)^2} \quad (4.19)$$

4.3.1.2 Efimtsov Model (1)

Efimtsov [38] proposed an empirical formula, which is dependent on the Strouhal number, Reynolds number, and Mach number. This formula was developed based on experimental data captured during a series of tests with Mach number ranges of $M = 0.41$ to 2.1 and at $Re_x = (0.5 - 4.85) \times 10^8$. Efimtsov's single-point wall-pressure spectrum is given by

$$\Phi(\omega)U_\tau/\tau_w^2\delta = F(\omega\delta/U_\tau) \quad (4.20)$$

where

$$F = 0.01\left(1 + 0.02(\text{Sh})^{2/3}\right)^{-1}. \quad (4.21)$$

In the above equations, τ_w is the frictional stress on the flow surface, $U_\tau = (\tau_w / \rho_w)^{1/2}$ is the friction (dynamic) velocity, and ρ_w is the air density adjacent to the surface. The Strouhal number is defined by $\text{Sh} = \omega\delta/U_\tau$, which plays a key role in the Efimtsov model. The spectral density can be found by

$$W(f) = \frac{2\pi(0.01\tau_w^2\delta)}{U_\tau\left(1 + 0.02(2\pi f\delta/U_\tau)^{2/3}\right)}. \quad (4.22)$$

4.3.1.3 Efimtsov Model (2)

Efimtsov [121] updated his single-point wall-pressure spectrum model with additional results of low and high-speed TsAGI (central aero-hydrodynamic institute in Moscow, Russia) wind tunnels, Tu144, and Tu22 in-flight measurements. He proposed the following equation:

$$\Phi(\omega) = 2\pi\alpha U_\tau^3 \rho^2 \delta \frac{\beta}{(1 + 8\alpha^3 \text{Sh}^2)^{1/3} + \alpha\beta \text{Re}_\tau (\text{Sh}/\text{Re}_\tau)^{10/3}} \quad (4.23)$$

with the following definitions and variables:

$$\beta = \left(1 + (\text{Re}_{\tau 0}/\text{Re}_\tau)^3\right)^{1/3} \quad (4.24)$$

$$\text{Re}_\tau = \delta U_\tau / \nu_w \quad (4.25)$$

$$\text{Re}_{\tau 0} = 3000 \quad (4.26)$$

$$\alpha = 0.01 \quad (4.27)$$

$$\nu_w = \nu_\infty \frac{\rho_\infty}{\rho_w} \left(\frac{T_w}{T_\infty}\right)^\gamma \quad (4.28)$$

$$T_w = T_\infty \left(1 + r \frac{\kappa - 1}{2} M^2 \right); \quad r = 0.89, \quad \kappa = 1.4 \quad (4.29)$$

$$\rho_w = \rho_\infty \frac{T_\infty}{T_w} \quad (4.30)$$

The spectral density can be calculated by

$$W(f) = \frac{(2\pi)^2 \alpha U_\tau^3 \rho_\infty^2 \delta \beta}{\left(1 + 8\alpha^3 (2\pi f \delta / U_\tau)^2 \right)^{1/3} + \alpha \beta \text{Re}_\tau \left((2\pi f \delta / U_\tau) / \text{Re}_\tau \right)^{10/3}}. \quad (4.31)$$

4.3.1.4 Chase-Howe Model

Howe [122] proposed the following model and attributed it to Chase [123]:

$$\frac{\Phi(\omega) U_\infty}{\tau_w^2 \delta^*} = \frac{2(\omega \delta^* / U_\infty)^2}{\left((\omega \delta^* / U_\infty)^2 + 0.0144 \right)^{3/2}}. \quad (4.32)$$

This model is a simplification of the previous model developed by Chase, which was more comprehensive for the wavevector-frequency p spectrum. Based on the above equation, the model spectrum is proportional to ω^2 at low frequencies and varies as ω^{-1} at higher frequencies [64]. The spectral density based on the Chase-Howe model can be calculated by

$$W(f) = \frac{4\pi(\delta^* / U_\infty)^3 (2\pi f \tau_w)^2}{\left((2\pi f \delta^* / U_\infty)^2 + 0.0144 \right)^{3/2}}. \quad (4.33)$$

4.3.1.5 Goody Model

Goody [64] proposed a single-point wall-pressure spectrum model based on the Chase-Howe model. Goody found that at low frequencies, the spectral levels of the Chase-Howe model are low and do not decay as quickly at high frequencies. Therefore, Goody [64] added a term to the denominator so that spectral levels decay as ω^{-5} as $\omega \rightarrow \infty$. The final form of his proposed empirical model is

$$\frac{\Phi(\omega)U_\infty}{\tau_w^2\delta} = \frac{3.0(\omega\delta/U_\infty)^2}{[(\omega\delta/U_\infty)^{0.75} + 0.5]^{3.7} + [(1.1R_T^{-0.57})(\omega\delta/U_\infty)]^7} \quad (4.34)$$

where the ratio of the outer-layer to inner-layer timescale (R_T) is defined as

$$R_T = \frac{\delta}{U_\infty} \cdot \frac{U_\tau^2}{\nu} = \left(\frac{U_\tau \delta}{\nu} \right) \sqrt{\frac{C_f}{2}}. \quad (4.35)$$

It can be seen that by increasing the Reynolds number, the timescale ratio is increased. This model covers a large range of Reynolds numbers, 1.4×10^3 to 2.34×10^4 . However, the model can be extrapolated to higher Reynolds number flows because it follows the Reynolds number independence, high-frequency, inner-layer scaling. The spectral density is calculated by

$$W(f) = \frac{6\pi(\delta/U_\infty)^3(2\pi f\tau_w)^2}{\left((2\pi f\delta/U_\infty)^{0.75} + 0.5\right)^{3.7} + \left((1.1R_T^{-0.57})(2\pi f\delta/U_\infty)\right)^7}. \quad (4.36)$$

In all of the models, the boundary layer parameters are calculated from the following equations, which are obtained from turbulent boundary layer equations [70]:

$$\delta = 0.37x \left(\frac{U_\infty x}{\nu} \right)^{-1/5} \quad (4.37)$$

$$C_f = \frac{2\tau_w}{\rho U_\infty^2} = 0.045 \left(\frac{U_\infty \delta}{\nu} \right)^{-1/4} \quad (4.38)$$

and the displacement thickness for the turbulent boundary layer is calculated by

$$\delta_{turb}^* = \frac{0.0174x}{\text{Re}_x^{0.139}}. \quad (4.39)$$

As Goody explained in Ref. [70], no universal scaling exists for boundary layers that collapse the pressure spectra with different Reynolds number at all frequencies. Traditionally, the two-layer model was applied to pressure fluctuations [64], i.e., the boundary layer was divided into two regions: inner- and outer-layer. It is generally agreed that the inner region is responsible

for the high-frequency behavior of the single-point wall-pressure spectrum [116]. The outer region determines the pressure levels at low frequencies.

Based on recent investigations, frequency regions have been categorized into three ranges; inner-layer, outer-layer, and overlap ranges. The inner-layer region (flow near the wall) has a set of velocity, pressure, and length that is different from the outer-layer region (flow away from the wall) set. The majority of the latest investigators show that an overlap region exists whereby both inner-layer and outer-layer scaling collapse the pressure spectrum [70]. Theoretical research shows that, in this frequency range, the pressure spectrum decreases as ω^{-1} . Goody et al. [70] explained this as universal turbulent motion within the logarithmic region of the boundary layer. However, experimental investigations show that the slope is in the range of $\omega^{-0.7} - \omega^{-0.8}$. It should be noted that by increasing the Reynolds number, the size of the overlap frequency range increases. Goody and other investigators such as Farabee and Casarella [45] showed that the spectrum increased as ω^2 in the low-frequency range, which was consistent with the previously performed theoretical results [124, 125]. Goody [64] also showed that the spectrum decayed as ω^{-5} in the high-frequency range. Goody [126] reviewed the experimental results of six research groups and showed that the spectra collapse at low- and mid-frequencies when τ_w was employed as the pressure scale and either δ^*/U_∞ , δ/U_∞ , δ^*/U_τ , or δ/U_τ was used as the timescale. However, Schewe [39] estimated this as $\omega^{-7/3}$. Table 4-1 shows a summary of frequency range slope predictions by several researchers.

TABLE 4-1

SUMMARY OF FREQUENCY RANGE SLOPE PREDICTIONS [116]

Model	Low-Frequency Range	Overlap Region	High-Frequency Range
Ffowcs-Williams [127]	ω^2		
Bradshaw [124]		ω^{-1}	
Panton and Linebarger [128]	ω^2	ω^{-1}	
Schewe [39]			$\omega^{-7/3}$
Farabee and Casarella [45]	ω^2	ω^{-1}	
Bull [49]		ω^{-1}	
Goody et al. [70]	ω^2		ω^{-5}

In the current research, employed scaling variables are as shown in Table 4-2.

TABLE 4-2

SCALING VARIABLES

Scaling Variables	Spectrum Scaling	Frequency Scaling
Inner	$\Phi(\omega)U_\tau^2 / \tau_w^2 \nu$	$\omega \nu / U_\tau^2$
Mixed	$\Phi(\omega)U_\tau / \tau_w^2 \delta$	$\omega \delta / U_\tau$
Outer	$\Phi(\omega)U_\infty / \tau_w^2 \delta$	$\omega \delta / U_\infty$

4.3.2 Mean Square Pressure Empirical Models

In order to predict the overall energy of the wall-pressure fluctuations in a turbulent boundary layer, mean square pressure fluctuation models can be employed. The available models have been developed based on experimental efforts.

4.3.2.1 Kraichnan Model

This model was one of the earliest empirical models for calculating the mean square wall-pressure fluctuations, which was developed by Kraichnan in 1956 [129]. His model was only a function of wall shear stress, which is formulated as

$$\overline{P'^2} = (6\tau_w)^2. \quad (4.40)$$

4.3.2.2 Lilley and Hodgson Model

This model, proposed in 1960 [130] based on experiments performed in a wind tunnel, was found to be a function of dynamic pressure:

$$\overline{P'^2} = (0.008q)^2. \quad (4.41)$$

4.3.2.3 Bull Model

In 1967, Bull [131] proposed a model based on the Reynolds number as follows:

$$\overline{P'^2} = (2.11\tau_w)^2 \quad \text{for} \quad \text{Re}_\theta = 6400 \quad (M = 0.3) \quad (4.42)$$

$$\overline{P'^2} = (2.80\tau_w)^2 \quad \text{for} \quad \text{Re}_\theta = 33800 \quad (M = 0.5). \quad (4.43)$$

4.3.2.4 Lowson Model

In 1967, Lowson [120] proposed an empirical model, based on Mach number and dynamic pressure. His model is valid for Mach numbers up to about 3.

$$\overline{P'^2} = \left(\frac{0.006q}{1 + 0.14M^2} \right)^2 \quad (4.44)$$

4.3.2.5 Schewe Model

Another empirical model was proposed by Schewe [39] in 1983, which was similar to the developed equation by Lilley and Hodgson.

$$\overline{P'^2} = (0.0102q)^2 \quad (4.45)$$

4.3.2.6 Lauchle and Daniels Model

Similar to Schewe's results, Lauchle and Daniels [132] found a mean square wall-pressure model based on dynamic pressure.

$$\overline{P'^2} = (0.0106q)^2 \quad (4.46)$$

4.3.2.7 Farabee and Casarella Model

In 1991, Farabee and Casarella [45] proposed their model based on a different approach. Here, the mean square wall-pressure fluctuations were obtained by numerically integrating the spectra. The lower-frequency limit was picked to be 50 (Hz), because the noise related to the facility dominated below this amount. The high-frequency integration limit was set to 20,000 (Hz), which was set by the upper frequency limit of response of the pinhole microphone system.

$$\begin{aligned} \overline{P'^2} &= 6.5\tau_w^2 & (\text{Re}_\tau = U_\tau \delta / \nu) \leq 333 \\ \overline{P'^2} &= [6.5 + 1.86 \ln(\text{Re}_\tau / 333)] \tau_w^2 & (\text{Re}_\tau = U_\tau \delta / \nu) > 333 \end{aligned} \quad (4.47)$$

4.3.2.8 Lueptow Model

Lueptow [47] developed his model in 1995 and showed that nondimensionalization with q reduces sensitivity to the Reynolds number. His data suggests:

$$\overline{P'^2} = (0.012q)^2 \quad (4.48)$$

CHAPTER 5

SIMULATION OF WALL-PRESSURE FLUCTUATIONS WITH NUMERICAL TECHNIQUES

5.1 Turbulence Modeling

Turbulent flows may be simulated by several approaches. Turbulence modeling is the most common method in this field, which plays an important role in simulation regarding the accuracy and efficiency. The goal of turbulence modeling is optimal combination of these two factors, which is difficult to achieve. Finding a universal model, which works well for all flow cases, is not possible. High quality grid system, proper time marching, and suitable turbulence model can lead to an improved result.

Turbulent flows are irregular, unstable, and random, which mostly occur at high Reynolds numbers. Turbulence is always three dimensional with high rate of diffusivity. The variables in turbulent flows are divided in one time-averaged part, which is independent of time, and one fluctuating part.

In the current research, numerical simulations are conducted using the CFD code Cobalt, which is an unstructured finite-volume code, using a cell-centered formulation. It is based on Godunov's first-order accurate, finite-volume, exact Riemann solution method. The exact Riemann solver is replaced with an inviscid flux function, which alleviates the inherent deficiencies of Riemann methods, namely the 'slowly-moving shock' and 'carbuncle' problems, while retaining their inherent advantages, most notably the exact capture of stationary contact surfaces. Second-order spatial accuracy is achieved via upwind-biased reconstruction based on least-squares gradients [133]. This compressible flow solver has been validated on several problems by Strang et al. [134], Forsythe et al. [135], Viswanathan et al. [136], and Boydston et

al. [137]. Cobalt software provides both RANS and LES approaches for turbulent flows. Several turbulence models available in Cobalt are listed in Table 5-1 [138].

TABLE 5-1
COBALT SOFTWARE TURBULENCE MODELS

No.	Turbulence Model	Description
1	Spalart-Allmaras (SA)	RANS, 1-equation
2	DDES based on SA	Hybrid RANS/LES, 1-equation
3	Menter's baseline (BSL)	RANS, 2-equation
4	Menter's shear stress transport (SST)	RANS, 2-equation
5	DDES based on SST	Hybrid RANS/LES, 2-equation
6	Wilcox's 1998 $k-\omega$ model	RANS, 2-equation
7	SA with rotation/curvature correction (SARC)	RANS, 1-equation
8	DDES based on SARC	Hybrid RANS/LES, 1-equation

Total variation diminishing limiters are used to limit extremes at cell faces. ParMETIS is used to implement parallel processing and domain decomposition and the message passing interface manages communication between processors.

In the current research, RANS-SA, delayed detached-eddy simulation based on RANS-SA (DDES-SA), RANS-SST, and DDES-SST are investigated.

5.1.1 RANS-SA (Spalart-Allmaras) Turbulence Model

The Spalart-Allmaras turbulence model was proposed by Spalart and Allmaras [139] in 1994. This model is a transport equation model for eddy viscosity [140].

This one-equation model is given by

$$\frac{D\bar{\nu}}{Dt} = \frac{1}{\sigma} \left[\nabla \cdot \left((\nu + \bar{\nu}) \nabla \bar{\nu} \right) + c_{b2} (\nabla \bar{\nu})^2 \right] + c_{b1} \bar{S} \bar{\nu} - c_{w1} f_w \left[\frac{\bar{\nu}}{d} \right]^2 \quad (5.1)$$

where $c_{b1}\bar{S}\bar{\nu}$ is the production term, $\frac{1}{\sigma}\left[\nabla\cdot((\nu+\bar{\nu})\nabla\bar{\nu})+c_{b2}(\nabla\bar{\nu})^2\right]$ is the diffusion term,

$c_{w1}f_w\left[\frac{\bar{\nu}}{d}\right]^2$ is the destruction term, and ν is the molecular viscosity. The turbulent eddy viscosity

is computed by

$$\nu_t = \bar{\nu} f_{v1} \quad (5.2)$$

where

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}; \quad \chi = \frac{\bar{\nu}}{\nu} \quad (5.3)$$

$$\bar{S} = S + \frac{\bar{\nu}}{\kappa^2 d^2} f_{v2} \quad (5.4)$$

and

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}. \quad (5.5)$$

In the above equations, d is the distance from the field point to the nearest wall, κ is the Von Karman constant, and S is the magnitude of the vorticity

$$S = \left| \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right|. \quad (5.6)$$

The function f_w is defined by

$$f_w(r) = g \left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right)^{1/6} \quad (5.7)$$

where

$$g = r + c_{w2}(r^6 - r); \quad r = \min \left[\frac{\bar{\nu}}{\bar{S}\kappa^2 d^2}, 10 \right]. \quad (5.8)$$

The constants are set as follows

$$\sigma = 2/3, \quad c_{b1} = 0.1355, \quad c_{b2} = 0.622, \quad \kappa = 0.41, \\ c_{v1} = 7.1, \quad c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{1+c_{b2}}{\sigma}, \quad c_{w2} = 0.3, \quad c_{w3} = 2. \quad (5.9)$$

The RANS-SA model has been shown to provide acceptable results for a wide variety of situations. However, some drawbacks have been observed and therefore, several modifications have been applied to the original equation.

5.1.2 Delayed Detached-Eddy Simulation based on RANS-SA (DDES-SA) Turbulence Model

The detached-eddy simulation (DES) is a hybrid technique proposed by Spalart et al. [141] in 1997 as a numerically feasible and reasonably accurate approach for predicting massively separated flows [142]. This turbulence model, which is also denoted as DES97, was defined as a “three-dimensional unsteady numerical solution using a single turbulence model, which functions as a subgrid-scale model in regions where the grid density is sufficiently fine for a large-eddy simulation, and as a Reynolds-averaged model in regions where it is not” [143]. This model is a combination of RANS and LES, whereby the model switches to a sub-grid scale formulation in regions sufficiently fine for LES calculations. Traditional LES wall models may have difficulty in accurately resolving separated flows, and grid refinement could demand excessive computing power. Since RANS models are known to be cost effective and have been fairly well developed to provide accurate boundary-layer calculations, they are used as a wall model in the near-wall region. However, RANS models cannot be employed in the prediction of problems with large separation regions such as flow past a cavity, vehicles, and so on [144].

DES does not have an explicit filter operator similar to LES. The switching between RANS and LES depends on grid spacing, velocity gradient, and eddy viscosity and can be controlled during the preprocessing stage.

However, DES97 shows an incorrect result in thick boundary layers and shallow separation regions [145]. It occurs when the grid spacing that is parallel to the wall becomes less than the boundary layer thickness. In this situation, the grid spacing is sufficiently fine for the DES length scale to follow the LES part. However, “resolved Reynolds stresses deriving from velocity fluctuations (“LES content”) have not replaced the modeled Reynolds stresses” [145]. Consequently, because the resolution is not fine to support the LES part and also because of delays in its generation, the stresses may reduce the skin friction and therefore premature separation may occur. The addressed shortcoming was solved by Spalart in the next version of DES97 called DDES. In DDES, the length scale $\tilde{d}$ is corrected as

$$\tilde{d} = d - f_d \max(0, d - C_{\text{DES}}\Delta) \quad (5.10)$$

where

$$f_d = 1 - \tanh((8r_d)^3), \quad r_d = \frac{\nu_t + \nu}{\sqrt{U_{i,j}U_{i,j}}\kappa^2 d^2}. \quad (5.11)$$

In the above equation, r_d is a modified version of r in the RANS-SA model and $U_{i,j}$ is the velocity gradients. Similar to the RANS-SA model, r_d is 1 in a logarithmic layer and gradually 0 toward the edge of the boundary layer. In equation (5.10), setting $f_d = 0$ yields RANS ($\tilde{d} = d$), while $f_d = 1$ provides DES97 ($\tilde{d} = \min(d, C_{\text{DES}}\Delta)$).

5.1.3 RANS-SST (Standard Menter Shear Stress Transport) Turbulence Model

The RANS-SST model is an eddy-viscosity turbulence model, which was introduced by Menter in 1994 [146]. It is combination of a $k-\omega$ model in the inner region of the boundary layer and a $k-\varepsilon$ model in the outer region and outside of the boundary layer. Both of these turbulence models have some deficiencies. The $k-\varepsilon$ model overpredicts the shear stress in adverse pressure gradient flows and requires near-wall modifications. The $k-\omega$ model is dependent on the free stream value of ω . The combined model has the following set of equations:

$$\frac{D\rho k}{Dt} = \tau_{ij} \frac{\partial u_i}{\partial x_j} - \beta^* \rho \omega k + \frac{\partial}{\partial x_j} \left[(\mu + \sigma_k \mu_t) \frac{\partial k}{\partial x_j} \right] \quad (5.12)$$

$$\frac{D\rho \omega}{Dt} = \frac{\gamma}{\nu_t} \tau_{ij} \frac{\partial u_i}{\partial x_j} - \beta \rho \omega^2 + \frac{\partial}{\partial x_j} \left[(\mu + \sigma_\omega \mu_t) \frac{\partial \omega}{\partial x_j} \right] + 2\rho(1-F_1)\sigma_{\omega 2} \frac{1}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j}. \quad (5.13)$$

If any constant is defined by ϕ , then the blended constant for the combined model is calculated by

$$\phi = F_1 \phi_1 + (1-F_1) \phi_2 \quad (5.14)$$

where the constants of set 1 (ϕ_1) are

$$\begin{aligned} \sigma_{k1} &= 0.85, \quad \sigma_{\omega 1} = 0.5, \quad \beta_1 = 0.0750, \quad a_1 = 0.31 \\ \beta^* &= 0.09, \quad k = 0.41, \quad \gamma_1 = \beta_1 / \beta^* - \sigma_{\omega 1} k^2 / \sqrt{\beta^*} \end{aligned} \quad (5.15)$$

and the constants of set 2 (ϕ_2) are

$$\begin{aligned} \sigma_{k2} &= 1.0, \quad \sigma_{\omega 2} = 0.856, \quad \beta_2 = 0.0828 \\ \beta^* &= 0.09, \quad k = 0.41, \quad \gamma_2 = \beta_2 / \beta^* - \sigma_{\omega 2} k^2 / \sqrt{\beta^*}. \end{aligned} \quad (5.16)$$

In the above equations, F_1 is introduced as

$$F_1 = \tanh(\arg_1^4) \quad (5.17)$$

where

$$\arg_1 = \min \left[\max \left(\frac{\sqrt{k}}{0.09\omega y}, \frac{500\nu}{y^2\omega} \right), \frac{4\rho\sigma_{\omega_2}k}{CD_{k\omega}y^2} \right] \quad (5.18)$$

and

$$CD_{k\omega} = \max \left(2\rho\sigma_{\omega_2} \frac{1}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j}, 10^{-20} \right). \quad (5.19)$$

where F_1 is the switching factor with the value of one in the near-wall region to activate the $k-\omega$ model, and with the value of zero at the boundary layer edge to activate the $k-\varepsilon$ model.

Menter proposed the turbulent eddy viscosity as

$$\nu_t = \frac{a_1 k}{\max(a_1\omega, \Omega F_2)} \quad (5.20)$$

where $F_2 = \tanh(\arg_2^2)$ with

$$\arg_2 = \max \left(2 \frac{\sqrt{k}}{0.09\omega y}, \frac{500\nu}{\omega y^2} \right) \quad (5.21)$$

and Ω is the absolute value of vorticity. F_2 is a function with the value of one for boundary layer flows and zero for shear layers.

5.1.4 Delayed Detached-Eddy Simulation based on RANS-SST (DDES-SST) Turbulence Model

Similar to section 5.1.2, the DDES model acts as the main part of the turbulence model, and the Menter RANS-SST model is employed as subgrid-scale (SGS) model. Therefore, the same processes for switching RANS and LES occur as presented in section 5.1.2.

5.2 Direct Numerical Simulation Technique

5.2.1 Inflow Generation Method

The three-dimensional simulation of turbulent flow past a flat plate requires realistic simulation of turbulent inflow condition at the upstream boundary. It is vital to specify a reasonable time series of turbulent fluctuations, which are in equilibrium with the mean flow [147]. The method should be highly accurate and efficient and can simulate the three-dimensional nature of the turbulent flow. The simplest technique is to superpose random fluctuations on a turbulent mean velocity profile. In the current research, the inflow generation technique is adapted from the work of Gloerfelt and Garrec [148]. The boundary layer thickness at about three-quarters of the length of the flat plate is taken as the reference length. A random excitation for the three components of the velocity and a harmonic forcing for the vertical velocity are superimposed on the mean profile as follows

$$\begin{aligned} u' &= \varepsilon U_{\infty} \text{rand}(y, z) e^{-\left(\frac{y-\delta_m/2}{1.2\delta_m/3}\right)^2} \\ v' &= \varepsilon U_{\infty} \cos(\omega t) e^{-\left(\frac{y-\delta_m/2}{1.2\delta_m/10}\right)^2} + \varepsilon U_{\infty} \text{rand}(y, z) e^{-\left(\frac{y-\delta_m/2}{1.2\delta_m/3}\right)^2} \\ w' &= \varepsilon U_{\infty} \text{rand}(y, z) e^{-\left(\frac{y-\delta_m/2}{1.2\delta_m/3}\right)^2} \end{aligned} \quad (5.22)$$

In equation (5.22), ω is introduced as the pulsation frequency which can be calculated by

$$\omega = 2\pi U_{\infty} / (8\delta_m). \quad (5.23)$$

Small amplitudes are preserved for the perturbations as $\varepsilon = 5 \times 10^{-3}$. The term indicated by “rand” generates a random number between -1 and 1. More details about this technique can be found in Ref. [148].

5.2.2 Three-Dimensional DNS Code

The simulation of the wall-pressure fluctuations due to a turbulent boundary layer is also carried out with the proposed scheme. The three-dimensional code is an extension of two-dimensional one, which employs the inflow generation technique. The β value in equation (2.46) remains as 0.65 and the value of κ in equation (2.37) is set as 0.05. Results obtained by the DNS code (MWWS scheme) are presented in next chapter, which are compared to the LES/RANS and empirical/semi-empirical models results.

CHAPTER 6

RESULTS

In this chapter, the numerical results obtained by LES/RANS models and the proposed scheme are compared with each other as well as with the empirical/semi-empirical models. The numerical results are analyzed using a Matlab code to obtain the power spectral density at each location.

6.1 Three-Dimensional Grid System and Flow Conditions

In order to calculate the pressure fluctuations due to a turbulent boundary layer, a flat plate ($0.3(m) \times 0.05(m)$) is considered as shown in Figure 6-1.

Figure 6-1. General configuration of flat plate

A free-stream Mach number of 0.5, which is relevant for cabin noise, and atmospheric conditions are used for flow simulations.

6.1.1 Grid System for Turbulence Modeling Methods

Figure 6-2 illustrates the three-dimensional grid system. The grid system is generated by Gridgen [149]. Grid clustering near the wall has been implemented. The grid is composed of about 850,000 cells.

a) General grid system

b) Clustering near the flat plate

Figure 6-2. Three-dimensional grid system of flat plate for turbulence modeling methods

6.1.2 Grid System for Direct Numerical Simulation Scheme

The three-dimensional grid system for the MWWS scheme is shown in Figure 6-3. In order to capture all scales of the flow, the number of grid points for direct numerical simulation should be in order of $Re^{9/4}$ for three-dimensional problems. In the current study the Reynolds number is on order of 10^6 . Therefore, the grid size should be on order of 10^{13} , which is beyond

available resources. For the present computation, a much smaller grid system composed of 5×10^6 grid points ($500 \times 100 \times 100$) are used. Grid clustering near the wall and along the center line of the flat plate is employed.

a) General grid system

b) Clustering near the flat plate

Figure 6-3. Three-dimensional grid system of flat plate for DNS technique

The time increment, Δt , used in these simulations is 10^{-7} (sec). In order to capture data, ten sensor points, as shown in Figure 6-4, are located along the center line. In order to obtain the mean flow properties and to prevent any effects of the initial flow conditions, the calculations were performed from $t = 0.01$ (sec). The solution proceeds up to 0.105 (sec) for all numerical methods.

Figure 6-4. Sensor points locations of flat plate along the center line

6.2 DDES-SA Turbulence Modeling Results

In this section, results obtained by the DDES-SA turbulence model simulation are presented. On the wall boundary (flat plate), no-slip boundary condition is imposed. For this boundary all components of the velocity are set to zero. Right and left boundaries are set as inflow and outflow boundary condition. Top boundary is set as farfield and two side boundaries are specified as periodic boundaries. For most external aerodynamics problems, a portion of the spatial domain is typically unbounded. Thus, it is essential to introduce an artificial boundary to make the computational domain finite and, thereby, reduce the computational effort and improve efficiency. In most cases, the boundary condition on the boundaries of the truncated domain is specified as periodic boundary condition. An accurate specification of such a boundary condition is important so that it correctly simulates the eliminated domain and the physical phenomena that occur therein.

The turbulent boundary layer (TBL) thickness can be calculated by

$$\delta = 0.37x \left(\frac{U_{\infty}x}{\nu} \right)^{-1/5}. \quad (6.1)$$

In this case, TBL thickness is estimated to be about 0.0054 (m) at $x = 0.29(m)$. The DDES-SA simulation predicts it about 0.0055 (m), which is fairly good estimate. Figure 6-5 illustrates U-velocity contours, which is showing the boundary layer development.

Figure 6-5. DDES-SA model simulation of flat plate, boundary layer development

Figure 6-6 shows instantaneous view of pressure contours of flat plate. The pressure field shows the generated sound on the flat plate. In this radiated field, acoustic waves are observed all over the flowfield.

Figure 6-6. DDES-SA model simulation of flat plate, pressure contours

Velocity data were recorded every 100 iterations at two vertical planes. The locations of sensor points are shown in Figure 6-7.

Figure 6-7. Locations of sensor points for numerical simulations

Velocity fluctuations at $x = 0.07(m)$ are illustrated in Figure . Except the locations away from the boundary layer, the velocity fluctuations (typically U-velocity) can be observed. At this plane the velocity is fluctuating between -3 to 3 (m/s). Y- and Z-components of velocity also show fluctuations between -1 to 1 (m/s). The range of fluctuations for both components is small and roughly zero. However, in some locations Z-component of the velocity shows more fluctuations than V-velocity, which shows the three-dimensional nature of the turbulence.

X-components of velocity fluctuations at
(0.07,0.0006,0.0)

Y-components of velocity fluctuations at
(0.07,0.0006,0.0)

Z- components of velocity fluctuations at (0.07,0.0006,0.0)

X-components of velocity fluctuations at
(0.07,0.0009,0.0)

Y-components of velocity fluctuations at
(0.07,0.0009,0.0)

Figure 6-8. DDES-SA model simulation, velocity fluctuations at different locations

Z-components of velocity fluctuations at (0.07,0.0009,0.0)

X-components of velocity fluctuations at
(0.07,0.0014,0.0)

Y-components of velocity fluctuations at
(0.07,0.0014,0.0)

Z-components of velocity fluctuations at (0.07,0.0014,0.0)

Figure 6-8. (continued)

X-components of velocity fluctuations at
(0.07,0.027,0.0)

Y-components of velocity fluctuations at
(0.07,0.027,0.0)

Z-components of velocity fluctuations at (0.07,0.027,0.0)

Figure 6-8. (continued)

Figure 6-9 illustrates the pressure fluctuations and power spectral density of the flat plate at (0.25,0,0). It can be seen that the pressure fluctuates between -200 to 200 (Pa) at this location, which is a fairly large range. Power spectral density plot shows agreement with Goody model at low-frequency region. At higher-frequency region, the spectrum falls within the Goody and Efimtsov (1) models.

a) Pressure fluctuations

b) Single-point wall-pressure spectra

Figure 6-9. DDES-SA model simulation, pressure fluctuations and power spectral density plots at (0.25,0,0)

Figure 6-10 and Figure 6-11 illustrate the single-point wall-pressure spectrum scaled by inner and outer variables. In these figures, single-point wall-pressure spectra are compared with the Goody and Efimtsov (1) models. Employing outer variables allows a good collapse at mid-frequency range. The spectrum matches well with the Goody model even at low-frequency

range. Power spectral density plot normalized by the mixed layer variables is depicted in Figure 6-12. The intermediate ω^{-1} scaling associated with turbulent activity in the log layer is observed in this plot. It should be noted that near-wall turbulence is the main source of high-frequency wall-pressure fluctuations and low-frequency wall-pressure fluctuations are caused by outer flow turbulence [72]. Consequently, in case of focusing on simulating the collapse of the spectra at high-frequency region, inner flow variable scaling is recommended. Otherwise, simulating the collapse at low-frequency region can be carried out with outer and mixed variables scaling.

Figure 6-10. DDES-SA model simulation, power spectral density at (0.25,0,0) normalized by inner boundary layer variables

Figure 6-11. DDES-SA model simulation, power spectral density at (0.25,0,0) normalized by outer boundary layer variables

Figure 6-12. DDES-SA model simulation, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables

Figure 6-13 depicts spectrum slope predicted by DDES-SA model compared with Goody model. Mixed layer variables scaling is employed in this figure. As can be observed, the DDES-SA model result is in fairly good agreement with the Goody model's prediction.

Figure 6-13. DDES-SA model simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)

The RMS wall-pressure fluctuations obtained by the DDES-SA model are presented in Figure 6-14. The values are compared with some of the mean square pressure empirical models and the average value is in agreement with the Lowson model.

Figure 6-14. RMS wall-pressure fluctuations (numerical model: DDES-SA)

Figure 6-15 illustrates mean square wall-pressure values along the center line of the flat plate. The Kraichnan model is at the high end of the predictions, which is not in agreement with other models. The Lowson model is at the low end of the predictions. The DDES-SA results falls within the Lowson, and Lilley and Hodgson models.

Figure 6-15. Mean square wall-pressure values along the center line of flat plate (numerical model: DDES-SA)

It is interesting to compare the mean square pressure empirical models with the single-point wall-pressure models as well as numerical results. The mean square pressure can be found by integrating the single-point wall-pressure spectrum over all the frequencies. Unfortunately, developing analytical solution for single-point wall-pressure spectrum model is not feasible due to complex nature of the flowfield [116].

Figure 6-16 illustrates the comparison between some of the empirical/semi-empirical models with the DDES-SA model results. Mean square pressure values of different locations along the center line of flat plate are shown in this figure. Kraichnan and Lowson Models are added to plot as highest and lowest end of the mean square pressure empirical models. The

Efimtsov model (2) values are too far from the results of other models. The DDES-SA model results are in agreement with Lowson, Lilley and Hodgson, and Goody models.

Figure 6-16. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: DDES-SA)

6.3 RANS-SA Turbulence Modeling Results

Second simulation is carried out with the RANS-SA turbulence model. Grid system and operating conditions are similar to the DDES-SA simulation. The RANS-SA simulation predicts the turbulent boundary layer thickness at about 0.0055 (m), which is in good agreement with equation (6.1). Figure 6-17 shows boundary layer development.

Figure 6-17. RANS-SA model simulation of flat plate, boundary layer development

Figure 6-18 illustrates instantaneous view of pressure contours of flat plate with the RANS-SA model simulation. The pressure field shows the generated sound on the flat plate.

Figure 6-18. RANS-SA model simulation of flat plate, pressure contours

Velocity fluctuations at $x = 0.07(m)$ are shown in Figure . The fluctuations can be clearly observed within the boundary layer. At this plane the velocity fluctuates roughly between -3 to 3 (m/s), which shows a good range of fluctuations. The range of fluctuations for Y- and Z-components of velocity is small and around zero.

X-components of velocity fluctuations at
(0.07,0.0006,0.0)

Y-components of velocity fluctuations at
(0.07,0.0006,0.0)

Z-components of velocity fluctuations at (0.07,0.0006,0.0)

X-components of velocity fluctuations at
(0.07,0.0009,0.0)

Y-components of velocity fluctuations at
(0.07,0.0009,0.0)

Figure 6-19. RANS-SA model simulation, velocity fluctuations at different locations

Z-components of velocity fluctuations at (0.07,0.0009,0.0)

X-components of velocity fluctuations at
(0.07,0.0014,0.0)

Y-components of velocity fluctuations at
(0.07,0.0014,0.0)

Z-components of velocity fluctuations at (0.07,0.0014,0.0)

Figure 6-19. (continued)

X-components of velocity fluctuations at
(0.07,0.027,0.0)

Y-components of velocity fluctuations at
(0.07,0.027,0.0)

Z-components of velocity fluctuations at (0.07,0.027,0.0)

Figure 6-19. (continued)

Pressure fluctuations and power spectral density plots of the flat plate at $x = 0.25(m)$ are illustrated in Figure 6-20. At this location, the pressure fluctuates between -200 to 200 (Pa). As can be seen in Figure 6-20 (b), power spectral density fails at low-frequency region. However, in higher-frequency range the plot shows agreement with the Goody model.

Figure 6-21 and Figure 6-22 show the single-point wall-pressure spectrum scaled by inner and outer variables. In these figures, single-point wall-pressure spectra are compared with the Goody and Efimtsov (1) models. The numerical results fluctuate between Efimtsov (1) and

Goody models. At this location the spectrum is flat in the low-frequency range. Power spectral density plot normalized by the mixed layer variables is illustrated in Figure 6-23. The intermediate ω^{-1} scaling associated with turbulent activity in the log layer is observed in this plot. Because of the limitation on execution time and time increment, the numerical results are shown in very small portion of time. Therefore, it is hard to elaborate the difference between inner, outer, and mixed variables scaling.

a) Pressure fluctuations

b) Single-point wall-pressure spectra

Figure 6-20. RANS-SA model simulation, pressure fluctuations and power spectral density plots at (0.25,0,0)

Figure 6-21. RANS-SA model simulation, power spectral density at (0.25,0,0) normalized by inner boundary layer variables

Figure 6-22. RANS-SA model simulation, power spectral density at (0.25,0,0) normalized by outer boundary layer variables

Figure 6-23. RANS-SA model simulation, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables

Figure 6-24 shows spectrum slope predicted by RANS-SA model compared with Goody model. Mixed layer variables scaling is employed in this figure. It is observed that the RANS-SA model fails to predict the ω^2 slope in low frequency region.

Figure 6-24. RANS-SA model simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)

The RMS wall-pressure fluctuations obtained by the RANS-SA model are presented in Figure 6-25 and compared with three empirical mean square pressure models. As it is observed, the average value is in agreement with the Lowson model.

Figure 6-25. RMS wall-pressure fluctuations (numerical model: RANS-SA)

Figure 6-26 depicts mean square pressure values by different models along the center line of the flat plate. As can be seen, results of the RANS-SA model fluctuate between the Lowson and the Lilley and Hodgson models. However, at most of the locations the numerical values tend to the Lowson model.

Figure 6-26. Mean square wall-pressure values along the center line of flat plate (numerical model: RANS-SA)

Figure 6-27 illustrates the comparison between some of the empirical/semi-empirical models with the RANS-SA turbulence model results. Mean square pressure values of different locations along the center line of flat plate are shown in this figure. The RANS-SA model results are in agreement with the Lawson and Goody models.

Figure 6-27. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: RANS-SA)

6.4 DDES-SST Turbulence Modeling Results

Third simulation is carried out with the DDES-SST turbulence model. The same grid system and operating conditions are employed for the current simulation. Turbulent boundary layer thickness estimated by this model is about 0.0056 (m) that is in good agreement with equation (6.1). Figure 6-28 shows boundary layer development.

Figure 6-28. DDES-SST model simulation of flat plate, boundary layer development

Figure 6-29 shows instantaneous view of pressure contours of flat plate with the DDES-SST model simulation. Similar to previous investigated turbulence models, the pressure field shows the generated sound on the flat plate.

Figure 6-29. DDES-SST model simulation of flat plate, pressure contours

Velocity fluctuations at $x = 0.07(m)$ are illustrated in Figure . Similar to the results of previous simulations, the velocity fluctuations can be observed within the boundary layer. At this plane the velocity is fluctuating roughly between -3 to 3 (m/s), which is similar to the DDES-SA model results. The Y- and Z-components of velocity fluctuate between -1 to 1 (m/s), which shows a small range.

X-components of velocity fluctuations at
(0.07,0.0006,0.0)

Y-components of velocity fluctuations at
(0.07,0.0006,0.0)

Z-components of velocity fluctuations at (0.07,0.0006,0.0)

X-components of velocity fluctuations at
(0.07,0.0009,0.0)

Y-components of velocity fluctuations at
(0.07,0.0009,0.0)

Figure 6-30. DDES-SST model simulation, velocity fluctuations at different locations

Z-components of velocity fluctuations at (0.07,0.0009,0.0)

X-components of velocity fluctuations at
(0.07,0.0014,0.0)

Y-components of velocity fluctuations at
(0.07,0.0014,0.0)

Z-components of velocity fluctuations at (0.07,0.0014,0.0)

Figure 6-30. (continued)

X-components of velocity fluctuations at
(0.07,0.027,0.0)

Y-components of velocity fluctuations at
(0.07,0.027,0.0)

Z-components of velocity fluctuations at (0.07,0.027,0.0)

Figure 6-30. (continued)

Pressure fluctuations and power spectral density of the flat plate at (0.25,0,0) are illustrated in Figure 6-31. Similar to the DDES-SA and RANS-SA models results, pressure fluctuates between -200 to 200 (Pa) at $x = 0.25(m)$. As observed in Figure 6-31 (b), power spectral density fails at low-frequency region. However, in higher-frequency range the plot shows agreement with the Goody and Efimtsov (1) models.

a) Pressure fluctuations

b) Single-point wall-pressure spectra

Figure 6-31. DDES-SST model simulation, pressure fluctuations and power spectral density plots at (0.25,0,0)

Figure 6-32 and Figure 6-33 illustrate the single-point wall-pressure spectrum at $x = 0.25(m)$ scaled by inner and outer scaling variables. At this location the spectrum follows the Robertson model in the low-frequency range. Power spectral density plot normalized by the

mixed layer variables is illustrated in Figure 6-34. The intermediate ω^{-1} scaling associated with turbulent activity in the log layer is observed in this plot.

Figure 6-32. DDES-SST model simulation, power spectral density at (0.25,0,0) normalized by inner boundary layer variables

Figure 6-33. DDES-SST model simulation, power spectral density at (0.25,0,0) normalized by outer boundary layer variables

Figure 6-34. DDES-SST model simulation, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables

Figure 6-35 illustrates spectrum slope predicted by DDES-SST model compared with Goody model. Mixed layer variables scaling is employed in this figure. The DDES-SST model

shows poor result in low frequency range. The slope of the spectrum is not in agreement with the Goody model's prediction.

Figure 6-35. DDES-SST model simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)

The RMS wall-pressure fluctuations obtained by the DDES-SST model are presented in Figure 6-36. As it is observed, the average value is in agreement with the Lilley and Hodgson model.

Figure 6-36. RMS wall-pressure fluctuations (numerical model: DDES-SST)

Figure 6-37 depicts mean square pressure values by different models along the center line of the flat plate. As can be seen, results of the DDES-SST model vary between Lawson, Lilley and Hodgson, and Farabee and Casarella models. However, at most of the locations the numerical values tend to the Lawson model.

Figure 6-37. Mean square wall-pressure values along the center line of flat plate (numerical model: DDES-SST)

Figure 6-38 illustrates the comparison between empirical/semi-empirical models with the DDES-SST turbulence model results. Mean square pressure values of different locations along the center line of flat plate are shown in this figure. It is observed that the DDES-SST model results are in agreement with Lowson, Lilley and Hodgson, and Goody models.

Figure 6-38. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: DDES-SST)

6.5 RANS-SST Turbulence Modeling Results

In order to elaborate the effects of different turbulence models, another simulation is performed with the RANS-SST turbulence model. For consistency reason, the grid system and operating conditions are similar to the previously performed simulations. The boundary layer developed by the RANS-SST turbulence model is shown in Figure 6-39. The thickness of the turbulent boundary layer is 0.0056 (m).

Figure 6-39. RANS-SST model simulation of flat plate, boundary layer development

Figure 6-40 presents instantaneous view of pressure contours of flat plate with the RANS-SST model simulation. The radiated sound can be clearly observed in this figure.

Figure 6-40. RANS-SST model simulation of flat plate, pressure contours

Velocity fluctuations at $x = 0.07(m)$ are shown in Figure . The range of fluctuations is similar to the previous turbulence models simulations.

X-components of velocity fluctuations at
(0.07,0.0006,0.0)

Y-components of velocity fluctuations at
(0.07,0.0006,0.0)

Z-components of velocity fluctuations at (0.07,0.0006,0.0)

X-components of velocity fluctuations at
(0.07,0.0009,0.0)

Y-components of velocity fluctuations at
(0.07,0.0009,0.0)

Figure 6-41. RANS-SST model simulation, velocity fluctuations at different locations

Z-components of velocity fluctuations at (0.07,0.0009,0.0)

X-components of velocity fluctuations at
(0.07,0.0014,0.0)

Y-components of velocity fluctuations at
(0.07,0.0014,0.0)

Z-components of velocity fluctuations at (0.07,0.0014,0.0)

Figure 6-41. (continued)

X-components of velocity fluctuations at
(0.07,0.027,0.0)

Y-components of velocity fluctuations at
(0.07,0.027,0.0)

Z-components of velocity fluctuations at (0.07,0.027,0.0)

Figure 6-41. (continued)

Pressure fluctuations and power spectral density plots of the flat plate are shown in Figure 6-42. The pressure fluctuates between -200 to 200 (Pa). As can be observed, the RANS-SST turbulence model cannot predict the correct power spectral density at low-frequency region. In higher-frequency region, plot shows agreement with the Goody and Efimtsov (1) models.

a) Pressure fluctuations

b) Single-point wall-pressure spectra

Figure 6-42. RANS-SST model simulation, pressure fluctuations and power spectral density plots at (0.25,0,0)

Figure 6-43 to Figure 6-45 illustrate the single-point wall-pressure spectrum scaled by inner, outer, and mixed layer variables. As can be seen, the numerical spectrum does not follow the trend of the Goody or Efimtsov (1) models spectra slope in the low-frequency range. The

intermediate ω^{-1} scaling associated with turbulent activity in the log layer is clearly observed in these plots.

Figure 6-43. RANS-SST model simulation, power spectral density at (0.25,0,0) normalized by inner boundary layer variables

Figure 6-44. RANS-SST model simulation, power spectral density at (0.25,0,0) normalized by outer boundary layer variables

Figure 6-45. RANS-SST model simulation, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables

Figure 6-46 depicts spectrum slope predicted by RANS-SST model compared with Goody model. Mixed layer variables scaling is employed in this figure. As can be observed, this turbulence model cannot predict the Goody model's predicted slope in low frequency range.

Figure 6-46. RANS-SST model simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)

The RMS wall-pressure fluctuations obtained by the RANS-SST model at different locations along the center line of the flat plate are shown in Figure 6-47. The average value is in agreement with the Lilley and Hodgson model.

Figure 6-47. RMS wall-pressure fluctuations (numerical model: RANS-SST)

Mean square pressure values obtained by different models along the center line of the flat plate are presented in Figure 6-48. The values are in agreement with Lowson, Farabee and Casarella, and Lilley and Hodgson models. As can be observed, Kraichnan model values are much higher at all locations and ruled out.

Figure 6-48. Mean square wall-pressure values along the center line of flat plate (numerical model: RANS-SST)

Figure 6-49 shows the comparison between empirical/semi-empirical models with the RANS-SST turbulence model results. Mean square pressure values obtained by the RANS-SST model are in agreement with Goody, Lowson, and Lilley and Hodgson models.

Figure 6-49. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: RANS-SST)

6.6 DNS Results (MWWS Scheme)

In this section, results obtained by direct numerical simulation (MWWS scheme) are presented and discussed. The operating conditions are similar to LES/RANS simulations.

Velocity fluctuations at $x=0.07(m)$ are shown in Figure . As can be observed, the velocity fluctuations range is a little larger than LES/RANS simulations results. The inflow turbulence seeding has the key role in generating the fluctuations. It can be seen that fluctuations are captured in all directions.

X-components of velocity fluctuations at
(0.07,0.0006,0.0)

Y-components of velocity fluctuations at
(0.07,0.0006,0.0)

Z-components of velocity fluctuations at (0.07,0.0006,0.0)

X-components of velocity fluctuations at
(0.07,0.0009,0.0)

Y-components of velocity fluctuations at
(0.07,0.0009,0.0)

Figure 6-50. MWWS scheme, velocity fluctuations at different locations

Z-components of velocity fluctuations at (0.07,0.0009,0.0)

X-components of velocity fluctuations at (0.07,0.0014,0.0)

Y-components of velocity fluctuations at (0.07,0.0014,0.0)

Z-components of velocity fluctuations at (0.07,0.0014,0.0)

Figure 6-50. (continued)

X-components of velocity fluctuations at
(0.07,0.027,0.0)

Y-components of velocity fluctuations at
(0.07,0.027,0.0)

Z-components of velocity fluctuations at (0.07,0.027,0.0)

Figure 6-50. (continued)

Pressure fluctuations and power spectral density of the flat plate at $x=0.25(m)$ are depicted in Figure 6-51. The pressure fluctuates between -200 to 200 (Pa) at this location. The power spectral density plot shows very good agreement with Goody model at low-frequency region. The plot shows that the DNS scheme (MWS scheme) is capable of simulating the slope as of Goody's in the low-frequency region. In higher-frequency region, the numerical spectrum falls between the Goody and Efimtsov (1) model.

a) Pressure fluctuations

b) Single-point wall-pressure spectra

Figure 6-51. MWWS scheme, pressure fluctuations and power spectral density plots at (0.25,0,0)

Figure 6-52 to Figure 6-54 show the single-point wall-pressure spectra scaled by inner, outer, and mixed layer scaling variables, respectively. The numerical spectrum follows the Goody spectrum trend in the low-frequency region. In mid-frequency range the spectrum fluctuates between the Goody and Efimtsov (1) models.

Figure 6-52. MWWS scheme, power spectral density at (0.25,0,0) normalized by inner boundary layer variables

Figure 6-53. MWWS scheme, power spectral density at (0.25,0,0) normalized by outer boundary layer variables

Figure 6-54. MWWS scheme, power spectral density at (0.25,0,0) normalized by mixed boundary layer variables

Figure 6-55 shows spectrum slope predicted by MWWS scheme compared with Goody model. Mixed layer variables scaling is employed in this figure. As can be observed, the MWWS scheme result is in very good agreement with the Goody model's prediction.

Figure 6-55. MWWS scheme simulation, comparison of spectrum slope prediction in low frequency range at (0.25,0,0)

RMS wall-pressure fluctuations along the center line of the flat plate obtained by the MWWS scheme are shown in Figure 6-56. The average value is in agreement with the Lowson model.

Figure 6-56. RMS wall-pressure fluctuations (numerical model: MWWS scheme)

Mean square pressure values by different empirical/semi-empirical models are compared with the MWWS scheme results in Figure 6-57. It is observed that the MWWS scheme results are in agreement with the Lowson model at most of the locations.

Figure 6-57. Mean square wall-pressure values along the center line of flat plate (numerical model: MWWS scheme)

Figure 6-58 presents the mean square pressure at different locations along the center line of the flat plate obtained by several mean square pressure models, MWWS scheme, and single-point wall-pressure spectrum models. The numerical results are in agreement with the Goody and Lowson models.

Figure 6-58. Comparison of single-point wall-pressure spectrum models with mean square pressure models at different locations along the center line of flat plate (numerical model: MWWS scheme)

6.7 Comparison of the Results

In order to compare all aspects of different models with each other, computation time for each of the turbulence models is investigated. A summary of computation time is shown in Table 6-1. Results are presented for 50,000 iterations with 32 processors employed. As can be seen, the DDES-SA model is more efficient than other models. This table presents the flow solution CPU rate at each time-step and also at each time-step for each cell. The RANS-SA model is in the next rank and the last one is the DDES-SST model. The DDES-SST model shows more than two times the computation time of other models.

TABLE 6-1
COMPUTATION TIME OF NUMERICAL SCHEMES

Turbulence Model	Flow Solution Computing Time (Sec)	Flow Solution CPU Rate	
		(Sec/Time-step)	(Micro-Sec/Cell / Time-step)
RANS-SA	2.95702E+05	5.91404	6.68310
DDES-SA	2.60378E+05	5.20755	5.88474
DDES-SST	5.75400E+05	11.50800	13.00450
RANS-SST	3.06331E+05	6.12661	6.92331

Mean square wall-pressure models were also investigated in the current study. As discussed previously, the mean square pressure values predict the overall energy of the wall-pressure fluctuations within a turbulent boundary layer. Mean square wall-pressure values of flat plate with different models are shown in Table 6-2. As can be seen, empirical models provide a wide range of values and it is not clear which model provides the most accurate result. Numerical results are compared at two different locations. As observed, results of all numerical methods are in agreement with the Lowson model at these two locations.

TABLE 6-2
MEAN SQUARE WALL-PRESSURE VALUES

Model	Date	At X = 0.09 (m), (Pa ²)	At X = 0.25 (m), (Pa ²)
Kraichnan	1956	150574.4	97184.2
Lilley and Hodgson	1960	20114.05	20114.05
Bull	1967	32791.76	21164.56
Lowson	1968	10561.88	10561.88
Schewe	1983	32697.9	32697.9
Lauchle and Daniels	1987	35312.72	35312.72
Farabee and Casarella	1991	6427.077	17547.4
Lueptow	1995	45256.61	45256.61
DDES-SA	2011	12087	9724

TABLE 6-2 (continued)

Model	Date	At X = 0.09 (m), (Pa ²)	At X = 0.25 (m), (Pa ²)
RANS-SA	2011	13359	10924
DDES-SST	2011	10783	8710.2
RANS-SST	2011	19188	8280.4
MWWS	2011	11065	10708

The single-point wall-pressure spectrum models were also integrated to provide mean square wall-pressure values.

Table 6-3 presents the comparison of numerical methods with empirical/semi-empirical models. RANS-SA and DDES-SA turbulence models show agreement with Lowson, Lilley and Hodgson, and Goody models. Results of RANS-SST and DDES-SST turbulence models compare well with Lowson and Goody models and in some locations Farabee and Casarella, and Lilley and Hodgson models. The MWWS scheme results show agreement with Lowson and Goody models. As can be seen, in mean square pressure models, all the numerical methods agree well with Lowson and Lilley and Hodgson models. Among the single-point wall-pressure models, Goody model provides the best result for mean square pressure values. It should be noted that a recent experimental investigation performed by Miller [116] does not provide any conclusions on which mean square pressure model was the best.

TABLE 6-3

COMPARISON OF THE MEAN SQUARE WALL-PRESSURE VALUES BY DIFFERENT MODELS

Model	Mean Square Pressure Model	Single-Point Wall-Pressure Spectrum Model
DDES-SA	Lowson, Lilley and Hodgson	Goody
RANS-SA	Lowson, Lilley and Hodgson	Goody
DDES-SST	Lowson, Farabee and Casarella, Lilley and Hodgson	Goody

TABLE 6-3 (continued)

Model	Mean Square Pressure Model	Single-Point Wall-Pressure Spectrum Model
RANS-SST	Lowson, Farabee and Casarella, Lilley and Hodgson	Goody
MWWS	Lowson	Goody

The power spectral density is also investigated in all the numerical methods. As discussed in the previous section, power spectral density is the frequency response of a random or periodic signal. It shows where the average power is distributed as a function of frequency. The summary of power spectral density results by different methods is shown in

Table 6-4. As it is observed, mid-frequency range is predicted by Goody and Efimtsov (1) models in all the methods. However, results show that the slope in the low-frequency range is not predicted as Goody model. The reason can be short execution time of the cases and/or the capability of resolving scales of motion by different turbulence models. As can be seen, DDES-SA model and MWWS scheme can predict the slope in the low-frequency range as of ω^2 . Results obtained by the DDES-SA turbulence model show better agreement with Goody and Efimtsov (1) models than other turbulence models. Results obtained by direct numerical simulation shows the best agreement with Goody model in the low-frequency region and Goody and Efimtsov (1) models in the mid-frequency region.

TABLE 6-4

COMPARISON OF POWER SPECTRAL DENSITY BY DIFFERENT METHODS

Numerical Method	Low-frequency Range	Mid-Frequency Range
DDES-SA	Goody model	Goody and Efimtsov (1) models
RANS-SA	Robertson model	Goody and Efimtsov (1) models
DDES-SST	Robertson model	Goody and Efimtsov (1) models

TABLE 6-4 (continued)

Numerical Method	Low-frequency Range	Mid-Frequency Range
RANS-SST	Robertson model	Goody and Efimtsov (1) models
MWWS	Goody model	Goody and Efimtsov (1) models

As discussed previously, there are three scaling variables for developing single-point wall-pressure spectrum models: Inner, outer and mixed variables. Because of short execution time, all the variables showed similar results. However, in case of focusing on simulating the collapse of the spectra at high-frequency region, inner flow variable scaling is recommended. Otherwise, simulating the collapse at low-frequency region can be carried out with outer or mixed variables scaling.

CHAPTER 7

SUMMARY AND CONCLUSIONS

The main purpose of this research was to investigate wall-pressure fluctuations due to a turbulent boundary layer. As discussed previously, the core goal of aeroacoustics is predicting the noise radiations caused by an unsteady flowfield. Pressure fluctuations are the key source of noise generation within unsteady turbulent flow fields, which can be estimated by different methods. In the current effort, numerical approaches were employed to predict the pressure and velocity fluctuations. Previous efforts have proven that direct numerical simulation is a good candidate for predicting pressure fluctuations. Some turbulence models have also shown good agreement with the experimental results. In order to evaluate the accuracy of different numerical schemes, four turbulence models and a proposed DNS scheme (MWWS) were employed. The investigated turbulence models were: Reynolds-averaged Navier-Stokes Spalart-Allmaras (RANS-SA), delayed detached-eddy simulation based on RANS-SA (DDES-SA), Menter's shear stress transport (RANS-SST), and DDES-SST. Several empirical/semi-empirical models were also reviewed and compared with numerical results.

Capturing the generated sound caused by unsteady turbulent flow requires the use of high order accurate or optimized finite difference schemes, which have a small amount of artificial dissipation. Therefore, a numerical scheme was developed based on the WENO and weighted compact schemes. It was validated by several one- and two-dimensional benchmark problems. Both Euler and Navier-Stokes equations were considered as the governing equations. The fundamentals of the scheme in one- and two-dimensional flowfields were similar. However, some modifications were required for two-dimensional flowfields. Results of the one- and two-dimensional cases showed higher accuracy and good resolution in comparison with traditional

schemes. The three-dimensional DNS code (MWWS) was an extension of its two-dimensional version. Inflow turbulence seeding was employed to generate the turbulent flowfield.

Single-point wall-pressure spectrum models were used to show the distribution of the mean square pressure fluctuations with frequency. The employed empirical/semi-empirical models were: Robertson, Chase-Howe, Goody, Efimtsov (1), and Efimtsov (2). With the exception of the Goody model, which was developed for marine applications, other models were the results of high-speed data. Inner, outer and mixed boundary layer variables have been employed for single-point wall-pressure models generation. The frequency regions have been categorized to low, overlap, and high ranges. In low frequency region, results obtained by DDES-SA model and the MWWS scheme were in agreement with Goody model, while RANS-SA, RANS-SST, and DDES-SST turbulence models showed agreement with Robertson model. In High frequency region, all investigated numerical methods were in agreement with Goody and Efimtsov (1) models. Values of the Efimtsov (2) model were much higher than numerical results and other empirical/semi-empirical models. The spectral level of the Chase-Howe model was low and did not decay quickly at high frequencies.

Mean square wall-pressure models were also investigated in the current study. These models predict the overall energy of the wall-pressure fluctuations in a turbulent boundary layer. The employed models were: Kraichnan, Lilley and Hodgson, Bull, Lowson, Schewe, Lauchle and Daniels, Farabee and Casarella, and Lueptow. All these models were developed based on experimental efforts. The single-point wall-pressure spectrum models were also integrated to provide mean square wall-pressure values. One-equation turbulence models (RANS-SA and DDES-SA) showed agreement with the Lowson, Lilley and Hodgson, and Goody models. The results of two-equation models (RANS-SST and DDES-SST) were in agreement with the

Lowson, Farabee and Casarella, Lilley and Hodgson, and Goody models. MWWS results also compared well with the Lowson and Goody models. Among all investigated turbulence models, DDES-SA provided the most accurate results in association with slope prediction in low frequency range. Moreover, computation time of the DDES-SA model was shorter than other turbulence models. Between DDES-SA model and MWWS scheme, the MWWS scheme provided more accurate results, especially in the low-frequency region.

REFERENCES

LIST OF REFERENCES

- [1] Hoffmann, K. A. and Chiang, S. T., "*Computational Fluid Dynamics*," 4th ed. Vol. 3, Aug 2000.
- [2] Godunov, S. K., "A Difference Scheme for Numerical Solution of Discontinuous Solution of Hydrodynamic Equations," *Math. Sbornik*, Vol. 47, translated US Joint Publ. Res. Service, JPRS 7226, 1969, pp. 271-306.
- [3] Leer, B. V., "Towards the Ultimate Conservative Difference Scheme, V. A Second Order Sequel to Godunov's Method," *Journal of Computational Physics*, Vol. 32, 1979, pp. 101-136.
- [4] Roe, P. L., "Approximate Riemann Solver, Parameter Vectors and Difference Schemes," *Journal of Computational Physics*, Vol. 43, 1981, pp. 357-372.
- [5] Harten, A., "High Resolution Schemes for Hyperbolic Conservation Laws," *Journal of Computational Physics*, Vol. 49, 1983, pp. 357-393.
DOI:10.1006/jcph.1997.5713
- [6] Harten, A., Engquist, B., Osher, S., and Chakravarthy, S. R., "Uniformly High Order Accurate Essentially Non-oscillatory Schemes, III," *Journal of Computational Physics*, Vol. 71, 1987, pp. 231-303.
DOI: 10.1016/0021-9991(87)90031-3
- [7] Shu, C. W. and Osher, S., "Efficient Implementation of Essentially Non-Oscillatory Shock-Capturing Schemes," *Journal of Computational Physics*, Vol. 77, 1988, pp. 439-471.
- [8] Shu, C. W. and Osher, S., "Efficient Implementation of Essentially Non-Oscillatory Shock-Capturing Schemes, II," *Journal of Computational Physics*, Vol. 83, July 1989, pp. 32-78.
DOI: 10.1016/0021-9991(89)90222-2
- [9] Liu, X. D., Osher, S., and Chan, T., "Weighted Essentially Non-Oscillatory Schemes," *Journal of Computational Physics*, Vol. 115, 1994, pp. 200-212.
- [10] Jiang, G.-S. and Shu, C.-W., "Efficient Implementation of Weighted ENO Scheme," *Journal of Computational Physics*, Vol. 126, June 1996, pp. 202-228.
- [11] Yee, H. C., Sandham, N. D., and Djomehri, M. J., "Low Dissipative High Order Shock Capturing Methods Using Characteristic-Based Filters," *Journal of Computational Physics*, Vol. 150, 1999, pp. 199-238.

- [12] Oliveria, M., Lu, P., Liu, X., and Liu, C., "Universal High Order Subroutine with New Shock Detector for Shock Boundary Layer Interaction," in 47th AIAA Aerospace Sciences Meeting Including The New Horizons Forum and Aerospace Exposition, AIAA 2009-1139, Jan 5-8, 2009, Orlando, FL.
- [13] Xie, P. and Liu, C., "Weighted Compact and Non-Compact Scheme for Shock Tube and Shock Entropy Interaction," in 45th AIAA Aerospace Sciences Meeting and Exhibit, AIAA 2007-509, Jan 8-11, 2007, Reno, NV.
- [14] Lele, S. K., "Compact Finite Difference Schemes with Spectral-like Resolution," *Journal of Computational Physics*, Vol. 103, 1992, pp. 16-42.
- [15] Kim, J. W. and Lee, D. J., "Optimized Compact Finite Difference Schemes with Maximum Resolution," *AIAA Journal*, Vol. 34, No. 5, May 1996, pp. 887-893.
- [16] Deng, X. and Maekawa, H., "A Uniform Fourth-Order Compact Scheme for Discontinuities Capturing," in AIAA, Fluid Dynamics Conference, 27th, AIAA Meeting Papers on Disc, A9636454, AIAA Paper 96-1974, June 17-20, 1996, New Orleans, LA.
- [17] Deng, X. and Mao, M., "Weighted Compact High-Order Nonlinear Schemes for the Euler Equations," in 13th AIAA Computational Fluid Dynamics Conference, AIAA 1997-1941, 1997, Snowmass.
- [18] Shu, C.-W., "Essentially Non-Oscillatory and Weighted Essentially Non-Oscillatory Schemes for Hyperbolic Conservation Laws," NASA/CR-97-206253, ICASE Report No. 97-65, Nov 1997.
- [19] Hixon, R., "A New Class of Compact Schemes," in 36th Aerospace Sciences Meeting and Exhibit, AIAA-1998-367, 12-15 Jan, 1998, Reno, NV.
- [20] Jiang, L., Shan, H., and Liu, C., "Weighted Compact Scheme for Shock Capturing," *International Journal of Computational Fluid Dynamics*, Vol. 15, 2001, pp. 147-155.
- [21] Deng, X., "High-Order Accurate Dissipative Weighted Compact Nonlinear Schemes," *Science in China*, Vol. 45, No. 3, March 2002, pp. 356-370.
- [22] Sengupta, T. K., Guntaka, A., and Dey, S., "Navier–Stokes Solution by New Compact Scheme for Incompressible Flows," *Journal of Scientific Computing*, Vol. 21, No. 3, Dec 2004, pp. 269-282.
- [23] Mawlood, M. K., et al., "Solution of Navier-Stokes Equations by Fourth-Order Compact Schemes and AUSM Flux Splitting," *International Journal of Numerical Methods for Heat and Fluid Flow*, Vol. 16, May 2005, pp. 107-120.
DOI: 10.1108/09615530610636982

- [24] Liu, C., Jiang, L., Visbal, M., and Xie, P., "Smart Weighted Compact Scheme for Shock Tube and Shock-Entropy Interaction," in 36th AIAA Fluid Dynamics Conference and Exhibit, AIAA 2006-3708, June 5-8, 2006, San Francisco, CA.
- [25] Xie, P., Jiang, L., and Liu, C., "Weighted Compact Scheme and Smart-Filter for 2-D Unsteady Shock/Boundary Layer Interaction," in 44th AIAA Aerospace Science Meeting and Exhibit, 9-12 Jan, 2006, Reno, NV.
- [26] Shen, Y. and Yang, G., "Hybrid Finite Compact-WENO Schemes for Shock Calculation," *International Journal for Numerical Methods in Fluids*, Vol. 53, 2007, pp. 531-560.
DOI: 10.1002/fld.1286
- [27] Xin, L., Xiaogang, D., and Meiliang, M., "High-Order Behaviors of Weighted Compact Fifth-Order Nonlinear Schemes," *AIAA Journal*, Vol. 45, No. 8, Aug 2007.
DOI: 10.2514/1.23797
- [28] Su, J., Oliveira, M., Xie, P., and Liu, C., "Error Analysis for Weighted Higher Order Compact Finite Difference Scheme," *Applied Mathematical Sciences*, Vol. 1, No. 58, 2007, pp. 2863-2881.
- [29] Nonomura, T., Iizuka, N., and Fujii, K., "Increasing Order of Accuracy of Weighted Compact Non-Linear Scheme," in 45th AIAA Aerospace Sciences Meeting and Exhibit, AIAA 2007-893, Jan 8-11, 2007, Reno, NV.
- [30] Willmarth, W. W. and Wooldridge, C. E., "Measurements of the Fluctuating Pressure at the Wall beneath a Thick Turbulent Boundary Layer," *Journal of Fluid Mechanics*, Vol. 14, April 1962, pp. 187-210.
DOI: 10.1017/S0022112062001160
- [31] Williams, J. E. F. and Lyon, R. H., "The sound Radiated from Turbulent Flows near Flexible Boundaries," *Bolt Beranek and Newman Inc.*, Report No. 1054, August 1963.
- [32] Corcos, G. M., "Resolution of Pressure in Turbulence," *The Journal of Acoustical Society of America*, Vol. 35, No. 2, February 1963, pp. 192-199.
- [33] Corcos, G. M., "The Structure of the Turbulent Pressure Field in Boundary-Layer Flows," *Journal of Fluid Mechanics*, Vol. 18, No. 3, 1964, pp. 353-378.
DOI: 10.1017/S002211206400026X
- [34] Willmarth, W. W. and Roos, F. W., "Resolution and Structure of the Wall Pressure Field beneath a Turbulent Boundary Layer," *Journal of Fluid Mechanics*, Vol. 22, July 1964, pp. 81-94.

- [35] Williams, J. E. F., "Surface-Pressure Fluctuations Induced by Boundary-Layer Flow at Finite Mach Number," *Journal of Fluid Mechanics*, Vol. 22, No. 3, March 2006, pp. 507-519.
DOI: 10.1017/S0022112065000939
- [36] Chase, D. M., "Turbulent-Boundary-Layer Pressure Fluctuations and Wavenumber Filtering by Nonuniform Spatial Averaging," *The Journal of Acoustical Society of America*, Vol. 46, No. 5 (Part 2), 1969, pp. 1350-1365.
- [37] Mellen, R. H., "Wave-Vector Filter Analysis of Turbulent Flow," *The Journal of Acoustical Society of America*, Vol. 95, No. 3, March 1994, pp. 1671-1673.
- [38] Efimtsov, B. M., "Characteristics of the Field of Turbulent Wall Pressure Fluctuations at Large Reynolds Numbers", *Soviet Physics Acoustics*, Vol. 28, No. 4, July-August 1982, pp. 289-292.
- [39] Schewe, G., "On the Structure and Resolution of Wall-Pressure Fluctuations Associated with Turbulent Boundary-Layer Flow," *Journal of Fluid Mechanics*, Vol. 134, 1983, pp. 311-328.
- [40] Chandiramani, K. L., "Response of Underwater Structures to Convective Component of Flow Noise," *The Journal of Acoustical Society of America*, Vol. 73, No. 3, March 1983, pp. 835-839.
- [41] Kim, J., "On the Structure of Pressure Fluctuations in Simulated Turbulent Channel Flow," *Journal of Fluid Mechanics*, Vol. 205, 1989, pp. 421-451.
DOI: 10.1017/S0022112089002090
- [42] Mellen, R. H., "On Modeling Convective Turbulence," *The Journal of Acoustical Society of America*, Vol. 88, No. 6, December 1990, pp. 2891-2893.
- [43] Choi, H. and Moin, P., "On the Space-Time Characteristics of Wall-Pressure Fluctuations," *Journal of Physics of Fluids*, Vol. 2, August 1990, pp. 1450-1460.
DOI: 10.1063/1.857593
- [44] Chase, D. M., "The Wave-Vector-Frequency Spectrum of Pressure on a Smooth Plane in Turbulent Boundary-Layer Flow at Low Mach Number," *The Journal of Acoustical Society of America*, August 1991, pp. 1032-1040.
- [45] Farabee, T. and Casarella, M. J., "Spectral Features of Wall Pressure Fluctuations beneath Turbulent Boundary Layers," *Journal of Physics of Fluids*, Vol. 3, No. 10, October 1991, pp. 2410-2420.
DOI: 10.1063/1.858179

- [46] Keith, W. L., Hurdis, D. A., and Abraham, B. M., "A Comparison of Turbulent Boundary Layer Wall-Pressure Spectra," *Journal of Fluids Engineering*, Vol. 114, September 1992, pp. 338-347.
- [47] Lueptow, R. M., "Transducer Resolution and the Turbulent Wall Pressure Spectrum," *The Journal of Acoustical Society of America*, Vol. 97, No. 1, January 1995, pp. 370-378.
- [48] Manoha, E., "The Wavenumber-Frequency Spectrum of the Wall Pressure Fluctuations beneath a Turbulent Boundary Layer " in 2nd AIAA and CEAS, Aeroacoustics Conference, AIAA-1996-1758, May 6-8, 1996, State College, PA.
- [49] Bull, M. K., "Wall Pressure Fluctuations beneath Turbulent Boundary Layers: Some Reflections on Forty Years of Research," *Journal of Sound and Vibration*, Vol. 190, No. 3, 1996, pp. 299-315.
- [50] Sawada, H., Kohno, T., and Kunimasu, T., "Pressure Fluctuation Measurements in the NAL 0.2-m Supersonic Wind Tunnel," in 36th Aerospace Sciences Meeting and Exhibit, AIAA-1998-636 Jan. 12-15, 1998, Reno. NV.
- [51] Abraham, B. M. and Keith, W. L., "Direct Measurements of Turbulent Boundary Layer Wall Pressure Wavenumber-Frequency Spectra," *Journal of Fluids Engineering*, Vol. 120, March 1998, pp. 29-39.
- [52] Efimtsov, B. M., Kozlov, N. M., Kravchenko, S. V., and Andersson, A. O., "Wall Pressure-Fluctuation Spectra at Small Forward-Facing Steps " in 5th AIAA/CEAS Aeroacoustics Conference and Exhibit, Collection of Technical Papers. Vol. 2 (A99-27801 06-71), AIAA-99- 1964, May 10-12, 1999, Bellevue, WA.
- [53] Efimtsov, B. M., Kozlov, N. M., Kravchenko, S. V., and Andersson, A. O., "Wall Pressure-Fluctuation Spectra at Small Backward-Facing Steps " in 6th AIAA/CEAS, Aeroacoustics Conference and Exhibit (21st AIAA Aeroacoustics Conference), AIAA 2000-2053 , June 12-14, 2000, Seattle, WA.
- [54] Smol'yakov, A. V., "Calculation of the Spectra of Pseudosound Wall-Pressure Fluctuations in Turbulent Boundary Layers," *Journal of Acoustical Physics*, Vol. 46, No. 3, 2000, pp. 342-347.
- [55] Tkachenko, V. M., "Conditions of Similarity of the Cross-Spectra of Pseudosound Pressures in Turbulence," *Journal of Acoustical Physics*, Vol. 46, No. 3, 2000, pp. 348-356.
- [56] Wang, M., "Dynamic Wall Modeling for LES of Complex Turbulent Flows," *Annual Research Briefs, CTR, Stanford University*, 2000, pp. 241-250.

- [57] Smol'yakov, A. V., "Noise of a Turbulent Boundary Layer Flow over Smooth and Rough Plates at Low Mach Numbers," *Journal of Acoustical Physics*, Vol. 47, No. 2, 2001, pp. 218–225.
- [58] Efimtsov, B. M., Rizzi, S., Andersson, A. O., and Andrianov, E. V., "Influence of Small Steps on Wall Pressure Fluctuation Spectra Measured on Tu-144LL Flying Laboratory " in 8th AIAA/CEAS Aeroacoustics Conference and Exhibit, AIAA-2002-2605, June 17-19, 2002, Breckenridge, Colorado.
- [59] Kim, J., Choi, J., and Sung, H. J., "Relationship between Wall Pressure Fluctuations and Streamwise Vortices in a Turbulent Boundary Layer," *Journal of Physics of Fluids*, Vol. 14, No. 2, 2002, pp. 898-901.
DOI: 10.1063/1.1429964
- [60] Leclercq, D. J. J. and Bohineust, X., "Investigation and Modeling of the Wall Pressure Field beneath a Turbulent Boundary Layer at Low and Medium Frequencies," *Journal of Sound and Vibration*, Vol. 257, No. 3, 2002, pp. 477–501.
DOI: 10.1006/jsvi.5049
- [61] Efimtsov, B. M., Kozlov, N. M., and Andersson, A. O., "Wall Pressure Fluctuations in a Local Supersonic Region," in 9th AIAA/CEAS Aeroacoustics Conference and Exhibit, AIAA 2003-3221, 12-14 May 2003, Hilton Head, South Carolina.
- [62] Efimtsov, B. M., Zosimov, V. V., Romashov, A. V., and Rybak, S. A., "Correlation of Pressure Fluctuations with Tangential Stresses in a Turbulent Boundary Layer," *Journal of Acoustical Physics*, Vol. 49, No. 1, 2003, pp. 113–115.
- [63] Smol'yakov, A. V. and Tkachenko, V. M., "A Note on "Investigation and Modeling of the Wall Pressure Field beneath a Turbulent Boundary Layer at Low and Medium Frequencies", " *Journal of Sound and Vibration*, Vol. 274, 2004, pp. 403-406.
DOI: 10.1016/j.jsv.2003.09.038
- [64] Goody, M., "Empirical Spectral Model of Surface Pressure Fluctuations," *AIAA Journal*, Vol. 42, No. 9, September 2004, pp. 1788-1794.
- [65] Lee, Y. T., Blake, W. K., and Farabee, T., "Prediction of Wall Pressure Spectrum using a RANS Calculation," in 43rd AIAA Aerospace Sciences Meeting and Exhibit, AIAA 2005-802, 10 -13 January, 2005, Reno, Nevada.
- [66] Efimtsov, B. M., Kozlov, N. M., and Andersson, A. O., "Wall Pressure Fluctuations in the Interaction Region of Shock and Boundary Layer," in 43rd AIAA Aerospace Sciences Meeting and Exhibit, AIAA 2005-801, 10 - 13 January, 2005, Reno, Nevada.
- [67] Efimtsov, B. M. and Lazarev, L. A., "Analysis of the Sound Transmission Loss of Panels with Resonant Systems on the Basis of Equivalent Representations," *Journal of Acoustical Physics*, Vol. 51, No. 3, 2005, pp. 300-305.

- [68] Smol'yakov, A. V., "A New Model for the Cross Spectrum and Wavenumber–Frequency Spectrum of Turbulent Pressure Fluctuations in a Boundary Layer," *Journal of Acoustical Physics*, Vol. 52, No. 3, 2006, pp. 331–337.
DOI: 10.1134/S1063771006030146
- [69] Peltier, L. J. and Hambric, S. A., "Estimating Turbulent-Boundary-Layer Wall-Pressure Spectra from CFD RANS Solutions," *Journal of Fluids and Structures*, Vol. 23, April 2007, pp. 920-937.
DOI: 10.1016/j.jfluidstructs.2007.01.003
- [70] Goody, M., Farabee, T., and Lee, Y. T., "Unsteady Pressures on the Surface of a Ship Hull," in Proceedings of IMECE2007, 2007 ASME International Mechanical Engineering Congress and Exposition, November 11-15, 2007, Seattle, Washington.
- [71] Dietiker, J. F. and Hoffmann, K. A., "Predicting Wall Pressure Fluctuation over a Backward-Facing Step Using Detached Eddy Simulation," *Journal of Aircraft*, Vol. 46, No. 6, Nov–Dec 2009, pp. 2115-2120.
DOI: 10.2514/1.43912
- [72] Lee, Y. T., Farabee, T., and Blake, W. K., "Turbulence Effects of Wall-Pressure Fluctuations for Reattached Flow," *Journal of Computers and Fluids*, Vol. 38, No. 5, May 2009, pp. 1033-1041.
DOI: 10.1016/j.compfluid.2008.01.012
- [73] Pirozzoli, S. and Bernardini, M., "Wall Pressure Fluctuations in Transonic Shock/Boundary Layer Interaction," in IUTAM Symposium on Computational Aero-Acoustics for Aircraft Noise Prediction, 2010.
- [74] Weirs, V. G. and Candler, G. V., "Optimization of Weighted ENO Schemes for DNS of Compressible Turbulence," in 13th AIAA Computational Fluid Dynamics Conference, AIAA 1997-1940, June 29-July 2, 1997, Snowmass Village, CO.
- [75] Martín, M. P., Taylor, E. M., Wu, M., and Weirs, V. G., "A Bandwidth-Optimized WENO Scheme for the Effective Direct Numerical Simulation of Compressible Turbulence," *Journal of Computational Physics*, Vol. 220, Issue 1, Dec 2006, pp. 270-289.
- [76] Adams, N. A. and Shariff, K., "A High-Resolution Hybrid Compact-ENO Scheme for Shock-Turbulence Interaction Problems," *Journal of Computational Physics*, Vol. 127, Aug. 1996, pp. 27 - 51.
DOI: 10.1006/jcph.1996.0156
- [77] Pirozzoli, S., "Conservative Hybrid Compact-WENO Schemes for Shock-Turbulence Interaction," *Journal of Computational Physics*, Vol. 178, May 2002, pp. 81-117.
DOI: 10.1006/jcph.2002.7021

- [78] Ren, y. x., Liu, m., and Zhang, h., "A Characteristic-Wise Compact-WENO Scheme for Solving Hyperbolic Conservation Laws," *Journal of Computational Physics*, Vol. 192, Dec. 2003, pp. 365-386.
DOI: 10.1016/j.jcp.2003.07.006
- [79] Hill, D. J. and Pullin, D. I., "Hybrid Tuned Center-Difference-WENO Method for Large-Eddy Simulation in the Presence of Strong Shocks," *Journal of Computational Physics*, Vol. 194, March 2004, pp. 435-450.
DOI: 10.1016/j.jcp.2003.07.032
- [80] Kim, D. and Kwon, J. H., "A Higher-Order accurate Hybrid Scheme Using a Central Flux Scheme and a WENO Scheme for Compressible Flowfield Analysis," *Journal of Computational Physics*, Vol. 210, Dec. 2005, pp. 554-583.
DOI: 10.1016/j.jcp.2005.04.023
- [81] Henrick, A. K., Aslam, T. D., and Powers, J. M., "Mapped weighted essentially non-oscillatory schemes: Achieving optimal order near critical points," *Journal of Computational Physics*, Vol. 207, 2005, pp. 542-567.
DOI:10.1016/j.jcp.2005.01.023
- [82] Hoffmann, K. A. and Chiang, S. T., "*Computational Fluid Dynamics*," 4th ed. Vol. 1, Aug 2000.
- [83] Harten, A., "The Artificial Compression Method for Computation of Shocks and Contact Discontinuities: III. Self-Adjusting Hybrid Schemes," *Mathematics of Computation*, Vol. 32, No. 142, Apr. 1978, pp. 363-389.
- [84] Jiang, G. S. and Shu, C. W., "Efficient Implementation of Weighted ENO Schemes," *Journal of Computational Physics*, Vol. 126, 1996, pp. 202-228.
- [85] Hoffmann, K. A. and Chiang, S. T., "*Computational Fluid Dynamics*," 4th ed. Vol. 2, Aug 2000.
- [86] Laney, C. B., "*Computational Gasdynamics*," Cambridge University Press, 1998.
- [87] Davoudzadeh, F., McDonald, H., and Thompson, B. E., "Accuracy Evaluation of Unsteady CFD Numerical Schemes by Vortex Preservation," *Journal of Computers and Fluids*, Vol. 24, No. 8, 1995, pp. 883-895.
- [88] Gerritsen, M., "*Designing an Efficient Solution Strategy for Fluid Flows*," PhD Thesis, SCCM Program, Stanford University, Oct. 1996.
- [89] Arshed, G. M. and Hoffmann, K. A., "Implementation of Improved WENO Scheme to Higher Dimensions in Relation to Shock-Turbulence Interaction," in 19th AIAA Computational Fluid Dynamics, AIAA 2009-3654, 22-25 June, 2009, San Antonio, TX.

- [90] Lo, S. C., Blaisdell, G. A., and Lyrintzis, A. S., "High-order Shock Capturing Schemes for Turbulence Calculations," in 45th AIAA Aerospace Sciences Meeting and Exhibit, AIAA 2007-827, Jan 8-11, 2007, Reno, NV.
- [91] Inoue, O. and Hattori, Y., "Sound Generation by Shock-Vortex Interactions," *Journal of Fluid Mechanics*, Vol. 380, 1999, pp. 81-116.
- [92] Grasso, F. and Pirozzoli, S., "Shock Wave-Vortex Interactions: Shock and Vortex Deformations, and Sound Production," *Journal of Theoretical and Computational Fluid Dynamics*, Vol. 13, 2000, pp. 421-456.
- [93] Ellzey, J. L. and Henneke, M. R., "The Shock-Vortex Interaction: The Origins of the Acoustic Wave," *Journal of Fluid Dynamics Research*, Vol. 21, 1997, pp. 171-184.
- [94] Bogey, C., Cacqueray, N. D., and Bailly, C., "Self-Adjusting Shock-Capturing spatial Filtering for High-Order Non-Linear Computations," in 14th AIAA/CEAS Aeroacoustics Conference (29th AIAA Aeroacoustics Conference), AIAA 2008-2968, 5-7 May, 2008, Vancouver, BC, Canada.
- [95] Dietiker, J.-F. and Hoffmann, K. A., "Computations of Turbulent Flow over a Backstep," in 37th AIAA Fluid Dynamics Conference and Exhibit, AIAA 2007-3862, 25-28 June, 2007, Miami, Florida.
- [96] Woodward, P. and Colella, P., "The Numerical Simulation of Two-Dimensional Fluid Flow with Strong Shocks," *Journal of Computational Physics*, Vol. 54, 1984, pp. 115-173.
- [97] Shi, J., Hu, C., and Shu, C.-W., "A Technique of Treating Negative Weights in WENO Schemes," *Journal of Computational Physics*, Vol. 175, 2002, pp. 108-127.
DOI:10.1006/jcph.2001.6892
- [98] Shu, C. W., "High Order Finite Difference and Finite Volume WENO Schemes and Discontinuous Galerkin Methods for CFD," *NASA/CR-2000-210865, ICASE Report No. 2001-11*, May 2001.
- [99] Shi, J., Zhang, Y. T., and Shu, C. W., "Resolution of High Order WENO Schemes for Complicated Flow Structures," *Journal of Computational Physics*, Vol. 186, 2003, pp. 690-696.
DOI: 10.1016/S0021-9991(03)00094-9
- [100] Xu, Z. and Shu, C. W., "Anti-Diffusive Flux Corrections for High Order Finite Difference WENO Schemes," *Journal of Computational Physics*, Vol. 205, Issue 2, May 2005, pp. 458-485.
DOI: 10.1016/j.jcp.2004.11.014

- [101] Zaide, D. W. and Roe, P. L., "Entropy-Based Mesh Refinement, II: A New Approach to Mesh Movement," in 19th AIAA Computational Fluid Dynamics, AIAA 2009-3791, 22-25 June, 2009, San Antonio, Texas.
- [102] Kuhl, A. L., et al., "Turbulent Wall Jet in a Mach Reflection Flow," *Progress in Aeronautics and Astronautics*, Vol. 13, 1990, pp. 201-232.
- [103] Colella, P. and Glaz, H. M., "Numerical Calculation of Complex Shock Reflections in Gases," *Proceedings, 9th International Conference on Numerical Methods in Fluid Dynamics, Saclay, France, Springer Lecture Notes in Physics*, Vol. 218, 1984, pp. 154-158.
- [104] Glaz, H. M., Colella, P., Glass, I. I., and Deschambault, R. L., "A Numerical Study of Oblique Shock-Wave Reflections with Experimental Comparisons," *Proceedings, Royal Society of London A*, Vol. 398, 1985, pp. 117-140.
- [105] Li, H. and Ben-Dor, G., "Analysis of Double-Mach-Reflection Wave Configurations with Convexly Curved Mach Stems," *Journal of Shock Waves*, Vol. 9, 1999, pp. 319-326.
- [106] Sandham, N. D. and Yee, H. C., "Performance of Low Dissipative High Order Shock-Capturing Schemes for Shock-Turbulence Interactions," *NASA/CR 208236, RIACS Technical Report 98.10, NASA Ames Research Center*, April 1998.
- [107] Corcos, G. M. and Sherman, F. S., "The Mixing Layer: Deterministic Models of a Turbulent Flow. Part 1. Introduction and the Two-Dimensional Flow," *Journal of Fluid Mechanics*, Vol. 139, 1984, pp. 29-65.
- [108] Yee, H. C., Vinokur, M., and Djomehri, M. J., "Entropy Splitting and Numerical Dissipation," *Journal of Computational Physics*, Vol. 162, July 2000, pp. 33-81.
DOI: 10.1006/jcph.2000.6517
- [109] Schetz, J. A., "Boundary Layer Analysis, Revised," Reston, Virginia, American Institute of Aeronautics and Astronautics, Inc., 2010.
- [110] Schlichting, H. and Gersten, K., "Boundary Layer," New York, Springer, 2000.
- [111] Crocker, M. J., "Handbook of Noise and Vibration Control," Hoboken, New Jersey, John Wiley and Sons, Inc, 2007.
- [112] Fleming, N. J., "The Silencing of Aircraft," *Journal of the Royal Aeronautical Society*, 50, 1946, pp. 639-676.
- [113] Hayden, R. E., Murray, B. S., and Theobald, M. A., "Boundary Layer Induced Noise in the Interior of Aircraft," *NASA CR-172152*, 1983.

- [114] Berland, J. and Gloerfelt, X., "Investigation of Noise Radiation from a Turbulent Boundary Layer," in 14th AIAA/CEAS Aeroacoustics Conference (29th AIAA Aeroacoustics Conference), 5-7 May, 2008, British Columbia, Canada.
- [115] Lighthill, M. J., "On Sound Generated Aerodynamically. I. General Theory," *Proceeding the Royal of Society*, London, Vol. 211, No. 1107, 1952, pp. 564-587.
- [116] Miller, T. S., "Turbulent Boundary Layer Models for Acoustic Analysis," Dissertation, Aerospace Engineering department, Wichita State university, Wichita, KS, USA, 2011.
- [117] Miller, T. S., Gallman, J. M., and Moeller, M. J., "Review of Turbulent Boundary Layer Models for Acoustic Analysis," in 49th AIAA Aerospace Sciences Meeting, 4-7 Jan, 2011, Orlando, Florida.
- [118] Newland, D. E., "An Introduction to Random Vibrations, Spectral and Wavelet Analysis," Third Ed., Longman Scientific and Technical, Essex, UK, 1993.
- [119] Robertson, J. E., "Prediction of In-Flight Fluctuating Pressure Environments Including Protuberance Induced Flow," *Wyle Laboratories Research Staff Report*, WR71-10, March 1971.
- [120] Lowson, M. V., "Prediction of Boundary Layer Pressure Fluctuations," *Wyle Laboratories Research Staff Report*, WR67-15, Oct. 1967.
- [121] Efimtsov, B. M., "The Prediction of the Pressure Fluctuation Field Characteristics of the TBL," *Contract No. BTRC-101F, Document No. BTRC101F-5/95*, 1995.
- [122] Howe, M. S., "Acoustics of Fluid-Structure Interactions," *Cambridge Univ. Press, Cambridge, England, U.K.*, P. 208, 1998.
- [123] Chase, D. M., "Modeling the Wavevector-Frequency Spectrum of Turbulent Boundary Layer Wall Pressure," *Journal of Sound and Vibration*, Vol. 70, No. 1, 1980, pp. 29-67.
- [124] Bradshaw, P., "Inactive Motion and Pressure Fluctuations in Turbulent Boundary Layers," *Journal of Fluid Mech.*, Vol. 30, 1967, pp. 241-258.
- [125] Blake, W. K., "Mechanics of Flow-Induced Sound and Vibration," Academic Press, Newyork, Chap. 7, 1986.
- [126] Goody, M., "An Experimental Investigation of Pressure Fluctuations in Three-Dimensional Turbulent Boundary Layers," Ph.D. Dissertation, Aerospace Engineering, Virginia Polytechnic Institute and State University, 1999.
- [127] Ffowcs-Williams, J. E., "Surface-Pressure Fluctuations Induced by Boundary-Layer Flow at Finite Mach Number," *Journal of Fluid Mechanics*, Vol. 22, No. 3, 1965, pp. 507-519. DOI: 10.1017/S0022112065000939

- [128] Panton, R. L. and Linebarger, J. H., "Wall Pressure Spectra Calculations for Equilibrium Boundary Layers," *Journal of Fluid Mechanics*, Vol. 65, No. 2, John H. Linebarger, pp. 261-287.
DOI: 10.1017/S0022112074001388
- [129] Kraichnan, R. H., "Pressure Fluctuations in Turbulent Flow Over a Flat Plate," *Journal of the Acoustical Society of America*, Vol. 28, No. 1, May 1956, pp. 378-390.
- [130] Lilley, G. M. and Hodgson, T. H., "On Surface Pressure Fluctuations in Turbulent Boundary Layers," *Advisory Group for Aeronautical Research and Development (AGARD), Report 276*, April 1960.
- [131] Bull, M. K., "Wall-Pressure Fluctuations Associated with Subsonic Turbulent Boundary Layer Flow," *Journal of Fluid Mech.*, Vol. 28, 1967, pp. 719-754.
- [132] Lauchle, G. C. and Daniels, M. A., "Wall-Pressure Fluctuations in Turbulent Pipe Flow," *Journal of Physics of Fluids*, Vol. 30, No. 10, Oct. 1987, pp. 3019-3024.
- [133] <http://www.cobaltcfd.com/index.php/site/software/cobalt>, Sept. 2011
- [134] Strang, W. Z., Tomaro, R. F., and Grismer, M. J., "The Definition Methods of Cobalt60: a Parallel, Implicit, Unstructured Euler/Navier-Stokes Flow Solver," in, AIAA 99-0786, January, 1999.
- [135] Forsythe, J. R., Strang, W. Z., and Hoffmann, K. A., "Validation of Several Reynolds-Averaged Turbulence Models in a 3D Unstructured Grid Code," in Fluids 2000 Conference and Exhibit, AIAA 2000-2552, 19-22 June, 2000, Denver, Colorado.
- [136] Viswanathan, A. K., Squires, K. D., and Forsythe, J. R., "Detached Eddy Simulation of the Flow over an Axisymmetric Cavity," in 41st Aerospace Sciences Meeting and Exhibit, AIAA 2003-265, 6-9 January 2003, 2003, Reno, Nevada.
- [137] Boydston, J. D. and Squires, K. D., "Detached Eddy Simulation of High Reynolds Number Flow Over a Rectangular Cavity," in 46th AIAA Aerospace Sciences Meeting and Exhibit, AIAA 2008-606, 7-10 January, 2008, Reno, Nevada.
- [138] *Cobalt User's Manual, Ver. 5.0, Cobalt Solution, LLC, Springfield, OH, 2010.*
- [139] Spalart, P. and Allmaras, S. R., "A One-Equation Turbulence Model for Aerodynamic Flows," *La Recherche Aerospatiale*, Vol. 1, 1994, pp. 5-21.
- [140] Deck, S., Duveau, P., d'Espiney, P., and Guillen, P., "Development and Application of Spalart-Allmaras One Equation Turbulence Model to Three-Dimensional Supersonic Complex Configurations," *Journal of Aerospace Science and Technology*, Vol. 6, 2002, pp. 171-183.

- [141] Spalart, P. R., Jou, W. H., Strelets, M., and Allmaras, S. R., "Comments on the Feasibility of LES for Wings, and on a Hybrid RANS/LES Approach," in Advance in DNS/LES, 1st AFOSR International Conference on DNS/LES, 1997, Greyden, Columbus, Ohio.
- [142] Squires, K. D., "Detached-Eddy Simulation: Current Status and Perspectives," *Proceedings of Direct and Large-Eddy Simulation-5*, 2004.
- [143] Travin, A., Shur, M., Strelets, M., and Spalart, P., "Detached-Eddy Simulations Past a Circular Cylinder," *Flow, Turbulence and Combustion*, Vol. 63, 2000, pp. 293-313.
DOI: 10.1023/A:1009901401183
- [144] Spalart, P. R., "Detached-Eddy Simulation," *Annual Review of Fluid Mechanics*, Vol. 41, 2008, pp. 181-202.
DOI: 10.1146/annurev.fluid.010908.165130
- [145] Spalart, P. R., Deck, S., Shur, M. L., and Squires, K. D., "A New Version of Detached-Eddy Simulation, Resistant to Ambiguous Grid Densities," *Journal of Theoretical and Computational Fluid Dynamics*, Vol. 20, 2006, pp. 181-195.
DOI: 10.1007/s00162-006-0015-0
- [146] Menter, F. R., "Two-Equation Eddy-Viscosity Turbulence Models for Engineering Applications," *AIAA Journal*, Vol. 32, No. 8, Aug. 1994, pp. 1598-1605.
- [147] Lund, T. S., Wu, X., and Squires, K. D., "Generation of Turbulent Inflow Data for Spatially-Developing Boundary Layer Simulations," *Journal of Computational Physics*, Vol. 140, 1998, pp. 233-258.
- [148] Gloorfelt, X. and Garrec, T. L., "Generation of Inflow Turbulence for Aeroacoustic Applications," in 14th AIAA/CEAS Aeroacoustics Conference (29th AIAA Aeroacoustics Conference), AIAA-2008-2926, May 5-7, 2008, Vancouver, BC, Canada.
- [149] <http://www.pointwise.com/gridgen>, Sept. 2011