ACHIEVABLE RATE FOR AMPLIFY-AND-FORWARD RELAY SYSTEM
AT LOW SNR

A Thesis by

Yu Bi

Bachelor of Science, Guilin University of Electronic Science and Technology, 2007

Submitted to the Department of Electrical Engineering and Computer Science
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
the requirements for the degree of
Master of Science

May 2011

© Copyright 2011 by Yu Bi

All Rights Reserved

ACHIEVABLE RATE FOR AMPLIFY-AND-FORWARD RELAY SYSTEM AT LOW SNR

The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Electrical Engineering.

Yanwu Ding, Committee Chair

Hyuck M. Kwon, Committee Member

Chunsheng Ma, Committee Member

DEDICATION

To my parents, Dr. Ding, Dr. Kwon, and my friends

ACKNOWLEDGMENTS

I would like to thank my advisor, Yanwu Ding, for her support, guidance, and patience.

Thanks are also due to my fellow lab workers who helped guide me in good and bad times.

ABSTRACT

In this thesis, the achievable rate at low SNR is established for Amplify-and-Forward (AF) cooperative system with a source node, a destination node and a relay node. To characterize the effect of relay locations, an aggregate channel model which consists of both long-term path loss and short-term path loss fading is used. To consider the effect of channel information at the relay, average and instantaneous power constraints are applied respectively when evaluating the achievable rate. As analytic solutions to the achievable rate seem difficult to obtain, approximations for the achievable rate, the optimal power loadings, the optimal amplification coefficients for both types of power constraints are derived in this thesis.

TABLE OF CONTENTS

Chapter		Page
1.	INTRODUCTION	1
2.	SYSTEM MODEL.....	2
2.1	System Model	2
2.2	Achievable Rate with Path Loss	4
2.3	Effective Signal-to-Noise Ratio (SNR) with Path Loss.....	7
3.	APPROXIMATION OF THE ACHIEVABLE RATE	10
3.1	Approximation of the Achievable Rate under Average Constraint.....	11
3.2	Approximation of the Achievable Rate under Instantaneous Constraint.....	13
4.	SIMULATION RESULT	16
4.1	Exact Achievable Rate with Path Loss	16
4.1.1	Exact Achievable Rate with Different Relay Locations	16
4.1.2	Effect of Power Distribution on Achievable rate	20
4.2	Approximations of the Achievable Rate with Path Loss	25
4.2.1	Approximations under the Average Constraint	25
4.2.2	Approximations under the Instantaneous Constraint	31
	REFERENCE.....	37
	APPENDIXES	41
I.	Proof of Theorem 1	42
II.	Derivations of Approximations.....	45
A.	Approximations under Average Constraint.....	45
B.	Approximations under Instantaneous Constraint.....	56
III	Proof for Optimization.....	65
A.	Optimization Problem of Lemma 1	65
B.	Optimization Problem of Lemma 4	67

LIST OF FIGURES

Figure	Page
1. System model.....	3
2. Achievable rate with different relay locations	17
3. b 's ratio with different relay locations	18
4. Optimum q_1 with different relay locations	19
5. Achievable rate with different relay locations	20
6. Achievable rate with different L for SNR=-10dB.....	21
7. Achievable rate with different L for SNR=-10dB.....	22
8. b 's ratio with different L for SNR=-10dB	22
9. Optimum q_1 with different L for SNR=-10dB.....	23
10. Achievable rate with different L for $\sigma^2=20$	23
11. b 's ratio with different L for $\sigma^2=20$	24
12. Optimum q_1 with different L for $\sigma^2=20$	24
13. Approximations of achievable rate under average constraint with $L=1$	25
14. b 's ratio of approximation under average constraint with $L=1$	26
15. Optimum q_1 of approximation under average constraint with $L=1$	27
16. Approximations of achievable rate under average constraint with $L=10$	28
17. b 's ratio of approximation under average constraint with $L=10$	28
18. Optimum q_1 of approximation under average constraint with $L=10$	29

LIST OF FIGURES (Continued)

Figure		Page
19.	Approximations of achievable rate under average constraint with $L=0.5$	29
20.	b 's ratio of approximation under average constraint with $L=0.5$	30
21.	Optimum q_1 of approximation under average constraint with $L=0.5$	31
22.	Approximations of achievable rate under instantaneous constraint with $L=1$	31
23.	b 's ratio of approximation under instantaneous constraint with $L=1$	32
24.	Optimum q_1 of approximation under instantaneous constraint with $L=1$	32
25.	Approximations of achievable rate under instantaneous constraint with $L=10$	33
26.	b 's ratio of approximation under instantaneous constraint with $L=10$	34
27.	Optimum q_1 of approximation under instantaneous constraint with $L=10$	34
28.	Approximations of achievable rate under instantaneous constraint with $L=0.5$	35
29.	b 's ratio of approximation under instantaneous constraint with $L=0.5$	35
30.	Optimum q_1 of approximation under instantaneous constraint with $L=0.5$	36

LIST OF ABBREVIATIONS/ NOMENCLATURE

AF Amplify-and-Forward

DF Decode and Forward

SNR Signal-to-Noise Ratio

CHAPTER 1

INTRODUCTION

This thesis focuses on the achievable rate at low signal-to-noise ratio (SNR) of amplify-and-forward (AF) system in which a relay node assists the transmission from source node to destination node. Based on the AF model, this thesis investigates the achievable rate with path loss. While the achievable rate for AF half –duplex cooperative systems is proven [1], the effect of the relay location on the achievable rate has remained unknown. To consider the effect of relay different location, an aggregate channel model is introduced, which consists of both long-term path loss and short-term path loss fading effects. The aim of this thesis is to derive the achievable rate with path loss for an AF protocol and characterize the covariance matrix of input signal to achieve the achievable rate with path loss. We consider the AF protocol proposed in [10], [11] for half-duplex relay systems in which transmission and reception are not carried out simultaneously [11], [14], [17], [18]-[23]. Such systems usually have lower complexity than full-duplex systems. In order to simply the system, the single relay system is considered in this thesis. In the system, it is shown that the optimal input covariance matrix to achieve the achievable rate is diagonal and the diagonal elements are obtained by solving optimization problem.

Notation: Matrices and column vectors are denoted by uppercase and lowercase boldface characters respectively (e.g., $\mathbf{A}$, $\mathbf{a}$). Notation x^* denotes the conjugate of complex number x . $Re(.)$ denotes the real part of the variable in the bracket. $E[.]$ is the expectation operator.

CHAPTER 2

SYSTEM MODEL

In this section, the system model for Amplify-and-Forward (AF) half duplex cooperative system is introduced, which consists of a source node, a relay node and a destination node and each node is equipped with a single antenna. In the system model, the channel gains are zero mean circular Gaussian and quasi-static, i.e., they will not change within one period of observation. In the thesis, the duration of one period of observation includes two time slots. It is assumed that the destination (receiver) has full knowledge of all channel gains while only their second order statistics are known at the source node (transmitter) [1].

To consider the effect of relay different location, the channel model which consists of both long-term path loss and short-term path loss fading effects is applied. The path loss between nodes A and B is denoted by $L_{AB} = K/d_{AB}^\alpha$, where K is the constant that depends on the environment, d_{AB} is the propagation distance and α is the path loss exponent. For free-space, we have $\alpha = 2, K = G_t G_r \lambda^2 / (4\pi^2)$, where G_t, G_r are antenna gains at T_x and R_x , and λ is the wavelength. As the distance from source-to-destination ($S \rightarrow D$), source-to-relay ($S \rightarrow R$), and relay-to-destination ($R \rightarrow D$) can be related to each other through a law of cosines, $d_{SD}^2 = d_{SR}^2 + d_{RD}^2 - 2d_{SR}d_{RD}\cos\theta$, where θ is the user-relay-destination angle between links $S \rightarrow R$ and $R \rightarrow D$. For convenience, a distance ratio $L = d_{RD}/d_{SR}$ is introduced [2]. When the value of ratio L is smaller than 1, the relay is more close to destination otherwise it is more close to source.

2.1 System Model

Fig.1. System model

The system model is shown in Fig.1. The relay node R assists the transmission from source node S to the destination D. The channel gain from the source to the destination is denoted by h_{sd} whereas those from the source to the relay and from the relay to the destination are denoted respectively by h_{sr} and h_{rd} . The signal is transmitted data as $\mathbf{x} = \begin{pmatrix} x(1) \\ x(2) \end{pmatrix}$, which means that $x(1)$ is transmitted from source to relay and destination respectively at the first time slot and $x(2)$ is transmitted from source to destination. During the first time slot, the source transmits $x(1)$ to both the destination and the relay node, and during the second time slot, the source transmits $x(2)$ to the destination, while the relay node simply amplifies and forwards what it receives from at first time slot to the destination. The signal transmission is also effected by path loss which is denoted by L_{SD} from source to destination whereas those from the source to the relay and the relay to destination are denoted by L_{SR} and L_{RD} . The noise at the relay node and destination node is assumed to be i.i.d. zero-mean circular Gaussian with variance σ^2 . Above all, the received signal $r_D(1)$ at the destination and the received signal $r_R(1)$ at relay at first time slot are:

$$\begin{cases} S \rightarrow D, & r_D(1) = \sqrt{L_{SD}}x(1) \cdot h_{sd} + n_D(1) \\ S \rightarrow R, & r_R(1) = \sqrt{L_{SR}}x(1) \cdot h_{sr} + n_R(1) \end{cases} \quad (1)$$

Where $n_D(1)$ and $n_R(1)$ are the noises at destination and relay at first time slot.

At 2nd time slot, the received signal at destination $r_D(2)$ is

$$r_D(2) = \sqrt{L_{SD}}x(2) \cdot h_{sd} + \sqrt{L_{RD}L_{SR}}bh_{rd}h_{sr} \cdot x(1) + n_D(2) \quad (2)$$

Where $n_D(2)$ is the noise at destination at the second time slot.

The relationship between input signal and output signal can be expressed by:

$$\mathbf{r} = \mathbf{H}\mathbf{x} + \mathbf{n} \quad (3)$$

Where the channel matrix $\mathbf{H}$ is shown as:

$$\mathbf{H} = \begin{bmatrix} \sqrt{L_{SD}}h_{sd} & 0 \\ b\sqrt{L_{RD}L_{SR}}h_{rd}h_{sr} & h_{sd} \end{bmatrix}$$

To simply the model in terms of path loss, we assume that $K=1$, $d_{SD}=1$, then

$$1 = d_{SR}^2 + d_{RD}^2 - 2d_{SR}d_{RD}\cos\theta$$

Where θ is the angle between SR and RD in Fig.1. So we can get new channel matrix as follow:

$$\mathbf{H} = \begin{bmatrix} h_{sd} & 0 \\ b\sqrt{L_{RD}L_{SR}}h_{rd}h_{sr} & h_{sd} \end{bmatrix} \quad (4)$$

Where b is the amplification coefficient at relay node which controls the signal strength forwarded from the relay node, and the equivalent noise vector at the destination $\mathbf{n} \sim (0, \sigma^2 \mathbf{\Sigma})$ with

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & L_{RD}b^2|h_{rd}|^2 + 1 \end{bmatrix} \quad (5)$$

2.2 Achievable Rate with Path Loss

In this section, the achievable rate with path loss is derived for the AF system. The distribution of signal $\mathbf{x}$ is given by zero mean Gaussian and its covariance is given by $\mathbf{Q}$ as

follows:

$$\begin{aligned}\mathbf{Q} &= \begin{bmatrix} x(1) & x(2) \end{bmatrix} \begin{bmatrix} x^*(1) \\ x^*(2) \end{bmatrix} = E \begin{bmatrix} |x(1)|^2 & |x^*(1)x(2)| \\ |x(1)x^*(2)| & |x(2)|^2 \end{bmatrix} \\ &= \begin{bmatrix} q_{11} & q_{12} \\ q_{12}^* & q_{22} \end{bmatrix}\end{aligned}\quad (6)$$

And the transmission power constraint is defined by $\text{tr}(\mathbf{Q}) \leq P_s$. We also denote the amplification constraint as b . So the Gaussian mutual information can be expressed as [3],

$$I(\mathbf{x}, \mathbf{r} | \mathbf{h}, b) = \log \det(\mathbf{I}_2 + \rho \mathbf{H}^H \mathbf{\Sigma}^{-1} \mathbf{H} \mathbf{Q})$$

The signal-to-noise ratio (SNR) is defined as $\frac{P_s}{2\sigma^2}$ and a variable ρ is introduced for convenience where the power of noise $\sigma^2 = \frac{1}{\rho}$ and the signal-to-noise ratio (SNR) can be expressed as $\frac{P_s}{2}\rho$. Above all, the achievable rate (bits/Hz) [4], [5] is got for the AF system with path loss as follow:

$$C = \frac{1}{2} \max_{w, \text{tr}(\mathbf{Q}) \leq P_s} E_{\mathbf{h}}[I(\mathbf{x}, \mathbf{r} | \mathbf{h}, b)] = \frac{1}{2} \max_{w, \text{tr}(\mathbf{Q}) \leq P_s} E_{\mathbf{h}}[\log \det(\mathbf{I}_2 + \rho \mathbf{H}^H \mathbf{\Sigma}^{-1} \mathbf{H} \mathbf{Q})] \quad (7)$$

Here $E_{\mathbf{h}}[\cdot]$ denotes the expectation taken over all the random channel realization $\mathbf{h}$, the maximum value is chosen from the feasible sets of the input covariance matrix and w is the ratio of $\check{b}^2/b_{max}^2$, where $\check{b}$ represents the optimal value of amplification coefficient while b_{max} means the maximum value of amplification coefficient. Suppose that the power constraint at relay is P_r and the transmission power available at the relay node is P_s . The amplification coefficient b is constrained such that the average power for relaying signals does not exceed the power budget available at relay node as follow:

$$b^2 * E[|\mathbf{r}_R(1)|^2] \leq P_R$$

Then we get the inequality as:

$$b^2 * [(E[L_{SR}|x(1)|^2|h_{sr}|^2]) + \sigma^2] \leq P_R \quad (8)$$

So we can define the amplification coefficient b by two cases:

- Average constraint: If just the second order information of h_{sr} is known at relay, the amplification coefficient b is expressed as (8) [10], [24]:

$$b \leq \sqrt{\frac{P_R}{L_{SR}E[|h_{sr}|^2]E[|x(1)|^2]+\sigma^2}} \quad (9)$$

Because $E[|x(1)|^2] = q_{11}$, $E[|h_{sr}|^2] = 1$ in the average case, the amplification coefficient b is simplified from (9):

$$b \leq \sqrt{\frac{P_R}{L_{SR}q_{11}+\sigma^2}} \quad (10)$$

- Instantaneous constraint: If full channel information of h_{sr} is known at relay, the amplification coefficient b is got from (8) [11], [12]:

$$b \leq \sqrt{\frac{P_R}{L_{SR}|h_{sr}|^2E[|x(1)|^2]+\sigma^2}} = \sqrt{\frac{P_R}{L_{SR}|h_{sr}|^2q_{11}+\sigma^2}} \quad (11)$$

Because the channel matrix $\mathbf{H}$ for AF system in (3) has zero entries and entries of product of two Gaussian variables, the channel matrix for relay system is no longer Gaussian [4]. And when the signal is transmitted, the path loss is applied so that a new method is used to derive the achievable rate with path loss for AF system. To evaluate the expectation in (7), the following theorem got under the average and instantaneous constraint is defined in (10) and (11). The details of the proof are shown in Appendix I.

Theorem 1: The achievable rate with path loss in (7) for the AF relay system in (3) is realized if and only if the covariance matrix of input signal $\mathbf{Q}$ in (6) is diagonal and the optimum q_{11} and q_{22} are obtained by solving the optimization problem as follows:

$$C = \frac{1}{2} \max_{w, q_{11}+q_{22} \leq P_s} E_{\mathbf{h}} \left[\log_2 \left((1 + \rho|h_{sd}|^2q_{11}) \left(1 + \frac{\rho|h_{sd}|^2q_{22}}{1+b^2L_{RD}|h_{rd}|^2} \right) + \right. \right.$$

$$\rho \frac{b^2 L_{SR} L_{RD} |h_{sr} h_{rd}|^2 q_{11}}{1 + b^2 L_{RD} |h_{rd}|^2} \Bigg) \Bigg] \quad (12)$$

2.3 Effective Signal-to-Noise Ratio (SNR) with Path Loss

As mentioned before, the SNR is expressed as $\frac{P_s}{2\sigma^2}$. Now, this thesis focuses on the effective SNR, which is the ratio of the effective power of signal to the effective power of noise.

From equation (3) and (4), the relation between input and output signal is expressed as:

$$\mathbf{r} = \mathbf{H}\mathbf{x} + \mathbf{n} = \begin{bmatrix} \frac{h_{sd}}{b\sqrt{L_{SR}L_{RD}}h_{sr}h_{rd}} & 0 \\ b\sqrt{L_{SR}L_{RD}}h_{sr}h_{rd} & h_{sd} \end{bmatrix} \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} + \begin{pmatrix} n_D(1) \\ n_D(2) \end{pmatrix} \quad (13)$$

To get the effective SNR, equation (13) is transferred as the following pattern:

$$\mathbf{r} = \mathbf{X}_{eff}\mathbf{h}_{eff} + \mathbf{n} = \begin{bmatrix} x(1) & 0 \\ x(2) & bx(1) \end{bmatrix} \begin{pmatrix} \frac{h_{sd}}{\sqrt{L_{SR}L_{RD}}h_{sr}h_{rd}} \\ \end{pmatrix} + \begin{pmatrix} n_D(1) \\ n_D(2) \end{pmatrix} \quad (14)$$

Where $\mathbf{X}_{eff}$ is the effective input signal matrix and $\mathbf{h}_{eff}$ is the effective channel vector. Thus

the power of the effective input signal is given by

$$\begin{aligned} P_{s_eff} &= E[\text{tr}(\mathbf{X}\mathbf{X}^H)] = E \left[\text{tr} \left(\begin{bmatrix} x(1) & 0 \\ x(2) & bx(1) \end{bmatrix} \begin{bmatrix} x^*(1) & x^*(2) \\ 0 & b^*x^*(1) \end{bmatrix} \right) \right] \\ &= \text{tr} \left[E \begin{bmatrix} |x(1)|^2 & x(1)x^*(2) \\ x^*(1)x(2) & |x(2)|^2 + |bx(1)|^2 \end{bmatrix} \right] \\ &= \text{tr} \begin{bmatrix} E|x(1)|^2 & E[x(1)x^*(2)] \\ E[x^*(1)x(2)] & E[|x(2)|^2] + E[|bx(1)|^2] \end{bmatrix} \end{aligned}$$

Based on the definition in equation (6), the power of the effective input signal can be expressed

as:

$$P_{s_eff} = \text{tr} \begin{bmatrix} q_{11} & q_{12}^* \\ q_{12} & q_{22} + b^2 q_{11} \end{bmatrix} = (1 + b^2)q_{11} + q_{22} \quad (15)$$

From equation (5), the power of the noise is given:

$$P_{n_eff} = E[\text{tr}(\mathbf{n}\mathbf{n}^H)] = \text{tr}[E(\mathbf{n}\mathbf{n}^H)]$$

$$= \text{tr} \begin{bmatrix} \sigma^2 & 0 \\ 0 & (L_{RD}b^2|h_{rd}|^2 + 1)\sigma^2 \end{bmatrix} = (L_{RD}b^2|h_{rd}|^2 + 2)\sigma^2 \quad (16)$$

Based on equation (15) and equation (16), the definition of the effective SNR is given as:

$$SNR_{eff} = \frac{P_{seff}}{P_{neff}} = \frac{(1+b^2)q_{11}+q_{22}}{(L_{RD}b^2|h_{rd}|^2+2)\sigma^2} \quad (17)$$

Based on the inequality (10) and (11), the effective SNR can be defined by the following two options:

- Under average constraint

$$\begin{aligned} SNR_{eff} &= \frac{P_{seff}}{P_{neff}} = E_{h_{rd}} \left[\frac{(1+b^2)q_{11}+q_{22}}{(L_{RD}b^2|h_{rd}|^2+2)\sigma^2} \right] \\ &= E_{h_{rd}} \left[\frac{\left(1 + \frac{wP_R}{L_{SR}q_{11}+\sigma^2}\right)q_{11}+q_{22}}{\left(L_{RD}\frac{wP_R}{L_{SR}q_{11}+\sigma^2}|h_{rd}|^2+2\right)\sigma^2} \right] = \frac{\left(1 + \frac{a}{L_{RD}}\right)q_{11}+q_{22}}{\sigma^2} \int_0^\infty \frac{1}{(ay+2)} e^{-y} dy \end{aligned} \quad (18)$$

Where we denote $y = |h_{rd}|^2$, $a = \frac{wL_{RD}P_R}{L_{SR}q_{11}+\sigma^2}$.

Assume $y = u - \frac{2}{a}$ and the range of u is $(\frac{2}{a}, \infty)$. So we can express equation (18) as:

$$\begin{aligned} SNR_{eff} &= \frac{\left(1 + \frac{a}{L_{RD}}\right)q_{11}+q_{22}}{\sigma^2} \int_0^\infty \frac{1}{\left(a\left(u - \frac{2}{a}\right) + 2\right)} e^{-\left(u - \frac{2}{a}\right)} dy \\ &= \frac{\left(1 + \frac{a}{L_{RD}}\right)q_{11}+q_{22}}{a\sigma^2} e^{\frac{2}{a}} \int_{\frac{2}{a}}^\infty \frac{1}{u} e^{-u} dy = \frac{\left(1 + \frac{a}{L_{RD}}\right)q_{11}+q_{22}}{a\sigma^2} e^{\frac{2}{a}} E_i\left(\frac{2}{a}\right) \end{aligned} \quad (19)$$

Where $E_i(x) = \int_x^\infty \frac{1}{u} e^{-u} du$ is the exponential integral [6].

- Under instantaneous constraint

$$\begin{aligned} SNR_{eff} &= \frac{P_{seff}}{P_{neff}} = E_h \left[\frac{(1+b^2)q_{11}+q_{22}}{(L_{RD}b^2|h_{rd}|^2+2)\sigma^2} \right] \\ &= E_h \left[\frac{\left(1 + \frac{wP_R}{L_{SR}|h_{sr}|^2q_{11}+\sigma^2}\right)q_{11}+q_{22}}{\left(L_{RD}\frac{wP_R}{L_{SR}|h_{sr}|^2q_{11}+\sigma^2}|h_{rd}|^2+2\right)\sigma^2} \right] \end{aligned}$$

$$= \int_0^\infty \int_0^\infty \frac{\left(1 + \frac{wP_R}{L_{SR}zq_{11} + \sigma^2}\right)q_{11} + q_{22}}{\left(L_{RD}\frac{wP_R}{L_{SR}zq_{11} + \sigma^2}y + 2\right)\sigma^2} e^{-(y+z)} dy dz \quad (20)$$

Where $y = |h_{rd}|^2, z = |h_{sr}|^2$

Chapter 3

Approximation of the Achievable Rate

Since it is very hard to get the closed-form solution to the achievable rate obtained from equation (20), the approximations for the achievable rate at low SNR are necessary and the results of derivation are shown in this chapter. The thesis focuses on the special case in which the channel gain and are *i.i.d.* zero -mean circular Gaussian variables with unit variance under both average constraint and instantaneous constraint. For notational convenience, we denote $x = |h_{sd}|^2$, $y = |h_{rd}|^2$, $z = |h_{sr}|^2$. The probability distribution functions of x , y and z are e^{-x} , e^{-y} and e^{-z} respectively. Thus, we can get the achievable rate for AF system with path loss from equation (20) as:

$$\begin{aligned}
 C &= \frac{1}{2} \max_{q_{11}+q_{22} \leq P_s} E_h \left[\log_2 \left((1 + \rho |h_{sd}|^2 q_{11}) \left(1 + \frac{\rho |h_{sd}|^2 q_{22}}{1 + b^2 L_{RD} |h_{rd}|^2} \right) + \rho \frac{b^2 L_{SR} L_{RD} |h_{sr} h_{rd}|^2 q_{11}}{1 + b^2 L_{RD} |h_{rd}|^2} \right) \right] \\
 &= \frac{1}{2} \max_{q_{11}+q_{22} \leq P_s} E_{x,y,z} \left[\log_2 \left((1 + \rho x q_{11}) \left(1 + \frac{\rho x q_{22}}{1 + b^2 L_{RD} y} \right) + \rho \frac{b^2 L_{SR} L_{RD} q_{11} y z}{1 + b^2 L_{RD} y} \right) \right] \\
 &= \frac{1}{2} \max_{q_{11}+q_{22} \leq P_s} J_C(\rho, q_{11}, q_{22}, b)
 \end{aligned}$$

Where

$$J_C(\rho, q_{11}, q_{22}, b) = E_{x,y,z} \left[\log_2 \left((1 + \rho x q_{11}) \left(1 + \frac{\rho x q_{22}}{1 + b^2 L_{RD} y} \right) + \rho \frac{b^2 L_{SR} L_{RD} q_{11} y z}{1 + b^2 L_{RD} y} \right) \right] \quad (21)$$

3.1 Approximation of the Achievable Rate under Average Constraint

In this part, this thesis will discuss the approximation of the achievable rate at low SNR with the amplification coefficient b at the relay node satisfying the average constraint. In order to simplification, q_{11} and q_{22} are replaced by q_1 and q_2 respectively. From equation (10), we

can define the amplification coefficient b as:

$$b^2 = \frac{wP_r}{L_{SR}q_1 + \sigma^2} = \frac{\rho P_r}{\rho L_{SR}q_1 + 1}, w \in [0,1] \quad (22)$$

Where w is the ratio of b^2/b_{max}^2 , which range is from 0 to 1. As we mentioned in chapter 2, ρ is direct proportion to the SNR, which means that ρ is very small at low SNR. Based on the average constraint in equation (22), we will show the approximations of the achievable rate at low SNR by three lemmas, the proof of which is given in Appendix II.

Lemma 1: Given the AF half-duplex relay system with path loss, in which the amplification coefficient satisfies average constraint defined in (22), the approximation of the achievable rate at low SNR with respect to a function up to ρ^2 can be expressed as:

$$C_{avg_1} \approx \frac{1}{2} \max_{\substack{w \\ q_1 + q_2 \leq P_s}} J_{C_1}(\rho, q_1, q_2, b)$$

Where

$$J_{C_1}(\rho, q_1, q_2, b) = \log_2(e) \left(\rho P_s - \rho^2(q_1^2 + q_2^2) + \frac{\rho^2 L_{RD} w P_r (L_{SR} q_1 - q_2)}{\rho L_{SR} q_1 + 1} \right) \quad (23)$$

In this thesis, the solutions of the achievable rate with path loss under the average constraint are defined by a function up to quadratic, cubic and quartic of ρ as C_{avg_1} , C_{avg_2} and C_{avg_3} respectively.

The optimization problem in **Lemma 1** can be solved by derivative with respect to q_1 and the details of proof are given in Appendix III. The optimum q_1 under average constraint is denoted as $\check{q}_{1_avg}$ and the optimum amplification coefficient is denoted as $\check{b}_{avg}$, and the ratio w is equal to $\check{b}_{avg}^2/b_{max_avg}^2$. Then the optimum value q_1 will satisfy equation (24a)

$$\check{q}_{1_avg} = \min \left\{ \frac{L_{RD}(L_{SR}+1) + \rho L_{SR} L_{RD} P_s}{4(\rho L_{SR} \check{q}_{1_avg} + 1)^2} + \frac{P_s}{2}, P_s \right\} \quad (24a)$$

And the optimum amplification coefficient b will satisfy equation (24b).

$$\check{b}_{avg} = b_{\max_avg}, \check{w}_{avg} = 1 \quad (24b)$$

Lemma 2: Given the AF half-duplex relay system with path loss, in which the amplification coefficient satisfies average constraint defined in (22), the approximation of the achievable rate at low SNR with respect to a function up to ρ^3 can be expressed as:

$$C_{avg_2} \approx \frac{1}{2} \max_{q_1+q_2 \leq P_s} J_{C_2}(\rho, q_1, q_2, b)$$

Where

$$J_{C_2} = \log_2(e) \left(\rho P_s - \rho^2(q_1^2 + q_2^2) + \frac{\rho^2 L_{RD} W P_r (L_{SR} q_1 - q_2)}{\rho L_{SR} q_1 + 1} + \frac{\rho^3 L_{RD} W P_r (2q_2^2 - L_{SR} P_s q_1)}{\rho L_{SR} q_1 + 1} - \frac{2\rho^3 L_{RD}^2 W^2 P_r^2 (L_{SR} q_1 - q_2)}{(\rho L_{SR} q_1 + 1)^2} + 2\rho^3(q_1^3 + q_2^3) \right) \quad (25)$$

Lemma 3: Given the AF half-duplex relay system with path loss, in which the amplification coefficient satisfies average constraint defined in (22), the approximation of the achievable rate at low SNR with respect to a function up to ρ^4 can be expressed as:

$$C_{avg_3} \approx \frac{1}{2} \max_{q_1+q_2 \leq P_s} J_{C_3}(\rho, q_1, q_2, b)$$

Where

$$\begin{aligned}
J_{C_3} = \log_2(e) & \left(\rho P_s - \rho^2(q_1^2 + q_2^2) + \frac{\rho^2 L_{RD} w P_r (L_{SR} q_1 - q_2)}{\rho L_{SR} q_1 + 1} + \frac{\rho^3 L_{RD} w P_r (2q_2^2 - L_{SR} P_s q_1)}{\rho L_{SR} q_1 + 1} \right. \\
& - \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 (L_{SR} q_1 - q_2)}{(\rho L_{SR} q_1 + 1)^2} + 2\rho^3(q_1^3 + q_2^3) - 6\rho^4(q_1^4 + q_2^4) - \frac{6\rho^4 L_{RD} w P_r q_2^3}{\rho L_{SR} q_1 + 1} \\
& + \frac{2\rho^4 L_{SR} L_{RD} w P_r q_1}{\rho L_{SR} q_1 + 1} (q_1 q_2 - 2P_s) - \frac{6\rho^4 L_{RD}^2 w^2 P_r^2 q_2^2}{(\rho L_{SR} q_1 + 1)^2} \\
& + \frac{2\rho^4 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (q_2 + P_s)}{(\rho L_{SR} q_1 + 1)^2} - \frac{2\rho^4 L_{SR}^2 L_{RD}^2 w^2 P_r^2 q_1^2}{(\rho L_{SR} q_1 + 1)^2} - \frac{6\rho^4 L_{RD}^3 w^3 P_r^3 q_2}{(\rho L_{SR} q_1 + 1)^3} \\
& \left. + \frac{6\rho^4 L_{SR} L_{RD}^3 w^3 P_r^3 q_1}{(\rho L_{SR} q_1 + 1)^3} \right)
\end{aligned} \tag{26}$$

3.2 Approximation of the Achievable Rate under Instantaneous Constraint

In this section, this thesis will focus on the approximation of the achievable rate at low SNR with the amplification coefficient b at the relay node satisfying the instantaneous constraint.

From equation (11), we can define the amplification coefficient b as:

$$b^2 = \frac{w P_R}{L_{SR} q_1 z + \sigma^2} = \frac{\rho w P_R}{\rho L_{SR} q_1 z + 1}, w \in [0,1] \tag{27}$$

Similar to the average case, the approximations of the achievable rate under the instantaneous constraint at low SNR are shown by two Lemmas, the proof of which is also given in Appendix II.

Lemma 4: Given the AF half-duplex relay system with path loss, in which the amplification coefficient satisfies instantaneous constraint defined in (27), the approximation of the achievable rate at low SNR with respect to a function up to ρ^2 can be expressed as:

$$C_{int_1} \approx \frac{1}{2} \max_{\substack{w \\ q_1+q_2 \leq P_s}} J_{C_4}(\rho, q_1, q_2, b)$$

Where

$$J_{C_4}(\rho, q_{11}, q_{22}, b) = \log_2(e) (\rho P_s - \rho^2(q_1^2 + q_2^2) + \rho^2 L_{RD} w P_r (L_{SR} q_1 - q_2)) \quad (28)$$

In this thesis, the solutions of the achievable rate with path loss under the instantaneous constraint are defined by a function up to quadratic and cubic of ρ as C_{int_1} and C_{int_2} respectively.

In instantaneous case, the optimization problem in **Lemma 4** can be solved by derivative with respect to q_1 and the details of proof are given in Appendix III. In thesis, optimum q_1 under instantaneous constraint is defined as $\check{q}_{1_int}$ and the optimum amplification coefficient is defined as $\check{b}_{int}$, and the ratio w is equal to $\check{b}_{int}^2 / b_{\max_int}^2$. Then the optimum value can be given as:

$$\check{q}_{1_int} = \left\{ \frac{P_r L_{RD} (L_{SR} + 1)}{4} + \frac{P_s}{2}, P_s \right\} \quad (29a)$$

$$\check{b}_{int} = b_{\max_int}, \quad \check{w}_{int} = 1 \quad (29b)$$

Lemma 5: Given the AF half-duplex relay system with path loss, in which the amplification coefficient satisfies instantaneous constraint defined in (22), the approximation of the achievable rate at low SNR with respect to a function up to ρ^3 can be expressed as:

$$C_{int_2} \approx \frac{1}{2} \max_{\substack{w \\ q_1+q_2 \leq P_s}} J_{C_5}(\rho, q_1, q_2, b)$$

Where,

$$J_{C_5}(\rho, q_{11}, q_{22}, b) = \log_2(e) (\rho P_s - \rho^2(q_1^2 + q_2^2) + \rho^2 L_{RD} w P_r (L_{SR} q_1 - q_2) + 2\rho^3(q_1^3 + q_2^3) + 2\rho^3 L_{RD} w P_r q_2 (L_{RD} w P_r + q_2) + \rho^3 L_{SR} L_{RD} w P_r q_1 (q_2 -$$

$$P_s) - 2\rho^3 L_{SR} L_{RD} w P_r q_1 (L_{RD} w P_r + L_{SR} q_1)) \quad (30)$$

CHAPTER 4

SIMULATION RESULT

In this section, the thesis will present the performance of our proposed achievable rate with path loss for amplify-and-forward (AF) half-duplex cooperative system, i.e., the achievable rate and the approximation of the achievable rate. In our model, we assume the angle θ between the link of source→relay and relay→destination is $\frac{\pi}{3}$ and the distance between source and destination $d_{SD}=1$. Because the locations of the relay will affect the performance of the achievable rate, this thesis discusses the several cases based on different distance ratios $L=d_{RD}/d_{SR}$.

4.1 Exact Achievable Rate with Path Loss

The equation (12) shows that there are several factors will affect the achievable rate. Here, the achievable rates got from equation (12) are denoted as exact achievable rate, which are shown as “exa avg” of “exa int” in the figures. And “avg” means we get the achievable rate under the average constraint and “int” means the achievable rate under the instantaneous constraint. Firstly, we consider the effect of the different relay locations in section 4.1.1.

4.1.1 Exact Achievable Rate with Different Relay Locations

From chapter 2, the power of noise is given as σ^2 and the effective SNR is also shown. Now this thesis will talk about the achievable rate by two ways: X-axis as $\rho = \frac{1}{\sigma^2}$ and X-axis as effective SNR. In the simulation, the thesis assumes the power of the signal $q_1 + q_2 = 2$ and the the power budget at relay node $P_r=1$.

- **Exact Achievable Rate with X-axis as $\rho = \frac{1}{\sigma^2}$**

Fig.2. Achievable rate with different relay locations

Fig.2. shows different achievable rates based on equation (12) with the different distance ratios L . For $L=1$, the distance from source to relay is same as the distance from relay to destination. And for $L=0.1$, relay is closer to the destination than it is to the source. Oppositely, the relay is much closer to the source for $L=10$. Thus, the conclusion that the achievable rate performs best when the relay location is closer to the destination and the worst relay location is in the middle of source and destination for $L=1$ can be seen in Fig.2. This observation is in agreement with the capacities for the conventional multiple-input-multiple-output Rayleigh fading systems, where the achievable rate for one transmitter two receiver system is higher than that for two transmitters and one receiver system [4]. Fig.2 also shows that the performance of achievable rate under the average constraint is almost same as the one under the instantaneous

case when we compare the all performances with $\rho = \frac{1}{\sigma^2}$ as X-axis. And the ratio $\check{b}^2/b_{max}^2$ is shown in Fig. 3. Here the optimum value of amplification coefficient b is denoted as $\check{b}$.

Fig.3. b 's Ratio with different relay Locations

In Fig.3, all the ratios are 1, which mean that the achievable rates can reach the maximum values when the amplification coefficients take the maximum values. Fig.4 shows that the optimum q_1 is about 1.5 for $L=1$ and the optimum q_1 is 2 when $L=0.1$, which means the achievable rate performs best when it just transmits the signal $x(1)$ when the relay is closer to the destination. And the optimum q_1 is decreasing as SNR increasing when the relay location is near the source.

Fig.4. Optimum q_1 with different relay locations

- **Exact Achievable Rate with X-axis as Effective SNR**

In Fig.5, the achievable rates are observed with effective SNR, which is defined in equation (19) and (20). The figure shows that the achievable rate under the instantaneous constraint performs better than under the average constraint for the distance ratio $L=1$ and $L=10$. This appearance means that the effective SNR has a better effort on the achievable rate under the instantaneous constraint than the average case while L is bigger than 1. But for the distance ratio $L=0.5$, which means the relay is closer to the destination than it is to the source, the achievable rate under the average constraint shows almost same as under the instantaneous constraint. Fig.5 also shows that when the distance ratio $L=0.5$ and $L=10$, the performances of the capacities are better than the case when $L=1$, which means that the relay's position affects the performance of

the achievable rate, which also offer a scheme to improve the achievable rate.

Fig.5. Achievable rate with different relay locations

4.1.2 Effect of Power Distribution between Source and Relay on Achievable Rate

The first effect on the achievable rate has been introduced in section 4.1.1. Now, another effect on the achievable rate: power distribution is considered. Firstly, the SNR of the input signal expressed as $\frac{P_s}{2\sigma^2}$ is assumed as -10dB and total power $P_s + P_r$ is fixed and it is equal to 3 in the simulation. So the power of the noise will change as the power of signal changing. Secondly, in this thesis it is assumed that the power of the noise is fixed, which means that the SNR will change as power of signal changing.

- SNR of the input sigal is -10dB

Fig.6. Achievable rate with different L for SNR=-10dB

Fig.6 shows the achievable rates based on equation (12) with different L for $SNR = \frac{P_s}{2\sigma^2} = -10\text{dB}$. Fig.6 shows that the achievable rate is decreasing as the signal power increasing when the SNR of the input signal is fixed. And the achievable rate performs better under the average constraint than under the instantaneous constraint for $L=1, 0.5$ and 10 . However, the achievable rate performs worse under the average constraint than under the instantaneous constraint for $L=0.1$.

Fig.7 shows the achievable rate based on equation (12) with $L=10$ for SNR=-10dB. Similarly, the achievable rate is decreasing as the signal power increasing. But the achievable rate under the average constraint doesn't show a big advantage compared with under the instantaneous constraint. And the optimization factors are shown in Fig.8 and Fig.9.

Fig.7. Achievable rate with $L=10$ for $\text{SNR}=-10\text{dB}$

Fig.8. b 's ratio with different L for $\text{SNR}=-10\text{dB}$

Fig.9. Optimum q_1 with different L for SNR=-10dB

- Power of noise is 20

Fig.10. Achievable rate with different L for $\sigma^2=20$

Fig.11. b 's ratio with different L for $\sigma^2=20$

Fig.12. Optimum q_1 with different L for $\sigma^2=20$

Fig.10 shows the achievable rates based on equation (12) with different L for the noise power=20. In Fig.10, the power of noise is fixed as $\sigma^2=20$. Thus, the achievable rate is

increasing as the signal power increasing. And Compared Fig.10 with Fig.7, it can be conclude that the effect of noise power on the achievable rate is more than the signal power. And the optimization problems in $\sigma^2=20$ case are shown in Fig.11 and Fig.12.

4.2 Approximations of the Achievable Rate with Path Loss

In this section, the simulation results of *Lemma 1* to *Lemma 5* are shown. And “avg appro” is denoted as the approximation of the achievable rate under average constraint while “int appro” means the approximation of the achievable rate under instantaneous constraint.

4.2.1 Approximations under the Average Constraint

- $L=1$

Fig.13. Approximations of achievable rate under average constraint with $L=1$

Fig.13 shows the achievable rate for AF system when the distance ratio $L=1$ ($L_{sr}=L_{rd}=1$). The exact achievable rate is simulated by equation (12). And the approximation

values is obtained numerically in equation by a quadratic, cubic and quartic functions of ρ , which is proportional to the input SNR. The Fig.13 shows that the approximation result is closer to exact achievable rate with the power of SNR increasing. And the result based on the quartic function of the SNR is almost matched to the exact achievable rate. And the path loss factors Lsr and Lrd don't affect the accuracy of the approximation result. Fig.14 and Fig.15 shows the optimum b and q_1 in *lemma 1, 2* and *3*.

Fig.14. b 's ratio of approximation under average constraint with $L=1$

Fig.15. Optimum q_1 of approximation under average constraint with $L=1$

- $L=10$

Because the path loss factors L_{sr} and L_{rd} are introduced, the approximation achievable rate doesn't follow the rule obtained by the figure of the approximation achievable rate with respect to higher power of SNR, the result will be closer to the exact achievable rate. For example, in Fig.16, the distance ratio $L=10$ ($L_{sr}=91$ and $L_{rd}=0.91$), even though the approximation of the achievable rate respected to the cubic function of SNR behaves better than the quadratic one, the achievable rate function when SNR's power is up to 4 has a big gap with the exact achievable rate when SNR is increasing because the path loss factor L_{sr} is big enough to affect the accuracy of the approximation result. In addition, the optimization problems are shown in Fig.17 and Fig.18.

Fig.16. Approximations of achievable rate under average constraint with $L=10$

Fig.17. b 's ratio of approximation under average constraint with $L=10$

Fig.18. Optimum q_1 of approximation under average constraint with $L=10$

- $L=0.5$

Fig.19. Approximation of achievable rate under average constraint with $L=0.5$

Fig.20. b 's ratio of approximation under average constraint with $L=0.5$

Fig.19 shows the approximations for the distance ratio $L=0.5$ and the path loss factors $Lsr=0.75$ and $Lrd=3.0$, which are moderate so that the approximations match the exact achievable rate better and better when the approximations with respect to the higher and power of ρ . And the optimum b and q_1 can be shown in Fig.20 and Fig.21.

Fig.21. Optimum q_1 approximation under average constraint with $L=0.5$

4.2.2 Approximations under the Instantaneous Constraint

- $L=1$

Fig.22. Approximations of achievable rate under instantaneous constraint with $L=1$

Fig.23. b 's ratio under instantaneous constraint with $L=1$

Fig.24. Optimum q_1 under instantaneous constraint with $L=1$

Similar to the case under the average constraint, the approximations of achievable rate

under the instantaneous constraint with $L=1$ follow the rule that the approximation results are closer to exact achievable rate with the power of SNR increasing, which is shown in Fig.22. And the optimization problem in this case is shown in Fig.23 and Fig.24.

- $L=10$

Fig.25. Approximations of achievable rate under instantaneous constraint with $L=10$

Fig.25. shows the approximations when the distance ratio $L=10$ and the path loss factors $Lsr=91$, which is big enough to affect the accuracy of the approximation result. So the achievable rate is considered in lower SNR, for example: from -25dB to -10dB. When SNR is low enough, it can offset the effect of the path loss on the achievable rate as shown in Fig.25. Fig.25 also shows that the SNR is not small enough to offset the effect of the path loss so that it appears a gap which is bigger and bigger with SNR increasing. And the approximation results also break the rule that approximation result is closer to exact achievable rate with the power of

SNR increasing. And the path loss factors L_{sr} and L_{rd} affect the accuracy of the approximation result. And the optimum b and q_1 in **lemma 4** and **5** are shown in Fig. 26 and Fig.27.

Fig.26. b 's ratio of approximation under instantaneous constraint with $L=10$

Fig.27. Optimum q_1 approximation under instantaneous constraint with $L=10$

- $L=0.5$

Fig.28. Approximations of achievable rate under instantaneous constraint with $L=0.5$

Fig.29. b 's ratio of approximation under instantaneous constraint with $L=0.5$

Fig.30. Optimum q_1 approximation under instantaneous constraint with $L=0.5$

The approximations of achievable rate under instantaneous constraint shown in Fig.28 is similar to the Fig.19 under the average constraint with $L=0.5$. And the optimization problem in this case can be shown as Fig.29 and Fig.30.

REFERENCES

REFERENCE

- [1] Yanwu Ding, Jian-Kang Zhang Wong, "Ergodic Channel Capacities for the Amplify-and-Forward Half-Duplex Cooperative Systems," *IEEE Trans. Inform. Theory*, vol. 55, no. 2, pp. 713-730, Feb. 2009.
- [2] H.Ochiai, P.Mitran, and V. Tarokh, "Variable-rate two-phase collaborative communication protocols for wireless networks," *IEEE Trans. Inform. Theory*, vol.52, no. 9, pp. 4299-4313, Sept.2006.
- [3] T.M. Cover and J.A. Thomas, *Elements of Information Theory*, John Wiley & Sons, New York,1991.
- [4] I.E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecom.*, vol.10, pp. 585-595, Nov. 1999.
- [5] G.J. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Labs Tech. J.*, vol. 1, no. 2, pp. 41-59, 1996.
- [6] N. N. Special Functions and their applications, Prentice-Hall, New York, 1965.
- [7] E.van der Meulen, "A survey of multi-way channels in information theory: 1961-1976," *IEEE Trans. Inform. Theory*, vol. 23, pp. 1-37, Jan. 1977.
- [8] T.M. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inform. Theory*, vol. IT-25, no. 5, pp. 572-584,1979.
- [9] A. Sendonaris, E. Erkip, and B. Aazhang, "User cooperation diversity —part I: System description ," *IEEE Trans. Comm.*, vol.51, pp. 1927-1938, Nov, 2003.
- [10] R.U.Nabar, H. Bölcskei, and F. W. Kneubühler, "Fading relay channel: Performance limits and space-time signal design," *IEEE J. Select. Areas Commun.*, vol.22, pp.1099-1109, Aug. 2004.
- [11] K. Azarian, H. El Gamal, and P. Schniter, "On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels," *IEEE Trans. Inform.Theory*, vol.51, pp. 4152-4172, Dec. 2005.
- [12] J.N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless

- networks: Efficient protocols and outage behavior,” *IEEE Trans. Inform. Theory*, vol. 49, pp. 3062-3080, Dec. 2004.
- [13] G. Kramer, M. Gastpar, and P. Gupta, “Cooperative strategies and achievable rate theorems for relay networks,” *IEEE Trans. Inform. Theory*, vol. 51, no. 9, Sept. 2005.
 - [14] A. Høst-Madsen and J. Zhang, “Capacity bounds and power allocation for wireless relay networks,” *IEEE Trans. Inform. Theory*, vol. 51, no. 6, pp. 2020-2040, June 2005.
 - [15] B. Hassibi and A. Dana, “On the power efficiency of sensory and ad-hoc wireless networks,” in *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, Yokohama, Japan, July 2003, p. 412.
 - [16] S. Yang and J.-C. Belfiore, “Optimal space-time codes for the MIMO amplify-and-forward cooperative channel,” *IEEE Trans. Inform. Theory*, vol. 53, pp. 647-663, Sept. 2007.
 - [17] S. Vishwanath, S. Jafar, and S. Sandhu, “Half-duplex relays: Cooperative communication strategies and outer bounds,” in *Proc. IEEE Int. Conf. Wireless Networks, Communications and Mobile Computing*, Maui, HI, USA, June 2005, pp. 1455-1459.
 - [18] M. A. Khojastepour, A. Sabharwal, and B. Aazhang, “On achievable rate of Gaussian ‘cheap’ relay channel,” in *Proc. IEEE Global Telecomm. Conf. (GLOBECOM)*, San Francisco, CA, Dec. 2003, pp. 1776-1780.
 - [19] P. Mitran, H. Ochiai, and V. Tarokh, “Space-time diversity enhancements using collaborative communications,” *IEEE Trans. Inform. Theory*, vol. 51, no. 6, pp. 2041-2057, June 2005.
 - [20] B. Rankov and A. Wittneben, “Spectral efficient signaling for half-duplex relay channels,” in *Proc. Asilomar Conf. Signals, Systems and Computers*, Pacific Grove, CA, USA, Oct./Nov. 2005, pp. 1066-1071.
 - [21] Z. Liu, V. Stanković, and Z. Xiong, “WYNER-ZIV coding for the half-duplex relay channel,” in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing*, Philadelphia, PA, USA, Mar. 2005, pp. v-1113-v-1116.
 - [22] Z. Zhang and T. M. Duman, “Capacity approaching Turbo coding for half duplex relaying,” in *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, Adelaide, Australia, Sept. 2005, pp. 1888-1892.

- [23] A.Chakrabarti, A. Sabharwal, and B. Aazhang, "Sensitivity of achievable rates for half-duplex relay channel," in *Proc. IEEE 6th Wksp Signal Processing Advances in Wireless Commun.*, New York, USA, June 2005, pp. 970-974.
- [24] H. Mheidat and M. Uysal, "Impact of receive diversity on the performance of amplify-and-forward relaying under APS and IPS power constraints," , *IEEE Trans. Inform.Theory*, vol. 49, pp. 3062-3080, Dec. 2004.

APPENDIXES

APPENDIX I

PROOF OF THEOREM 1

Since the channel matrix for AF signal relay system with path loss involves a product of two Gaussian random variables, which is no longer Gaussian. Now we use a new approach to get the achievable rate with path loss. To prove the **Theorem 1**, we need to prove the optimal input covariance matrix to achieve the achievable rate is diagonal firstly.

In equation (7), Σ is denoted by: $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & L_{RD}b^2|h_{rd}|^2 + 1 \end{bmatrix}$, so we can get

$$\Sigma^{-1} = \frac{1}{|L_{RD}b^2|h_{rd}|^2+1} \begin{bmatrix} L_{RD}b^2|h_{rd}|^2 + 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (31)$$

And we define as follow:

$$\begin{aligned} W &= \rho \mathbf{H}^H \Sigma^{-1} \mathbf{H} \\ &= \frac{\rho}{|L_{RD}b^2|h_{rd}|^2 + 1} \begin{bmatrix} h_{sd} & 0 \\ b\sqrt{L_{RD}L_{SR}}h_{rd}h_{sr} & h_{sd} \end{bmatrix}^H \begin{bmatrix} L_{RD}b^2|h_{rd}|^2 + 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_{sd} & 0 \\ b\sqrt{L_{RD}L_{SR}}h_{rd}h_{sr} & h_{sd} \end{bmatrix} \\ &= \begin{bmatrix} \rho|h_{sd}|^2 + \frac{\rho b^2 L_{RD}L_{SR}|h_{sr}h_{rd}|^2}{|L_{RD}b^2|h_{rd}|^2+1} & \frac{\rho\sqrt{L_{RD}L_{SR}}h_{sr}^*h_{rd}^*h_{sd}}{|L_{RD}b^2|h_{rd}|^2+1} \\ \frac{\rho\sqrt{L_{RD}L_{SR}}h_{sd}^*h_{sr}h_{rd}}{|L_{RD}b^2|h_{rd}|^2+1} & \frac{\rho|h_{sd}|^2}{|L_{RD}b^2|h_{rd}|^2+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned} \quad (32)$$

Therefore, we have

$$E_{\mathbf{h}}[\log \det(\mathbf{I}_2 + \mathbf{W}\mathbf{Q})] = E_{\mathbf{h}}[\log a_1] + E_{\mathbf{h}}[\log a_2] \quad (33)$$

Where a_1 and a_2 are determined by

$$\begin{aligned}
a_1 &= 1 + aq_{11} + dq_{22} + (ad - bc) \det Q \\
&= 1 + \left(\rho |h_{sd}|^2 + \frac{\rho b^2 L_{SR} L_{RD} |h_{sr} h_{rd}|^2}{L_{RD} b^2 |h_{rd}|^2 + 1} \right) q_{11} + \frac{\rho |h_{sd}|^2}{L_{RD} b^2 |h_{rd}|^2 + 1} q_{22} \\
&\quad + \left(\left(\rho |h_{sd}|^2 + \frac{\rho b^2 L_{SR} L_{RD} |h_{sr} h_{rd}|^2}{L_{RD} b^2 |h_{rd}|^2 + 1} \right) \left(\frac{\rho |h_{sd}|^2}{L_{RD} b^2 |h_{rd}|^2 + 1} \right) - \frac{\rho b \sqrt{L_{SR} L_{RD}} h_{sr}^* h_{rd}^* h_{sd}}{L_{RD} b^2 |h_{rd}|^2 + 1} \frac{\rho b \sqrt{L_{SR} L_{RD}} h_{sr} h_{rd} h_{sd}^*}{L_{RD} b^2 |h_{rd}|^2 + 1} \right) \det Q \\
&= 1 + \rho |h_{sd}|^2 q_{11} + \frac{1}{L_{RD} b^2 |h_{rd}|^2 + 1} \left(\rho b^2 L_{SR} L_{RD} |h_{sr} h_{rd}|^2 q_{11} + \rho |h_{sd}|^2 q_{22} + \rho^2 |h_{sd}|^4 \det Q \right) \\
a_2 &= 1 + \frac{1}{a_1} 2 \operatorname{Re}(cq_{12}) \\
&= 1 + 2 \operatorname{Re} \left(\frac{\rho b \sqrt{L_{SR} L_{RD}} h_{sr} h_{rd} h_{sd}^* q_{12}}{a_1 L_{RD} b^2 |h_{rd}|^2 + 1} \right)
\end{aligned} \tag{34}$$

In the following, we will prove the second term in (33) $E_{\mathbf{h}}[\log a_2] \leq 0$

Let $h_{sd} = |h_{sd}|e^{j\theta_{sd}}$, $h_{sr} = |h_{sr}|e^{j\theta_{sr}}$, $h_{rd} = |h_{rd}|e^{j\theta_{rd}}$ and $q_{12} = |q_{12}|e^{j\alpha}$, then we

have

$$\begin{aligned}
E_{\mathbf{h}}[a_2] &= E_{\mathbf{h}} \left[\log \left(1 + 2 \operatorname{Re} \left(\rho \frac{b |h_{sr} h_{sd} h_{rd}| |q_{12}|}{a_1 (1 + b_1^2 L_{RD} |h_{rd}|^2)} e^{-\theta_{sd} + \theta_{sr} + \theta_{rd} + \alpha} \right) \right) \right] \\
&= E_{\mathbf{h}}[\log(1 + \varepsilon \cos(-\theta_{sd} + \theta_{sr} + \theta_{rd} + \alpha))]
\end{aligned}$$

$$\text{Here, } \varepsilon = 2\rho \frac{b_1 \sqrt{L_{SR} L_{RD}} |h_{sr} h_{sd} h_{rd}| |q_{12}|}{a_1 (1 + b_1^2 L_{RD} |h_{rd}|^2)}.$$

Since channel gains h_{sd} , h_{sr} , h_{rd} are circular Gaussian variables, their magnitudes are Rayleigh distributed and their phases are uniformly distributed in $[0, 2\pi]$. Since b is independent of $\theta_{sd}, \theta_{sr}, \theta_{rd}$; $|q_{12}|$ and $e^{j\alpha}$ are deterministic; h_{rd} is independent of h_{sd}, h_{sr} , we can evaluate $E_{\mathbf{h}}[\log a_2]$ by taking expectation over random variable θ_{sd} first,

$$\begin{aligned}
E_{\mathbf{h}}[\log a_2] &= E_{\mathbf{h}} \left[\int_0^{2\pi} \log(1 + \varepsilon \cos(-\theta_{sd} + \theta_{sr} + \theta_{rd} + \alpha)) d\theta_{rd} \right] \\
&= E_{\mathbf{h}} \left[\int_0^{\pi} \log(1 - \varepsilon^2 \cos^2(-\theta_{sd} + \theta_{sr} + \theta_{rd} + \alpha)) d\theta_{rd} \right]
\end{aligned}$$

Since

$$1 - \varepsilon^2 \cos^2(-\theta_{sd} + \theta_{sr} + \theta_{rd} + \alpha) \leq 1$$

$$\log(1 - \varepsilon^2 \cos^2(-\theta_{sd} + \theta_{sr} + \theta_{rd} + \alpha)) \leq 0$$

$$E_{\mathbf{h}}[\log a_2] \leq 0$$

Combining with (14), we can get

$$E_{\mathbf{h}}[\log \det(\mathbf{I}_2 + \rho \mathbf{H}^H \boldsymbol{\Sigma}^{-1} \mathbf{H} \mathbf{Q})] \leq E_{\mathbf{h}}[\log a_1] \quad (35)$$

Where the equality in (16) holds if and only if $E_{\mathbf{h}}[\log a_2] = 0$, which is equivalent to $q_{12} = 0$.

On the other hand, we know:

$$\det(\mathbf{Q}) = q_{11}q_{22} - |q_{12}|^2 \leq q_{11}q_{22} \quad (36)$$

Where the equality in (36) holds if and only if $q_{12} = 0$. Combining (35) with (36), we have

$$\begin{aligned} & E_{\mathbf{h}}[\log \det(\mathbf{I}_2 + \rho \mathbf{H}^H \boldsymbol{\Sigma}^{-1} \mathbf{H} \mathbf{Q})] \\ & \leq E_{\mathbf{h}} \left[\log \left((1 + \rho |h_{sd}|^2 q_{11}) \left(1 + \rho \frac{|h_{sd}|^2 q_{22}}{1 + b^2 |L_{RD} h_{rd}|^2} \right) + \rho \frac{b^2 L_{SR} L_{RD} |h_{sr} h_{rd}|^2 q_{11}}{1 + b^2 L_{RD} |h_{rd}|^2} \right) \right] \end{aligned} \quad (37)$$

The equality in (37) holds if only if $q_{12} = 0$ and $\mathbf{Q}$ is diagonal. Then, $\text{tr}(\mathbf{Q}) = q_{11} + q_{22}$.

Above all, we can get the achievable rate as follows:

$$\begin{aligned} C &= \frac{1}{2} \max_{b, \text{tr}(\mathbf{Q}) \leq P_s} E_{\mathbf{h}}[\log \det(\mathbf{I}_2 + \rho \mathbf{H}^H \boldsymbol{\Sigma}^{-1} \mathbf{H} \mathbf{Q})] \\ &= \frac{1}{2} \max_{b, q_{11} + q_{22} \leq P_s} E_{\mathbf{h}} \left[\log_2 \left((1 + \rho |h_{sd}|^2 q_{11}) \left(1 + \frac{\rho |h_{sd}|^2 q_{22}}{1 + b^2 L_{RD} |h_{rd}|^2} \right) + \rho \frac{b^2 L_{SR} L_{RD} |h_{sr} h_{rd}|^2 q_{11}}{1 + b^2 L_{RD} |h_{rd}|^2} \right) \right] \end{aligned} \quad (38)$$

The optimal input covariance matrix to achieve the achievable rate with path loss for AF system is $\tilde{\mathbf{Q}} = \text{diag}(\check{q}_1, \check{q}_2)$, where $\check{q}_1$ and $\check{q}_2$ are obtained by solving the optimization problem in (38). Thus, we get the proof **Theorem 1** now.

APPENDIX II

DERIVATIONS OF APPROXIMATIONS

A. Approximation under Average Constraint

Firstly, we will prove the **Lemma 1**. Based on the equation (22), we can reorganize the equation (21) as:

$$\begin{aligned}
 J_c(\rho, q_1, q_2, b) &= E_{x,y,z} \left[\log \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2}{1 + b^2 L_{RD} y} \right) + \frac{\rho b^2 L_{SR} L_{RD} q_1 y z}{1 + b^2 L_{RD} y} \right) \right] \\
 &= \int_0^\infty \int_0^\infty \int_0^\infty \log \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2}{1 + b^2 L_{RD} y} \right) + \frac{\rho b^2 L_{SR} L_{RD} q_1 y z}{1 + b^2 L_{RD} y} \right) e^{-(x+y+z)} dx dy dz \\
 &= \log_2(e) \int_0^\infty \int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz
 \end{aligned}$$

Where $\ln(\cdot)$ is the natural logarithm. Next, we evaluate $J_c(\rho, q_{11}, q_{22}, b)$ at low SNR.

$$\begin{aligned}
 J_c(\rho, q_1, q_2, b) &= \log_2(e) \underbrace{\left[\int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) \right) e^{-(x+y)} dx dy \right]}_{L_1} \\
 &\quad + \underbrace{\int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz}_{L_2}
 \end{aligned} \tag{39}$$

First of all, we derive the first part of J_c :

$$\begin{aligned}
 L_1 &= \int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) \right) e^{-(x+y)} dx dy \\
 &= \int_0^\infty \ln(1 + \rho x q_1) e^{-x} dx + \int_0^\infty \int_0^\infty \ln \left(1 + \frac{\rho x q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y)} dx dy \\
 &= -\int_0^\infty \frac{\rho q_1}{1 + \rho x q_1} de^{-x} - \int_0^\infty \int_0^\infty \ln \left(\frac{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) de^{-x} e^{-y} dy \\
 &= \rho q_1 + \int_0^\infty \frac{\rho^2 q_1^2}{(1 + \rho x q_1)^2} de^{-x} + \int_0^\infty \int_0^\infty \frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} e^{-(x+y)} dx dy \\
 &= \rho q_1 - \rho^2 q_1^2 - \int_0^\infty \frac{2\rho^3 q_1^3}{(1 + \rho x q_1)^3} de^{-x} - \int_0^\infty \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} de^{-x} e^{-y} dy
 \end{aligned}$$

$$\begin{aligned}
&= \rho q_1 - \rho^2 q_1^2 + \underbrace{2\rho^3 q_1^3}_{o(\rho^3)} - \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} de^{-y} \\
&\quad + \int_0^\infty \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^2 de^{-x} e^{-y} dy
\end{aligned}$$

Where $f(x) \triangleq o(g(x))$, $g(x) > 0$ denotes that when x is small, there exists a positive constant c

such that $|f(x)| \leq cg(x)$.

$$\begin{aligned}
&\because - \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} de^{-y} \\
&= \frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + 1} + \int_0^\infty \frac{\rho^2 L_{RD} w P_r q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^2} de^{-y} \\
&= \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{SR} q_1 + 1)} - \underbrace{\int_0^\infty \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^3} de^{-y}}_{o(\rho^3)} \\
&= \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{SR} q_1 + 1)} + o(\rho^3) \\
&\because \int_0^\infty \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^2 e^{-y} de^{-x} dy \\
&= - \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^2 e^{-y} dy - \underbrace{\int_0^\infty \int_0^\infty 2 * \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^3 e^{-y} de^{-x} dy}_{o(\rho^3)} \\
&= \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^2 de^{-y} + o(\rho^3) \\
&= - \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + 1} \right)^2 - \underbrace{\int_0^\infty \frac{2\rho^3 L_{RD} w P_r q_2^2 (1 + \rho L_{SR} q_1)^2}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^3} de^{-y}}_{o(\rho^3)} + o(\rho^3) \\
&= -\rho^2 q_2^2 + o(\rho^3)
\end{aligned}$$

Thus, we can express L_1 as:

$$\begin{aligned}
\Rightarrow L_1 &= \rho q_1 - \rho^2 q_1^2 + \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{\rho L_{SR} q_1 + 1} - \rho^2 q_2^2 + o(\rho^3) \\
&= \rho(q_1 + q_2) - \rho^2(q_1^2 + q_2^2) - \frac{\rho^2 L_{RD} w P_r q_2}{\rho L_{SR} q_1 + 1} + o(\rho^3) \\
&= \rho P_s - \rho^2(q_1^2 + q_2^2) - \frac{\rho^2 L_{RD} w P_r q_2}{\rho L_{SR} q_1 + 1} + o(\rho^3)
\end{aligned} \tag{40}$$

Now, we derive the second part of J_C :

$$\begin{aligned}
&\because \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) \\
&= \frac{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} * \frac{d}{dz} \left(\frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z}
\end{aligned}$$

So we can express L_2 as:

$$\begin{aligned}
L_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz \\
&= - \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} e^{-(x+y)} de^{-z} dx dy \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1))} e^{-x} de^{-y} dx + o(\rho^3) \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1))} e^{-x} de^{-y} dx + o(\rho^3) \\
&= - \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + 1 + \rho x q_2(1 + \rho L_{SR} q_1))} de^{-x} + o(\rho^3) \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(\rho L_{SR} q_1 + 1)} + o(\rho^3)
\end{aligned} \tag{41}$$

Combing equation (40) and (41), we can get $J_{C,1}$ as:

$$\begin{aligned}
J_{c-1}(\rho, q_1, q_2, b) &= \log_2(e)(L_1 + L_2) \\
&= \log_2(e) \left(\rho P_s - \rho^2(q_1^2 + q_2^2) + \frac{\rho^2 L_{RD} w P_r (L_{SR} q_1 - q_2)}{\rho L_{SR} q_1 + 1} \right) + o(\rho^3)
\end{aligned} \tag{42}$$

Thus, we can get **Lemma 1** as equation (42).

Secondly, we prove the **Lemma 2** which is focus on the further approximation of achievable rate up to cubic of SNR. From equation (39), we can express L_1 as:

$$\begin{aligned}
L_1 &= \int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) \right) e^{-(x+y)} dx dy \\
&= \int_0^\infty \ln(1 + \rho x q_1) e^{-x} dx + \int_0^\infty \int_0^\infty \ln \left(1 + \frac{\rho x q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y)} dx dy \\
&= - \int_0^\infty \frac{\rho q_1}{1 + \rho x q_1} de^{-x} - \int_0^\infty \int_0^\infty \ln \left(\frac{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-y} de^{-x} dy \\
&= \rho q_1 + \int_0^\infty \frac{\rho^2 q_1^2}{(1 + \rho x q_1)^2} de^{-x} + \int_0^\infty \int_0^\infty \frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} e^{-(x+y)} dx dy \\
&= \rho q_1 - \rho^2 q_1^2 - \int_0^\infty \frac{2\rho^3 q_1^3}{(1 + \rho x q_1)^3} de^{-x} - \int_0^\infty \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} de^{-x} e^{-y} dy \\
&= \rho q_1 - \rho^2 q_1^2 + 2\rho^3 q_1^3 \\
&\quad - \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} de^{-y} + \int_0^\infty \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^2 e^{-y} de^{-x} dy \\
&\quad \because - \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} de^{-y} \\
&= \frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + 1} + \int_0^\infty \frac{\rho^2 L_{RD} w P_r q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^2} de^{-y} \\
&= \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{SR} q_1 + 1)} - \int_0^\infty \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^3} de^{-y} \\
&= \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{SR} q_1 + 1)} + \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(\rho L_{SR} q_1 + 1)^2} + \underbrace{\int_0^\infty \frac{24\rho^4 L_{RD}^3 w^3 P_r^3 q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^4} de^{-y}}_{o(\rho^4)} \\
&= \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{SR} q_1 + 1)} + \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(\rho L_{SR} q_1 + 1)^2} + o(\rho^4)
\end{aligned}$$

$$\begin{aligned}
& \because \int_0^\infty \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^2 e^{-y} de^{-x} dy \\
&= - \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^2 e^{-y} dy - \int_0^\infty \int_0^\infty 2^* \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^3 e^{-y} de^{-x} dy \\
&= \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^2 de^{-y} + \int_0^\infty 2^* \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^3 e^{-y} dy + o(\rho^4) \\
&= - \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + 1} \right)^2 - \int_0^\infty \frac{2 \rho^3 L_{RD} w P_r q_2^2 (1 + \rho L_{SR} q_1)^2}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^3} de^{-y} - 2 \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^3 de^{-y} + o(\rho^4) \\
&= -\rho^2 q_2^2 + \frac{2 \rho^3 L_{RD} w P_r q_2^2}{(\rho L_{SR} q_1 + 1)} + 2 \rho^3 q_2^3 + \underbrace{\int_0^\infty \frac{6 \rho^4 L_{RD}^2 w^2 P_r^2 q_2^2 (1 + \rho L_{SR} q_1)^2}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^4} de^{-y} + 6 \int_0^\infty \frac{\rho^4 L_{RD} w P_r q_2^3 (1 + \rho L_{SR} q_1)^3}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^4} de^{-y}}_{o(\rho^4)} \\
&\Rightarrow L_1 = \rho q_1 - \rho^2 q_1^2 + \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{\rho L_{SR} q_1 + 1} + \frac{2 \rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(\rho L_{SR} q_1 + 1)^2} - \rho^2 q_2^2 + \frac{2 \rho^3 L_{RD} w P_r q_2^2}{(\rho L_{SR} q_1 + 1)} + 2 \rho^3 (q_1^3 + q_2^3) + o(\rho^4) \\
&= \rho (q_1 + q_2) - \rho^2 (q_1^2 + q_2^2) - \frac{\rho^2 L_{RD} w P_r q_2}{\rho L_{SR} q_1 + 1} + \frac{2 \rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(\rho L_{SR} q_1 + 1)^2} + 2 \rho^3 (q_1^3 + q_2^3) + o(\rho^4) \\
&= \rho P_s - \rho^2 (q_1^2 + q_2^2) - \frac{\rho^2 L_{RD} w P_r q_2}{\rho L_{SR} q_1 + 1} + \frac{2 \rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(\rho L_{SR} q_1 + 1)^2} + 2 \rho^3 (q_1^3 + q_2^3) + o(\rho^4)
\end{aligned} \tag{43}$$

Now, we derive the approximation of L_2 , which is the second part in (39), as:

$$\begin{aligned}
L_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz \\
&= - \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1) (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} e^{-(x+y)} de^{-z} dx dy \\
&= \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1) (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))} e^{-(x+y)} dx dy + o(\rho^4) \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1) (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))} e^{-x} de^{-y} dx + o(\rho^4) \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1) (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))} e^{-x} de^{-y} dx \\
&\quad + \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 y}{(1 + \rho x q_1) (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))^2} e^{-x} de^{-y} dx + o(\rho^4)
\end{aligned}$$

$$\begin{aligned}
&= -\int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + 1 + \rho x q_2(1 + \rho L_{SR} q_1))} de^{-x} \\
&\quad + \int_0^\infty \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w y + 1 + \rho x q_2(1 + \rho L_{SR} q_1))^2} e^{-x} de^{-y} dx + o(\rho^4) \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(\rho L_{SR} q_1 + 1)} + \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1^2}{(1 + \rho x q_1)^2 (\rho L_{SR} q_1 + 1 + \rho x q_2(1 + \rho L_{SR} q_1))} de^{-x} \\
&\quad + \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 (1 + \rho L_{SR} q_1)}{(1 + \rho x q_1)(\rho L_{SR} q_1 + 1 + \rho x q_2(1 + \rho L_{SR} q_1))^2} de^{-x} \\
&\quad + \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + 1 + \rho x q_2(1 + \rho L_{SR} q_1))^2} de^{-x} + o(\rho^4) \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(\rho L_{SR} q_1 + 1)} - \frac{\rho^3 L_{SR} L_{RD} w P_r q_1^2}{(\rho L_{SR} q_1 + 1)} - \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2}{(\rho L_{SR} q_1 + 1)} - \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(\rho L_{SR} q_1 + 1)^2} + o(\rho^4) \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (1 - \rho q_1 - \rho q_2)}{(\rho L_{SR} q_1 + 1)} - \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(\rho L_{SR} q_1 + 1)^2} + o(\rho^4)
\end{aligned} \tag{44}$$

From equation (43) and (44), the achievable rate at low SNR can be approximated by a cubic function of ρ as in **Lemma 2**.

Finally, we will prove the **Lemma 3**, which is the approximation of achievable rate up to ρ^4 . From equation (39), we can express L_1 as:

$$\begin{aligned}
L_1 &= \int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) \right) e^{-(x+y)} dx dy \\
&= \int_0^\infty \ln(1 + \rho x q_1) e^{-x} dx + \int_0^\infty \int_0^\infty \ln \left(1 + \frac{\rho x q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y)} dx dy \\
&= -\int_0^\infty \frac{\rho q_1}{1 + \rho x q_1} de^{-x} - \int_0^\infty \int_0^\infty \ln \left(\frac{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-y} de^{-x} dy \\
&= \rho q_1 + \int_0^\infty \frac{\rho^2 q_1^2}{(1 + \rho x q_1)^2} de^{-x} + \int_0^\infty \int_0^\infty \frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} e^{-(x+y)} dx dy
\end{aligned}$$

$$\begin{aligned}
&= \rho q_1 - \rho^2 q_1^2 - \int_0^\infty \frac{2\rho^3 q_1^3}{(1 + \rho x q_1)^3} de^{-x} - \int_0^\infty \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} e^{-y} de^{-x} dy \\
&= \rho q_1 - \rho^2 q_1^2 + 2\rho^3 q_1^3 + \int_0^\infty \frac{6\rho^4 q_1^4}{(1 + \rho x q_1)^4} de^{-x} \\
&\quad - \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} de^{-y} + \int_0^\infty \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^2 e^{-y} de^{-x} dy \\
&= \rho q_1 - \rho^2 q_1^2 + 2\rho^3 q_1^3 - 6\rho^4 q_1^4 + o(\rho^5) \\
&\quad - \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} de^{-y} + \int_0^\infty \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^2 e^{-y} de^{-x} dy \\
&\because - \int_0^\infty \frac{\rho q_2 + \rho^2 L_{SR} q_1 q_2}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} de^{-y} \\
&= \frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + 1} + \int_0^\infty \frac{\rho^2 L_{RD} w P_r q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^2} de^{-y} \\
&= \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{SR} q_1 + 1)} - \int_0^\infty \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^3} de^{-y} \\
&= \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{SR} q_1 + 1)} + \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(\rho L_{SR} q_1 + 1)^2} + \int_0^\infty \frac{6\rho^4 L_{RD}^3 w^3 P_r^3 q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^4} de^{-y} \\
&= \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{SR} q_1 + 1)} + \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(\rho L_{SR} q_1 + 1)^2} - \frac{6\rho^4 L_{RD}^3 w^3 P_r^3 q_2}{(\rho L_{SR} q_1 + 1)^3} - \underbrace{\int_0^\infty \frac{24\rho^5 L_{RD}^4 w^4 P_r^4 q_2 (1 + \rho L_{SR} q_1)}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^5} de^{-y}}_{o(\rho^5)} \\
&\because \int_0^\infty \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^2 e^{-y} de^{-x} dy \\
&= - \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^2 e^{-y} dy - \int_0^\infty \int_0^\infty 2 * \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^3 e^{-y} de^{-x} dy \\
&= \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^2 de^{-y} + \int_0^\infty 2 * \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^3 e^{-y} dy \\
&\quad + \int_0^\infty \int_0^\infty 6 * \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 + \rho^2 L_{SR} q_1 q_2 x} \right)^4 e^{-y} de^{-x} dy \\
&= - \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + 1} \right)^2 - \int_0^\infty \frac{2\rho^3 L_{RD} w P_r q_2^2 (1 + \rho L_{SR} q_1)^2}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^3} de^{-y} - 2 \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^3 de^{-y} \\
&\quad - 6 \int_0^\infty \left(\frac{\rho q_2 (1 + \rho L_{SR} q_1)}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right)^4 e^{-y} dy + o(\rho^5)
\end{aligned}$$

$$\begin{aligned}
&= -\rho^2 q_2^2 + \frac{2\rho^3 L_{RD} w P_r q_2^2}{(\rho L_{SR} q_1 + 1)} + 2\rho^3 q_2^3 + \int_0^\infty \frac{6\rho^4 L_{RD}^2 w^2 P_r^2 q_2^2 (1 + \rho L_{SR} q_1)^2}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^4} de^{-y} \\
&\quad + 6 \int_0^\infty \frac{\rho^4 L_{RD} w P_r q_2^3 (1 + \rho L_{SR} q_1)^3}{(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1)^4} de^{-y} - 6\rho^4 q_2^4 + o(\rho^5) \\
&= -\rho^2 q_2^2 + \frac{2\rho^3 L_{RD} w P_r q_2^2}{(\rho L_{SR} q_1 + 1)} + 2\rho^3 q_2^3 - \frac{6\rho^4 L_{RD}^2 w^2 P_r^2 q_2^2}{(\rho L_{SR} q_1 + 1)^2} - \frac{6\rho^4 L_{RD} w P_r q_2^3}{(\rho L_{SR} q_1 + 1)} - 6\rho^4 q_2^4 + o(\rho^5)
\end{aligned}$$

Above all, we can get L_1 as:

$$\begin{aligned}
L_1 &= \rho q_1 - \rho^2 q_1^2 + \rho q_2 - \frac{\rho^2 L_{RD} w P_r q_2}{1 + \rho L_{SR} q_1} + \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(1 + \rho L_{SR} q_1)^2} - \rho^2 q_2^2 + \frac{2\rho^3 L_{RD} w P_r q_2^2}{(1 + \rho L_{SR} q_1)} + 2\rho^3 (q_1^3 + q_2^3) \\
&\quad - \frac{6\rho^4 L_{RD}^3 w^3 P_r^3 q_2}{(1 + \rho L_{SR} q_1)^3} - \frac{6\rho^4 L_{RD}^2 w^2 P_r^2 q_2^2}{(1 + \rho L_{SR} q_1)^2} - \frac{6\rho^4 L_{RD} w P_r q_2^3}{(1 + \rho L_{SR} q_1)} - 6\rho^4 (q_1^4 + q_2^4) + o(\rho^5) \\
&= \rho(q_1 + q_2) - \rho^2(q_1^2 + q_2^2) - \frac{\rho^2 L_{RD} w P_r q_2 - 2\rho^3 L_{RD} w P_r q_2^2}{1 + \rho L_{SR} q_1} + \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(1 + \rho L_{SR} q_1)^2} + 2\rho^3 (q_1^3 + q_2^3) \\
&\quad - \frac{6\rho^4 L_{RD} w P_r q_2^3}{(1 + \rho L_{SR} q_1)} - \frac{6\rho^4 L_{RD}^2 w^2 P_r^2 q_2^2}{(1 + \rho L_{SR} q_1)^2} - \frac{6\rho^4 L_{RD}^3 w^3 P_r^3 q_2}{(1 + \rho L_{SR} q_1)^3} - 6\rho^4 (q_1^4 + q_2^4) + o(\rho^5)
\end{aligned} \tag{45}$$

Now we derive the L_2 part:

$$\begin{aligned}
&\therefore \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) \\
&= \frac{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} * \frac{d}{dz} \left(\frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} \\
L_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz \\
&= - \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} e^{-(x+y)} de^{-z} dx dy \\
&= \underbrace{\int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))} e^{-(x+y)} dx dy}_A \\
&\quad + \underbrace{\int_0^\infty \int_0^\infty \int_0^\infty \left(\frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} \right)^2 e^{-(x+y)} de^{-z} dx dy}_B
\end{aligned} \tag{46}$$

$$\begin{aligned}
A &= -\int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))} e^{-x} de^{-y} dx \\
&= \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (\rho L_{SR} q_1 + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))^2} e^{-x} e^{-y} dx dy \\
&= -\int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (\rho L_{SR} q_1 + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))^2} e^{-x} de^{-y} dx \\
&= \int_0^\infty \underbrace{\frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))}}_{C_1} e^{-x} dx \\
&\quad - \underbrace{\int_0^\infty \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (\rho L_{SR} q_1 + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))^3} e^{-x} e^{-y} dx dy}_{C_2}
\end{aligned}$$

$$\begin{aligned}
C_1 &= -\int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(\rho L_{SR} q_1 + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))} de^{-x} \\
&= -\int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(1 + \rho x q_2)(1 + \rho L_{SR} q_1)} de^{-x} \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} - \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (2\rho^2 q_1 q_2 x + \rho q_1 + \rho q_2)}{(1 + \rho L_{SR} q_1)(1 + \rho x q_1)^2 (1 + \rho x q_2)^2} e^{-x} dx \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} + \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (2\rho^2 q_1 q_2 x + \rho P_s)}{(1 + \rho L_{SR} q_1)(1 + \rho x q_1)^2 (1 + \rho x q_2)^2} de^{-x} \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} - \frac{\rho^3 L_{SR} L_{RD} w P_r P_s q_1}{(1 + \rho L_{SR} q_1)} \\
&\quad + \underbrace{\frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} \int_0^\infty \frac{d}{dx} \left(\frac{(2\rho^2 q_1 q_2 x + \rho P_s)}{(1 + \rho x q_1)^2 (1 + \rho x q_2)^2} \right) e^{-x} dx}_{D_1}
\end{aligned}$$

$$\begin{aligned}
D_1 &= \int_0^\infty \frac{d}{dx} \left(\frac{(2\rho^2 q_1 q_2 x + \rho P_s)}{(1 + \rho x q_1)^2 (1 + \rho x q_2)^2} \right) e^{-x} dx \\
&= \int_0^\infty \frac{2\rho^2 q_1 q_2 (1 + \rho x q_1)(1 + \rho x q_2) - (2\rho^2 q_1 q_2 x + \rho P_s) [2\rho q_1 (1 + \rho x q_2) + 2\rho q_2 (1 + \rho x q_1)]}{(1 + \rho x q_1)^3 (1 + \rho x q_2)^3} e^{-x} dx \\
&= \int_0^\infty \frac{2\rho^2 q_1 q_2 (1 + \rho x q_1)(1 + \rho x q_2) - (2\rho^2 q_1 q_2 x + \rho P_s) [2\rho q_1 + 2\rho^2 q_1 q_2 x + 2\rho q_2 + 2\rho^2 q_1 q_2 x]}{(1 + \rho x q_1)^3 (1 + \rho x q_2)^3} e^{-x} dx \\
&= \int_0^\infty \frac{2\rho^2 q_1 q_2 (1 + \rho x q_1)(1 + \rho x q_2) - (2\rho^2 q_1 q_2 x + \rho P_s) [4\rho + 4\rho^2 q_1 q_2 x]}{(1 + \rho x q_1)^3 (1 + \rho x q_2)^3} e^{-x} dx
\end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \frac{2\rho^2 q_1 q_2 (1 + \rho x q_1)(1 + \rho x q_2) - (8\rho^3 q_1 q_2 x + 4\rho^2 P_s + 8\rho^4 q_1^2 q_2^2 x^2 + 4\rho^3 q_1 q_2 P_s x)}{(1 + \rho x q_1)^3 (1 + \rho x q_2)^3} e^{-x} dx \\
&= \int_0^\infty \frac{2\rho^2 q_1 q_2 (1 + \rho x q_1)(1 + \rho x q_2) - 4\rho^2 (2\rho q_1 q_2 x + P_s + 2\rho^2 q_1^2 q_2^2 x^2 + \rho q_1 q_2 P_s x)}{(1 + \rho x q_1)^3 (1 + \rho x q_2)^3} e^{-x} dx \\
&= \int_0^\infty \frac{2\rho^2 q_1 q_2 (1 + \rho x q_1)(1 + \rho x q_2) - 4\rho^2 P_s}{(1 + \rho x q_1)^3 (1 + \rho x q_2)^3} e^{-x} dx + o(\rho^3) \\
&= - \int_0^\infty \frac{2\rho^2 q_1 q_2}{(1 + \rho x q_1)^2 (1 + \rho x q_2)^2} de^{-x} + \int_0^\infty \frac{4\rho^2 P_s}{(1 + \rho x q_1)^3 (1 + \rho x q_2)^3} de^{-x} + o(\rho^3) \\
&= 2\rho^2 q_1 q_2 - \int_0^\infty \frac{2\rho^2 q_1 q_2 [2\rho q_1 (1 + \rho x q_2) + 2\rho q_2 (1 + \rho x q_1)]}{(1 + \rho x q_1)^3 (1 + \rho x q_2)^3} e^{-x} dx - 4\rho^2 P_s + o(\rho^3) \\
&= 2\rho^2 q_1 q_2 - 4\rho^2 P_s + o(\rho^3)
\end{aligned}$$

Take D_1 to C_1 , we get C_1 :

$$\begin{aligned}
C_1 &= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} - \frac{\rho^3 L_{SR} L_{RD} w P_r P_s q_1}{(1 + \rho L_{SR} q_1)} \\
&\quad + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} (2\rho^2 q_1 q_2 - 4\rho^2 P_s) \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} - \frac{\rho^3 L_{SR} L_{RD} w P_r P_s q_1}{(1 + \rho L_{SR} q_1)} + \frac{2\rho^4 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} (q_1 q_2 - 2P_s) + o(\rho^5)
\end{aligned}$$

Now we derive C_2 as follow:

$$\begin{aligned}
C_2 &= - \int_0^\infty \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (1 + \rho x q_2)(1 + \rho L_{SR} q_1)}{(1 + \rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1))^3} e^{-x} e^{-y} dx dy \\
&= \int_0^\infty \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (1 + \rho x q_2)(1 + \rho L_{SR} q_1)}{(1 + \rho x q_1)((1 + \rho x q_2)(1 + \rho L_{SR} q_1) + \rho L_{RD} w P_r y)^3} e^{-x} de^{-y} dx \\
&= - \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (1 + \rho x q_2)(1 + \rho L_{SR} q_1)}{(1 + \rho x q_1)((1 + \rho x q_2)(1 + \rho L_{SR} q_1))^3} e^{-x} dx \\
&\quad - \int_0^\infty \int_0^\infty \frac{6\rho^4 L_{SR} L_{RD}^3 w^3 P_r^3 q_1 (1 + \rho x q_1)(1 + \rho L_{SR} q_1)}{(1 + \rho x q_1)((1 + \rho x q_2)(1 + \rho L_{SR} q_1) + \rho L_{RD} w P_r y)^4} e^{-x} de^{-y} dx \\
&= \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1 + \rho x q_1)((1 + \rho x q_2)(1 + \rho L_{SR} q_1))^2} de^{-x} \\
&\quad + \int_0^\infty \frac{6\rho^4 L_{SR} L_{RD}^3 w^3 P_r^3 q_1}{(1 + \rho x q_1)((1 + \rho x q_2)(1 + \rho L_{SR} q_1))^3} e^{-x} dx + o(\rho^5)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1+\rho L_{SR} q_1)^2} - \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 \left[\rho q_1 (1+\rho x q_2)^2 + 2\rho q_2 (1+\rho x q_1)(1+\rho x q_2) \right]}{(1+\rho x q_1)^2 (1+\rho L_{SR} q_1)^2 (1+\rho x q_2)^4} de^{-x} \\
&\quad - \int_0^\infty \frac{6\rho^4 L_{SR} L_{RD}^3 w^3 P_r^3 q_1}{(1+\rho x q_1)((1+\rho x q_2)(1+\rho L_{SR} q_1))^3} de^{-x} + o(\rho^5) \\
&= -\frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1+\rho L_{SR} q_1)^2} + \frac{2\rho^4 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (q_2 + P_s)}{(1+\rho L_{SR} q_1)^2} + \frac{6\rho^4 L_{SR} L_{RD}^3 w^3 P_r^3 q_1}{(1+\rho L_{SR} q_1)^3} + o(\rho^5)
\end{aligned}$$

Based on the result C_1 and C_2 , we can express A as:

$$\begin{aligned}
A &= C_1 + C_2 \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1+\rho L_{SR} q_1)} - \frac{\rho^3 L_{SR} L_{RD} w P_r P_s q_1}{(1+\rho L_{SR} q_1)} - \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1+\rho L_{SR} q_1)^2} \\
&\quad + \frac{2\rho^4 L_{SR} L_{RD} w P_r q_1}{(1+\rho L_{SR} q_1)} (q_1 q_2 - 2P_s) + \frac{2\rho^4 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (q_2 + P_s)}{(1+\rho L_{SR} q_1)^2} + \frac{6\rho^4 L_{SR} L_{RD}^3 w^3 P_r^3 q_1}{(1+\rho L_{SR} q_1)^3} + o(\rho^5)
\end{aligned} \tag{47}$$

In the following, we derive the B part as follow:

$$\begin{aligned}
B &= \int_0^\infty \int_0^\infty \int_0^\infty \left(\frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1+\rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1+\rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} \right)^2 e^{-(x+y)} de^{-z} dx dy \\
&= -\int_0^\infty \int_0^\infty \frac{\rho^4 L_{SR}^2 L_{RD}^2 w^2 P_r^2 q_1^2 y^2}{(1+\rho x q_1)^2 (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1+\rho L_{SR} q_1))^2} e^{-(x+y)} dx dy \\
&\quad - \int_0^\infty \int_0^\infty \int_0^\infty \frac{2\rho^6 L_{SR}^3 L_{RD}^3 w^3 P_r^3 q_1^3 y^3}{((1+\rho x q_1)(\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1+\rho L_{SR} q_1)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)^3} e^{-(x+y)} de^{-z} dx dy \\
&= \int_0^\infty \int_0^\infty \frac{\rho^4 L_{SR}^2 L_{RD}^2 w^2 P_r^2 q_1^2 y^2}{(1+\rho x q_1)^2 (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1+\rho L_{SR} q_1))^2} e^{-x} de^{-y} dx + o(\rho^6) \\
&= \int_0^\infty \int_0^\infty \frac{2\rho^4 L_{SR}^2 L_{RD}^2 w^2 P_r^2 q_1^2 y}{(1+\rho x q_1)^2 (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1+\rho L_{SR} q_1))^2} e^{-x} de^{-y} dx \\
&\quad - \underbrace{\int_0^\infty \int_0^\infty \frac{2\rho^5 L_{SR}^2 L_{RD}^3 w^3 P_r^3 q_1^2 y^2}{(1+\rho x q_1)^2 (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1+\rho L_{SR} q_1))^3} e^{-x} de^{-y} dx + o(\rho^6)}_{o(\rho^5)} \\
&= \int_0^\infty \int_0^\infty \frac{2\rho^4 L_{SR}^2 L_{RD}^2 w^2 P_r^2 q_1^2}{(1+\rho x q_1)^2 (\rho L_{SR} q_1 + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1+\rho L_{SR} q_1))^2} e^{-x} de^{-y} dx + o(\rho^5) \\
&= \int_0^\infty \frac{2\rho^4 L_{SR}^2 L_{RD}^2 w^2 P_r^2 q_1^2}{(1+\rho x q_1)^2 (\rho L_{SR} q_1 + 1 + \rho x q_2 (1+\rho L_{SR} q_1))^2} de^{-x} + o(\rho^5) \\
&= -\frac{2\rho^4 L_{SR}^2 L_{RD}^2 w^2 P_r^2 q_1^2}{(\rho L_{SR} q_1 + 1)^2} + o(\rho^5)
\end{aligned} \tag{48}$$

Taking equation (47) and (48) in (46), we can get:

$$\begin{aligned}
L_2 &= A + B \\
&= \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} - \frac{\rho^3 L_{SR} L_{RD} w P_r P_s q_1}{(1 + \rho L_{SR} q_1)} - \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1 + \rho L_{SR} q_1)^2} \\
&+ \frac{2\rho^4 L_{SR} L_{RD} w P_r q_1}{(1 + \rho L_{SR} q_1)} (q_1 q_2 - 2P_s) + \frac{2\rho^4 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (q_2 + P_s)}{(1 + \rho L_{SR} q_1)^2} - \frac{2\rho^4 L_{SR}^2 L_{RD}^2 w^2 P_r^2 q_1^2}{(1 + \rho L_{SR} q_1)^2} + \frac{6\rho^4 L_{SR} L_{RD}^3 w^3 P_r^3 q_1}{(1 + \rho L_{SR} q_1)^3} + o(\rho^5)
\end{aligned} \tag{49}$$

Taking equation (45) and equation (49) in (39), we express the achievable rate at low SNR by a quartic unction of ρ as **Lemma 3**.

B. Approximation under Instantaneous Constraint

Similarly to the A part, in this section we will discuss the approximation of the achievable rate at low SNR with the amplification coefficient b at the relay node satisfying the instantaneous constraint. From (27) equation

$$\frac{1}{1 + b^2 L_{RD} y} = \frac{1 + \rho L_{SR} q_1 z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \tag{50}$$

Firstly, we prove **Lemma 4**. Based on equation (27) and (50), we can reorganize (21) under the instantaneous constraint as:

$$\begin{aligned}
J_c(\rho, q_1, q_2, b) &= E_{x,y,z} \left[\log \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2}{1 + b^2 L_{RD} y} \right) + \frac{\rho b^2 L_{SR} L_{RD} q_1 y z}{1 + b^2 L_{RD} y} \right) \right] \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \log \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2}{1 + b^2 L_{RD} y} \right) + \frac{\rho b^2 L_{SR} L_{RD} q_1 y z}{1 + b^2 L_{RD} y} \right) e^{-(x+y+z)} dx dy dz \\
&= \log_2(e) \int_0^\infty \int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1 z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) + \rho \frac{\rho w P_r}{\rho L_{SR} q_1 z + 1} L_{SR} L_{RD} q_1 y z \frac{1 + \rho L_{SR} q_1 z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) \\
&\quad e^{-(x+y+z)} dx dy dz
\end{aligned}$$

$$\begin{aligned}
&= \log_2(e) \int_0^\infty \int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1 z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz \\
&= \log_2(e) \underbrace{\left[\int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2}{\rho L_{RD} w P_r y + 1} \right) \right) e^{-(x+y)} dx dy \right]}_{L_1} \\
&\quad + \underbrace{\int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1 z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz}_{L_2}
\end{aligned} \tag{51}$$

Here, the L_1 part can be given as:

$$\begin{aligned}
L_1 &= \int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2}{\rho L_{RD} w P_r y + 1} \right) \right) e^{-(x+y)} dx dy \\
&= \int_0^\infty \ln(1 + \rho x q_1) e^{-x} dx + \int_0^\infty \int_0^\infty \ln \left(1 + \frac{\rho x q_2}{\rho L_{RD} w P_r y + 1} \right) e^{-(x+y)} dx dy \\
&= - \int_0^\infty \frac{\rho q_1}{(1 + \rho x q_1)} de^{-x} - \int_0^\infty \int_0^\infty \ln \left(\frac{\rho x q_2 + \rho L_{RD} w P_r y + 1}{\rho L_{RD} w P_r y + 1} \right) e^{-y} de^{-x} dy \\
&= \rho q_1 - \int_0^\infty \frac{\rho^2 q_1^2}{(1 + \rho x q_1)^2} de^{-x} + \int_0^\infty \int_0^\infty \frac{\rho q_2}{\rho x q_2 + \rho L_{RD} w P_r y + 1} e^{-(x+y)} dx dy \\
&= \rho q_1 - \underbrace{\rho^2 q_1^2 - \int_0^\infty \frac{2\rho^3 q_1^3}{(1 + \rho x q_1)^3} de^{-x}}_{o(\rho^3)} - \int_0^\infty \frac{\rho q_2}{\rho L_{RD} w P_r y + 1} de^{-y} + \int_0^\infty \int_0^\infty \left(\frac{\rho q_2}{\rho x q_2 + \rho L_{RD} w P_r y + 1} \right)^2 e^{-y} de^{-x} dy \\
&= \rho q_1 - \rho^2 q_1^2 + \rho q_2 + \int_0^\infty \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{RD} w P_r y + 1)^2} de^{-y} + o(\rho^3) \\
&\quad + \int_0^\infty \left(\frac{\rho q_2}{\rho L_{RD} w P_r y + 1} \right)^2 de^{-y} - \underbrace{\int_0^\infty \int_0^\infty \frac{2\rho^3 q_2^3}{(\rho x q_2 + \rho L_{RD} w P_r y + 1)^3} e^{-y} de^{-x} dy}_{o(\rho^3)} \\
&= \rho(q_1 + q_2) - \rho^2(q_1^2 + q_2^2) - \rho^2 L_{RD} w P_r q_2 + o(\rho^3) \\
&= \rho P_s - \rho^2(q_1^2 + q_2^2) - \rho^2 L_{RD} w P_r q_2 + o(\rho^3)
\end{aligned} \tag{52}$$

Now we derive the L_2 part.

$$\begin{aligned}
L_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1 z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1}{(1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} \\
&\quad * \rho^2 L_{SR} L_{RD} w q_1 y \frac{\rho q_2 x(1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1)}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} dx dy dz \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x(1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} \\
&\quad e^{-(x+y+z)} dx dy dz \\
&= - \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x(1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} \\
&\quad e^{-(x+y)} de^{-z} dx dy \\
&= \int_0^\infty \int_0^\infty \underbrace{\frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x(1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)(\rho L_{RD} w P_r y + 1)}}_{A_1} e^{-(x+y)} dx dy \\
&\quad + \underbrace{\int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} \left(\frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x(1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} \right)}_{\substack{\tilde{M} \\ A_2}} e^{-(x+y+z)} dx dy dz
\end{aligned}$$

The A_1 part can be expressed as follow:

$$\begin{aligned}
A_1 &= \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x(1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)(\rho L_{RD} w P_r y + 1)} e^{-(x+y)} dx dy \\
&= \int_0^\infty \int_0^\infty \underbrace{\frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 x y}{(\rho L_{RD} w P_r y + 1 + \rho x q_2)(\rho L_{RD} w P_r y + 1)}}_{o(\rho^3)} e^{-(x+y)} dx dy \\
&\quad + \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)} e^{-(x+y)} dx dy \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)} e^{-x} de^{-y} dx + o(\rho^3) \\
&= \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (\rho L_{RD} w P_r y + 1 + \rho x q_2) - \rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 y}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)^2} e^{-(x+y)} dx dy + o(\rho^3) \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)} e^{-x} dx de^{-y} + o(\rho^3) \\
&= \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(1 + \rho x q_2)} e^{-x} dx - \underbrace{\int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)^2} e^{-(x+y)} dx dy}_{o(\rho^3)} + o(\rho^3) \\
&= \rho^2 L_{SR} L_{RD} w P_r q_1 + o(\rho^3)
\end{aligned}$$

Now, we derive the A_2 part.

$$\begin{aligned}
\therefore M &= \frac{d}{dz} \left(\frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} \right) \\
&= - \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1)) ((1 + \rho x q_1)(L_{SR} q_1 + \rho L_{SR} q_1 q_2 x) + \rho L_{SR} L_{RD} w P_r q_1 y)}{\underbrace{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)^2}_{o(\rho^3)} (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} \\
&\quad - \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{\underbrace{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2}_{o(\rho^3)}} \\
&= o(\rho^3)
\end{aligned}$$

$$\therefore A_2 = \int_0^\infty \int_0^\infty \int_0^\infty M * e^{-(x+y+z)} dx dy dz = o(\rho^3)$$

Based on A_1 and A_2 , we can express L_2 as:

$$L_2 = A_1 + A_2 = \rho^2 L_{SR} L_{RD} w P_r q_1 + o(\rho^3) \quad (53)$$

Combing equation (52) and (53), the achievable rate under the instantaneous constraint can be

approximated by a quadratic function of the ρ as:

$$\begin{aligned}
C_{\text{int}_1} &= \frac{1}{2} \log_2(e) \max_w (L_1 + L_2) \\
&\approx \frac{1}{2} \log_2(e) \max_{q_{11} + q_{22} \leq P_s} \left(\rho P_s - \rho^2 (q_1^2 + q_2^2) + \rho^2 w P_r L_{RD} (L_{SR} q_1 - q_2) \right)
\end{aligned}$$

Thus, we show the proof of **Lemma 4**.

At last, we will show the proof of **Lemma 5**, which is the approximation function with respect to ρ^3 . Now we derive the two parts in equation (51), about the L_1 part:

$$\begin{aligned}
L_1 &= \int_0^\infty \int_0^\infty \ln \left((1 + \rho x q_1) \left(1 + \frac{\rho x q_2}{\rho L_{RD} w P_r y + 1} \right) \right) e^{-(x+y)} dx dy \\
&= \int_0^\infty \ln(1 + \rho x q_1) e^{-x} dx + \int_0^\infty \int_0^\infty \ln \left(1 + \frac{\rho x q_2}{\rho L_{RD} w P_r y + 1} \right) e^{-(x+y)} dx dy \\
&= - \int_0^\infty \frac{\rho q_1}{(1 + \rho x q_1)} d e^{-x} - \int_0^\infty \int_0^\infty \ln \left(\frac{\rho x q_2 + \rho L_{RD} w P_r y + 1}{\rho L_{RD} w P_r y + 1} \right) e^{-y} dy d e^{-x}
\end{aligned}$$

$$\begin{aligned}
&= \rho q_1 - \int_0^\infty \frac{\rho^2 q_1^2}{(1 + \rho x q_1)^2} de^{-x} + \int_0^\infty \int_0^\infty \frac{\rho q_2}{\rho x q_2 + \rho L_{RD} w P_r y + 1} e^{-(x+y)} dx dy \\
&= \rho q_1 - \rho^2 q_1^2 - \int_0^\infty \frac{2\rho^3 q_1^3}{(1 + \rho x q_1)^3} de^{-x} - \int_0^\infty \frac{\rho q_2}{\rho L_{RD} w P_r y + 1} de^{-y} + \int_0^\infty \int_0^\infty \left(\frac{\rho q_2}{\rho x q_2 + \rho L_{RD} w P_r y + 1} \right)^2 e^{-y} dy de^{-x} \\
&= \rho q_1 - \rho^2 q_1^2 + 2\rho^3 q_1^3 + \rho q_2 + \int_0^\infty \frac{\rho^2 L_{RD} w P_r q_2}{(\rho L_{RD} w P_r y + 1)^2} de^{-y} \\
&\quad + \int_0^\infty \left(\frac{\rho q_2}{\rho L_{RD} w P_r y + 1} \right)^2 de^{-y} - \int_0^\infty \int_0^\infty \frac{2\rho^3 q_2^3}{(\rho x q_2 + \rho L_{RD} w P_r y + 1)^3} e^{-y} dy de^{-x} \\
&= \rho q_1 - \rho^2 q_1^2 + 2\rho^3 q_1^3 + \rho q_2 - \rho^2 L_{RD} w P_r q_2 - \int_0^\infty \frac{2\rho^3 L_{RD}^2 w^2 P_r^2 q_2}{(\rho L_{RD} w P_r y + 1)^3} de^{-y} \\
&\quad - \rho^2 q_2^2 - \int_0^\infty \frac{2\rho^3 L_{RD} w P_r q_2^2}{(\rho L_{RD} w P_r y + 1)^3} de^{-y} - \int_0^\infty \frac{2\rho^3 q_2^3}{(\rho L_{RD} w P_r y + 1)^3} de^{-y} + o(\rho^4) \\
&= \rho(q_1 + q_2) - \rho^2(q_1^2 + q_2^2) - \rho^2 L_{RD} w P_r q_2 + 2\rho^3 q_1^3 + 2\rho^3 L_{RD}^2 w^2 P_r^2 q_2 + 2\rho^3 L_{RD} w P_r q_2^2 + 2\rho^3 q_2^3 + o(\rho^4) \\
&= \rho P_s - \rho^2(q_1^2 + q_2^2) - \rho^2 L_{RD} w P_r q_2 + 2\rho^3(q_1^3 + q_2^3) + 2\rho^3 L_{RD}^2 w^2 P_r^2 q_2 + 2\rho^3 L_{RD} w P_r q_2^2 + o(\rho^4)
\end{aligned} \tag{54}$$

About the L_2 part:

$$\begin{aligned}
&\because \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1 z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) \\
&= \frac{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1}{(1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2(1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} \\
&\times \left((1 + \rho x q_1) \frac{d}{dz} \left(\frac{\rho x q_2(1 + \rho L_{SR} q_1 z)}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) + \frac{d}{dz} \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right)
\end{aligned}$$

And here,

$$\begin{aligned}
&(1 + \rho x q_1) \frac{d}{dz} \left(\frac{\rho x q_2(1 + \rho L_{SR} q_1 z)}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) + \frac{d}{dz} \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \\
&= \rho x q_2(1 + \rho x q_1) \frac{\rho L_{SR} q_1(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1) - \rho L_{SR} q_1(1 + \rho L_{SR} q_1 z)}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} \\
&\quad + \rho^2 L_{SR} L_{RD} w P_r q_1 y \frac{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1) - \rho L_{SR} q_1 z}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2}
\end{aligned}$$

$$\begin{aligned}
&= \rho x q_2 (1 + \rho x q_1) \frac{\rho L_{SR} q_1 \times \rho L_{RD} w P_r y}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} + \rho^2 L_{SR} L_{RD} w P_r q_1 y \frac{(\rho L_{RD} w P_r y + 1)}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} \\
&= \rho^2 L_{SR} L_{RD} w P_r q_1 y \frac{\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1)}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} \\
L_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} \ln \left((1 + \rho x q_1) \left(1 + \rho x q_2 \frac{1 + \rho L_{SR} q_1 z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) + \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y z}{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1} \right) e^{-(x+y+z)} dx dy dz \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1}{(1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z} \\
&\quad \times \rho^2 L_{SR} L_{RD} w q_1 y \frac{\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1)}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} e^{-(x+y+z)} dx dy dz \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)} \\
&\quad \times \frac{1}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} e^{-(x+y+z)} dx dy dz \\
&= - \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)} \\
&\quad \times \frac{1}{(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} e^{-(x+y)} dx dy e^{-z} \\
&= \underbrace{\int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{(1 + \rho x q_1) (\rho L_{RD} w P_r y + 1 + \rho x q_2 (\rho L_{RD} w P_r y + 1))} e^{-(x+y)} dx dy}_{A_1} \\
&\quad + \underbrace{\int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} (M) e^{-(x+y+z)} dx dy dz}_{A_2}
\end{aligned}$$

Where,

$$M = \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)}$$

Now we derive the A_1 part as follow:

$$\begin{aligned}
A_1 &= \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{(1 + \rho x q_1) (\rho L_{RD} w P_r y + 1 + \rho x q_2 (\rho L_{RD} w P_r y + 1))} e^{-(x+y)} dx dy \\
&= \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 xy}{(\rho L_{RD} w P_r y + 1 + \rho x q_2) (\rho L_{RD} w P_r y + 1)} e^{-(x+y)} dx dy + \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1) (\rho L_{RD} w P_r y + 1 + \rho x q_2)} e^{-(x+y)} dx dy \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 xy}{(\rho L_{RD} w P_r y + 1)^2 + \rho x q_2 (\rho L_{RD} w P_r y + 1)} e^{-y} de^{-x} dy - \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y}{(1 + \rho x q_1) (\rho L_{RD} w P_r y + 1 + \rho x q_2)} e^{-x} de^{-y} dx
\end{aligned}$$

Where,

$$\begin{aligned}
&= \underbrace{\int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 y \left((\rho L_{RD} w P_r y + 1)^2 + \rho x q_2 (\rho L_{RD} w P_r y + 1) \right)}{\left((\rho L_{RD} w P_r y + 1)^2 + \rho x q_2 (\rho L_{RD} w P_r y + 1) \right)^2} e^{-(x+y)} dx dy + o(\rho^4)}_{E_1} \\
&+ \underbrace{\int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (\rho L_{RD} w P_r y + 1 + \rho x q_2) - \rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 y}{(1 + \rho x q_1) (\rho L_{RD} w P_r y + 1 + \rho x q_2)^2} e^{-(x+y)} dx dy}_{E_2} \\
&E_1 = \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 y}{\left((\rho L_{RD} w P_r y + 1)^2 + \rho x q_2 (\rho L_{RD} w P_r y + 1) \right)} e^{-(x+y)} dx dy \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 y}{\left((\rho L_{RD} w P_r y + 1)^2 + \rho x q_2 (\rho L_{RD} w P_r y + 1) \right)} e^{-y} dy de^{-x} \\
&= \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 y}{(\rho L_{RD} w P_r y + 1)^2} e^{-y} dy - \underbrace{\int_0^\infty \int_0^\infty \frac{\rho^4 L_{SR} L_{RD} w P_r q_1 q_2^2 y}{\left((\rho L_{RD} w P_r y + 1)^2 + \rho x q_2 (\rho L_{RD} w P_r y + 1) \right)^2} e^{-(x+y)} dx dy}_{o(\rho^4)} \\
&= - \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2 y}{(\rho L_{RD} w y + 1)^2} de^{-y} + o(\rho^4) \\
&= \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2}{(\rho L_{RD} w y + 1)^2} e^{-y} dy + o(\rho^4) \\
&= \rho^3 L_{SR} L_{RD} w P_r q_1 q_2 + o(\rho^4)
\end{aligned}$$

$$\begin{aligned}
E_2 &= \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (1 + \rho x q_2)}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)^2} e^{-(x+y)} dx dy \\
&= - \int_0^\infty \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1 (1 + \rho x q_2)}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)^2} e^{-x} dx dy e^{-y} \\
&= \int_0^\infty \frac{\rho^2 L_{SR} L_{RD} w P_r q_1}{(1 + \rho x q_1)(1 + \rho x q_2)} e^{-x} dx - \int_0^\infty \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 (1 + \rho x q_2)}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)^3} e^{-(x+y)} dx dy \\
&= \rho^2 L_{SR} L_{RD} w P_r q_1 - \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 ((1 + \rho x q_1) q_2 + (1 + \rho x q_2) q_1)}{(1 + \rho x q_1)^2 (1 + \rho x q_2)^2} e^{-x} dx \\
&\quad - \underbrace{\int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1 + \rho x q_1)(1 + \rho x q_2)^2} e^{-x} dx + \int_0^\infty \int_0^\infty \frac{6\rho^4 L_{SR} L_{RD}^3 w^3 P_r^3 q_1 (1 + \rho x q_2)}{(1 + \rho x q_1)(\rho L_{RD} w P_r y + 1 + \rho x q_2)^4} e^{-(x+y)} dx dy}_{o(\rho^4)} \\
&= \rho^2 L_{SR} L_{RD} w P_r q_1 + \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 q_2}{(1 + \rho x q_1)(1 + \rho x q_2)^2} de^{-x} + \int_0^\infty \frac{\rho^3 L_{SR} L_{RD} w P_r q_1^2}{(1 + \rho x q_1)^2 (1 + \rho x q_2)} de^{-x} \\
&\quad + \int_0^\infty \frac{2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1}{(1 + \rho x q_1)(1 + \rho x q_2)^2} de^{-x} + o(\rho^4) \\
&= \rho^2 L_{SR} L_{RD} w P_r q_1 - \rho^3 L_{SR} L_{RD} w P_r q_1 q_2 - \rho^3 L_{SR} L_{RD} w P_r q_1^2 - 2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 + o(\rho^4) \\
&= \rho^2 L_{SR} L_{RD} w P_r q_1 - \rho^3 L_{SR} L_{RD} w P_r P_s q_1 - 2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 + o(\rho^4)
\end{aligned}$$

Taking E_1 and E_2 in A_1 , we get:

$$\Rightarrow A_1 = \rho^2 L_{SR} L_{RD} w P_r q_1 + \rho^3 L_{SR} L_{RD} w P_r q_1 q_2 - \rho^3 L_{SR} L_{RD} w P_r P_s q_1 - 2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 + o(\rho^4)$$

Then, we derive the A_2 part.

$$\begin{aligned}
\frac{d}{dz}(M) &= \frac{d}{dz} \left(\frac{\rho^2 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} \right) \\
&= - \frac{\rho^3 L_{SR} L_{RD} w P_r q_1 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1)) ((1 + \rho x q_1)(L_{SR} q_1 + \rho L_{SR} q_1 q_2 x) + \rho L_{SR} L_{RD} w P_r q_1 y)}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)^2 (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} \\
&\quad - \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 y (\rho q_2 x (1 + \rho x q_1) + (\rho L_{RD} w P_r y + 1))}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} \\
&= - \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 y}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z)^2 (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} \\
&\quad - \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 y}{((1 + \rho x q_1)(\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} + o(\rho^4)
\end{aligned}$$

$$\begin{aligned}
A_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{d}{dz} (M) * e^{-(x+y+z)} dx dy dz \\
&= - \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 y}{\left((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z \right)^2 (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} e^{-(x+y+z)} dx dy dz \\
&\quad - \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 y}{\left((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z \right) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} e^{-(x+y+z)} dx dy dz \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 y}{\left((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z \right)^2 (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} e^{-(x+z)} dx dz de^{-y} \\
&\quad + \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 y}{\left((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z \right) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} e^{-(x+z)} dx dz de^{-y} \\
&\quad + o(\rho^4) \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2}{\left((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z \right)^2 (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)} e^{-(x+y)} dx dy de^{-z} \\
&\quad + \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2}{\left((1 + \rho x q_1) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1 + \rho x q_2 (1 + \rho L_{SR} q_1 z)) + \rho^2 L_{SR} L_{RD} w P_r q_1 y z \right) (\rho L_{SR} q_1 z + \rho L_{RD} w P_r y + 1)^2} e^{-(x+y)} dx dy de^{-z} \\
&\quad + o(\rho^4) \\
&= \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2}{\left((1 + \rho x q_1) (\rho L_{RD} w P_r y + 1 + \rho x q_2) \right)^2 (\rho L_{RD} w P_r y + 1)} e^{-x} dx de^{-y} \\
&\quad + \int_0^\infty \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2}{\left((1 + \rho x q_1) (\rho L_{RD} w P_r y + 1 + \rho x q_2) \right) (\rho L_{RD} w P_r y + 1)^2} e^{-x} dx de^{-y} + o(\rho^4) \\
&= \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2}{\left((1 + \rho x q_1) (1 + \rho x q_2) \right)^2} de^{-x} + \int_0^\infty \frac{\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2}{\left((1 + \rho x q_1) (1 + \rho x q_2) \right)} de^{-x} + o(\rho^4) \\
&= -2\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 + o(\rho^4)
\end{aligned}$$

Taking A_1 and A_2 into L_2 , we get:

$$\begin{aligned}
&\Rightarrow L_2 = A_1 + A_2 \\
&= \rho^2 L_{SR} L_{RD} w P_r q_1 + \rho^3 L_{SR} L_{RD} w P_r q_1 q_2 - 2\rho^3 L_{SR} L_{RD}^2 w^2 P_r^2 q_1 - \rho^3 L_{SR} L_{RD} w P_r P_s q_1 - 2\rho^3 L_{SR}^2 L_{RD} w P_r q_1^2 + o(\rho^4)
\end{aligned} \tag{55}$$

Based on (54) and (55), we get

$$\begin{aligned}
&\Rightarrow \frac{1}{\log_2(e)} J_c = L_1 + L_2 \\
&= \rho P_s - \rho^2 (q_1^2 + q_2^2) + \rho^2 L_{RD} w P_r (L_{SR} q_1 - q_2) + 2\rho^3 (q_1^3 + q_2^3) \\
&\quad + 2\rho^3 L_{RD} w P_r q_2 (L_{RD} w P_r + q_2) + \rho^3 L_{SR} L_{RD} w P_r q_1 (q_2 - P_s) - 2\rho^3 L_{SR} L_{RD} w P_r q_1 (L_{RD} w P_r + L_{SR} q_1) + o(\rho^4)
\end{aligned}$$

Thus, the **Lemma 5** is proved above.

APPENDIX III

PROOF FOR OPTIMIZATION

The optimization problem in equation (38) involves a three dimensional integral which is not a convex function with respect to amplification coefficient b , and it is hard to find the closed form solution. So we get the approximations of the achievable rate as mentioned in **Lemma 1** and **Lemma 4**, which are easier to get the optimum value of q_1 and b .

A. Optimization Problem of Lemma 1

From equation (23), we can derive the optimum value of q_1 and the ratio of b^2/b_{max}^2 : w.

Firstly, we define:

$$\begin{aligned} C_a &= \rho P_s - \rho^2 (q_1^2 + q_2^2) + \frac{\rho^2 L_{SR} L_{RD} w q_1}{(\rho L_{SR} q_1 + 1)} - \frac{\rho^2 L_{RD} w q_2}{(\rho L_{SR} q_1 + 1)} \\ &= \rho P_s - \rho^2 (q_1^2 + (P_s - q_2)^2) + \frac{\rho^2 L_{RD} w (L_{SR} q_1 + q_1 - P_s)}{(\rho L_{SR} q_1 + 1)} \end{aligned} \quad (56)$$

In equation (56), w is linear has range from 0 to 1. To get the maximum value of the achievable rate and the optimum value of q_1 , we need to derivative C_a with respect to q_1 as:

- The first derivative of C_a

$$\begin{aligned} \frac{dC_a}{dq_1} &= -\rho^2 (2q_1 - 2(P_s - q_1)) + \frac{\rho^2 w L_{RD} (L_{SR} + 1) (\rho L_{SR} q_1 + 1) - \rho^3 w L_{SR} L_{RD} (L_{SR} + 1) q_1}{(\rho L_{SR} q_1 + 1)^2} + \frac{\rho^3 w L_{SR} L_{RD} P_s}{(\rho L_{SR} q_1 + 1)^2} \\ &= -2\rho^2 (q_1 - (P_s - q_1)) + \frac{\rho^3 w L_{RD} L_{SR} (L_{SR} + 1) q_1 + \rho^2 w L_{RD} (L_{SR} + 1) - \rho^3 w L_{SR} L_{RD} (L_{SR} + 1) q_1}{(\rho L_{SR} q_1 + 1)^2} + \frac{\rho^3 w L_{SR} L_{RD} P_s}{(\rho L_{SR} q_1 + 1)^2} \\ &= -2\rho^2 (2q_1 - P_s) + \frac{\rho^2 w L_{RD} (L_{SR} + 1)}{(\rho L_{SR} q_1 + 1)^2} + \frac{\rho^3 w L_{SR} L_{RD} P_s}{(\rho L_{SR} q_1 + 1)^2} \end{aligned} \quad (57)$$

- The second derivative of C_a

$$\frac{d^2 C_a}{dq_1^2} = -4\rho^2 - \frac{2\rho^3 w L_{SR} L_{RD} (L_{SR} + 1)}{(\rho L_{SR} q_1 + 1)^3} - \frac{2\rho^4 w L_{SR}^2 L_{RD} P_s}{(\rho L_{SR} q_1 + 1)^3} < 0 \quad (58)$$

From equation (58), we know that the function of achievable rate is convex with respected to q_1 . So when $\frac{dC_a}{dq_1}$ is equal to zero, the achievable rate has the maximum value.

Based on equation (56), we can get the optimum value of q_1 by:

$$4q_1 - 2P_s = \frac{w L_{RD} (L_{SR} + 1) + \rho w L_{SR} L_{RD} P_s}{(\rho L_{SR} q_1 + 1)^2}$$

$$q_{1_avg_opt} = \frac{w L_{RD} (L_{SR} + 1) + \rho w L_{SR} L_{RD} P_s}{4(\rho L_{SR} q_{1_avg_opt} + 1)^2} + \frac{1}{2} P_s \quad (59)$$

Because q_1 has the range from 0 to P_s , we can get the optimum q_1 which satisfy (59) when the solution of (59) is smaller than P_s . Otherwise, the optimum q_1 is equal to P_s .

The first part in equation (59) can be easily prove that it is bigger and equal than zero, which means that $q_{1_avg_opt} \geq \frac{P_s}{2}$ and the equality holds if and only if the ratio w is equal to zero. And because the second and the third parts in equation (57) are always bigger and equal to zero, we will get the conclusion as:

$$\left. \frac{dC_a}{dq_1} \right|_{w=0} < \left. \frac{dC_a}{dq_1} \right|_{w=1} \quad (60)$$

The equation (60) means that the achievable rate under the average constraint is changing faster when the ratio w is equal to one than when the ratio w is equal to zero. Based on equation (58) and (59), we can get the inequality as:

$$C_a \left(q_1 = \frac{P_s}{2}, w=0 \right) < C_a \left(q_1 = \frac{P_s}{2}, w=1 \right) \quad (61)$$

From equation (59), we also can conclude that

$$C_a\left(q_1 = \frac{P_s}{2}, w=1\right) < C_a\left(q_1 = \frac{L_{RD}(L_{SR}+1) + \rho L_{SR} L_{RD} P_s}{4(\rho L_{SR} q_1 + 1)^2} + \frac{P_s}{2}, w=1\right) \quad (62)$$

Where the right side of the inequality (62) is the maximum value of the achievable rate when $w=1$. Combing the inequality (61) and (62), we get a new inequality as:

$$C_a\left(q_1 = \frac{P_s}{2}, w=0\right) < C_a\left(q_1 = \frac{L_{RD}(L_{SR}+1) + \rho L_{SR} L_{RD} P_s}{4(\rho L_{SR} q_1 + 1)^2} + \frac{P_s}{2}, w=1\right) \quad (63)$$

Based on inequality (63), we can get the conclusion that the achievable rate will have the maximum value if only if the ratio w is equal to one and the optimum value of q_1 should satisfy the equation (59). Thus, we can get the conclusion about the optimization problem with average amplification constraint in chapter 3.

B. Optimization Problem of *Lemma 4*

Similar to average case, we also need to derive the optimum value of q_1 and the ratio of b^2/b_{max}^2 : w from equation (28). We define,

$$C_i = \rho P_s - \rho^2 (q_1^2 + q_2^2) + \rho^2 w P_r L_{RD} (L_{SR} q_1 - q_2) \quad (64)$$

In equation (64), w is linear has range from 0 to 1. To get the maximum value of the achievable rate and the optimum value of q_1 , we need to derivative C_i respected to q_1 as:

- The first derivative of C_i

$$\begin{aligned} \frac{dC_i}{dq_1} &= -\rho^2 (2q_1 - 2(P_s - q_1)) + \rho^2 w P_r L_{RD} (L_{SR} + 1) \\ &= -\rho^2 (4q_1 - 2P_s) + \rho^2 w P_r L_{RD} (L_{SR} + 1) \end{aligned} \quad (65)$$

- The second derivative of C_i

$$\frac{d^2 C_i}{dq_1^2} = -4\rho^2 < 0 \quad (66)$$

From equation (66), we know that the function of achievable rate is convex with respected to q_1 . So when $\frac{dC_i}{dq_1}$ is equal to zero, the achievable rate has the maximum value. And we can get the optimum value of q_1 as:

$$-\rho^2(4q_1 - 2P_s) + \rho^2 w P_r L_{RD}(L_{SR} + 1) = 0, 0 \leq q_1 \leq P_s$$

$$q_{1_int_opt} = \left\{ \frac{w P_r L_{RD}(L_{SR} + 1)}{4} + \frac{1}{2} P_s, P_s \right\} \quad (67)$$

Similar as the average case we proved before, we also can prove that the achievable rate under the instantaneous constraint has the maximum value if and only if the ratio w is equal to one and the optimum value of q_1 will satisfy the equation (67).