MODELING AND CONTROL OF A FLAPPING WING MICRO AIR VEHICLE
AT HOVER CONDITION

A Thesis by

Zhuo Yan

Bachelor of Science in Aerospace Engineering, Wichita State University, 2014

Submitted to the Department of Aerospace Engineering
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
the requirements for the degree of
Master of Science

July 2016
MODELING AND CONTROL OF A FLAPPING WING MICRO AIR VEHICLE
AT HOVER CONDITION

The following faculty members have examined the final copy of this thesis for form and
content, and recommend that it be accepted in partial fulfillment of the requirement for the
degree of Master of Science, with a major in Aerospace Engineering.

______________________________
Animesh Chakravarthy, Committee Chair

______________________________
Zheng Chen, Committee Member

______________________________
James Steck, Committee Member
In this thesis a mathematical model of a flapping wing MAV is discussed. Aerodynamic forces and moments due to some key unsteady aerodynamic mechanisms are studied to derive the vehicle’s longitudinal equations of motion under symmetric flapping assumption. The dynamic model is then simplified and linearized about a hover condition. With the assumption that the frequency of wing flapping motion is much higher than the body’s natural frequency of motion, averaging theory is applied to the system. Two types of averaging methods are applied, full cycle averaging and quarter cycle averaging, to obtain a linear time invariant system (LTI) and a jump-style linear time varying (LTV) system respectively. Stability analysis and controller design are based on the linear time invariant system. A linear controller with eigenstructure assignment technique is designed and attached to the nonlinear system to stabilize the vehicle at hover condition under perturbations.
TABLE OF CONTENTS

Chapter | Page
---|---
1. INTRODUCTION ......................................................................................................... 1
  1.1 Preface .................................................................................................................. 1
  1.2 Unsteady Aerodynamic Mechanism ................................................................. 1
  1.3 Flapping Wing MAVs ........................................................................................... 5
  1.4 Thesis Outline ....................................................................................................... 8
2. DYNAMIC MODELING .............................................................................................. 9
  2.1 Reference Frames ................................................................................................. 9
  2.2 Wing Kinematics ................................................................................................ 10
  2.3 Aerodynamic Forces ........................................................................................... 13
  2.4 Nonlinear Model ................................................................................................. 16
3. LINEARIZED MODEL ............................................................................................... 23
  3.1 Simplified Model ................................................................................................ 23
  3.2 Averaged Aerodynamic Forces and Moments .................................................... 26
  3.3 Linearization ....................................................................................................... 37
4. CONTROLLER DESIGN ............................................................................................ 54
  4.1 Control Derivative .............................................................................................. 54
  4.2 Controller Design .............................................................................................. 57
  4.3 Simulation Results .............................................................................................. 58
5. CONCLUSION ............................................................................................................ 64
REFERENCES .................................................................................................................. 65
APPENDIXES ..................................................................................................................... 69
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Angle of Attack</td>
</tr>
<tr>
<td>$J_0(z)$</td>
<td>Bessel Function</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density of Air</td>
</tr>
<tr>
<td>$\hat{r}_2$</td>
<td>Dimensionless Distance of Center of Pressure from Wing Base</td>
</tr>
<tr>
<td>$\hat{x}_0$</td>
<td>Dimensionless Distance of the Rotation Axis from Wing Leading Edge</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>Duration of Wing Rotation</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Eigenvalue</td>
</tr>
<tr>
<td>$K$</td>
<td>Feedback Gain</td>
</tr>
<tr>
<td>$\phi_m$</td>
<td>Flapping Amplitude</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Flapping Angle</td>
</tr>
<tr>
<td>$f$</td>
<td>Flapping Frequency</td>
</tr>
<tr>
<td>$\Gamma(z)$</td>
<td>Gamma Function</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational Constant</td>
</tr>
<tr>
<td>$\alpha_{incl}$</td>
<td>Inclination Angle</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass of the Vehicle</td>
</tr>
<tr>
<td>$c_m$</td>
<td>Maximum Wing Chord Width</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>Mean Wing Chord</td>
</tr>
<tr>
<td>$\alpha_m$</td>
<td>Mid-Stroke Angle of Attack</td>
</tr>
<tr>
<td>$\alpha_{incl_m}$</td>
<td>Mid-Stroke Inclination Angle</td>
</tr>
</tbody>
</table>
\[ I_{yy} \] Moment of Inertia about \( Y_B \)-axis
\[ F_{tr,N} \] Normal Component of Translational Force
\[ c_N \] Normal Component of Translational Force Coefficient
\[ \hat{c} \] Normalized Rotational Chord
\[ T \] Period of a Flapping Cycle
\[ \theta \] Pitch Angle
\[ F_{rot} \] Rotational Force
\[ C_{rot} \] Rotational Force Coefficient
\[ H_0(z) \] Struve Function
\[ F_{tr,T} \] Tangential Component of Translational Force
\[ C_T \] Tangential Component of Translational Force Coefficient
\[ U_{cp} \] Velocity at Wing Centre of Pressure
\[ A \] Wing Area
\[ h \] Wing Joints Location on \( Z_B \)-axis
\[ L \] Wing Length
\[ \phi_0 \] Wing Offset Angle
\[ (X_B, Y_B, Z_B) \] Body Frame
\[ (X_E, Y_E, Z_E) \] Earth Frame
\[ (X_S, Y_S, Z_S) \] Stroke Plane Frame
\[ (X_W, Y_W, Z_W) \] Wing Frame
\[ (\Delta x, \pm \Delta y, \Delta z) \] Wing Joints Position in Body Frame
<table>
<thead>
<tr>
<th>(u, v, w)</th>
<th>Body Linear Velocities in Body Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p, q, r)</td>
<td>Body Angular Velocities in Body Frame</td>
</tr>
<tr>
<td>( R_{BS} )</td>
<td>Transformation Matrix from Body Frame to Stroke Plane Frame</td>
</tr>
<tr>
<td>( R_{SB} )</td>
<td>Transformation Matrix from Stroke Plane Frame to Body Frame</td>
</tr>
<tr>
<td>( R_{WS} )</td>
<td>Transformation Matrix from Wing Frame to Stroke Plane Frame</td>
</tr>
<tr>
<td>( R_{SW} )</td>
<td>Transformation Matrix from Stroke Plane Frame to Wing Frame</td>
</tr>
<tr>
<td>( (\cdot) )</td>
<td>Time Derivative of a Parameter</td>
</tr>
<tr>
<td>( (x_B,y_B,z_B) )</td>
<td>Components of a Parameter in Body Frame</td>
</tr>
<tr>
<td>( (x_S,y_S,z_S) )</td>
<td>Components of a Parameter in Stroke Plane Frame</td>
</tr>
<tr>
<td>( (x_W,y_W,z_W) )</td>
<td>Components of a Parameter in Wing Frame</td>
</tr>
<tr>
<td>( \overline{T(\cdot)} )</td>
<td>Cycle Averaged Value of a Parameter</td>
</tr>
<tr>
<td>( \overline{T/4(\cdot)} )</td>
<td>Quarter Cycle Averaged Value of a Parameter</td>
</tr>
<tr>
<td>( \Delta(\cdot) )</td>
<td>Small Perturbation of a Parameter</td>
</tr>
<tr>
<td>( (\cdot)^v )</td>
<td>Value induced by Change in Body Velocities</td>
</tr>
<tr>
<td>( (\cdot)^\alpha )</td>
<td>Value induced by Change in Angle of Attack</td>
</tr>
</tbody>
</table>
1.1 Preface

Interest in micro air vehicles (MAVs) has grown progressively in the past decade due to the various applications including aerial photography and videography, civil search and rescue, hazardous environment exploration, military surveillance missions, deliveries, etc. MAVs can be classified as fixed wing, rotary wing and flapping wing. Many fixed and rotary wing MAVs have been successfully designed and built with great stability and maneuverability. Most rotary wing MAVs (such as quadrotors) provide the ability of hovering, but they suffer from aerodynamic inefficiency (especially at small scale) due to high viscous losses at low Reynolds numbers [1, 2] which results in poor endurance. In addition, as the size of the MAV becomes as small as less than a few centimeters, lift generation and flight control become fundamentally challenging for fixed wing MAVs. Therefore, researchers have been looking for alternate solutions with flapping wing MAV as one of the approach.

Flapping wing MAVs are bio-inspired from ‘natural flight’. They try to mimic the motion of insects and birds. They offer many advantages in efficiency, maneuverability, lift generation and stability when compared to conventional fixed and rotary wing MAVs considering the size and weight of the vehicle. But at the same time, the kinematics of flapping flight is quite complex. With the additional difficulties in micro-fabrication, constraints in micro-technology for actuation and sensing, make the study of flapping wing MAVs challenging.

1.2 Unsteady Aerodynamic Mechanism

The flow associated with insect flapping flight is incompressible, laminar and unsteady. It occurs in the Reynolds number regime of 10 to 10000 [3, 4] with the range of 30 to 1000 being a very active area of research [5]. Instead of directly studying the insects, most researchers use mechanical models with dynamic scaling to match the actual insect in Reynolds number and reduced frequency. These models provide great help in identification and studying of the unsteady flapping mechanisms which are related to the formation and shedding of the vortices, and the vortex-wing interaction. Some of the key mechanisms are briefly introduced next.

1.2.1 Delayed Stall of Leading Edge Vortex

Ellington et al. discovered that the delayed stall and leading edge vortex significantly improve the lift generation for a flapping wing [6]. It is the result of the wing translational motion that starts from rest. As the wing travels several chord lengths and increases its angle of attack, the flow separates at the leading edge and reattaches before it reaches the trailing
edge. Large organized vortices form in the separation zone above the wing at the leading edge which generates a lower pressure area and induces an increased suction force on the upper surface of the wing which acts normal to the wing. As a result, the lift and drag are enhanced as shown in Figure 1.1. The leading edge vortex stays constant in shape and size for distance of several chord lengths due to the existence of a strong spanwise flow that stabilizes the vortex; after which turbulent vortices develop and cause the wing to stall. This mechanism is unique to the flapping wing flight because it applies only during the onset of wing motion and lasts for a few wing chord lengths at large angles of attack.

![Figure 1.1. Effect of suction force due to leading edge vortex [7].](image)

1.2.2 Rotational Circulation

According to Dickinson’s study, the mechanism of rotational circulation contributes to the aerodynamic force generation as the wing rapidly rotates and changes direction at the end of each stroke [8]. It results from the interaction of the wing’s translational and rotational velocity. This phenomenon is similar to the Magnus effect where a flying tennis ball with a backspin or topspin will be pulled upward or downward. Therefore, the timing of the wing rotation relative to the wing translation is important to the force generation. As demonstrated in Figure 1.2, the lift is enhanced if the wing flips before the stroke reversal, which is termed as advanced rotation; the lift is reduced if the wing flips after the stroke reversal, which is called delayed rotation; and the symmetrical rotation will cause a positive lift peak before the stroke reversal and a negative lift peak after the stroke reversal. Sane and Dickinson also found the lift peak at the end of stroke is proportional to the angular velocity of the wing using the quasi-steady theory [9, 10].
Figure 1.2. Effect of wing rotation on the lift generation [8]. Black lines show the side-view of wing flapping trajectory; red arrows show the lift generated.

1.2.3 Clap and Fling

One of the earliest unsteady aerodynamic mechanisms proposed by Weis-Fogh to explain how insects fly is known as Weis-Fogh mechanism [11]. The corresponding insects’
motion has been termed ‘clap and fling’ by Lighthill [12] and Ellington [13] who proved the hypothesis and analytically described the contribution of this motion to the lift generation using two-dimensional inviscid theory. Weis-Fogh's experiment was performed on a chalcid wasp, Encarsia Formosa. It was found the lift generated by the chalcid wasp was insufficient to maintain the flight using steady state approximation. To explain it, he observed the motion that a chalcid wasp claps the wings together and then flings open about the trailing edge. Figure 1.3 describes this procedure and the consequent vortex development. The clap and fling mechanism can be observed in many insects like wasps, butterflies, locustas, fruitflies, and hawkmoths. It is believed to be very beneficial in lift enhancing [11, 14, 15, 16].

Figure 1.3. Clap and fling mechanism and vortex development [7, 11]. Black lines with arrows show the flow; dark blue arrows show the induced velocities; light blue arrows show the resultant aerodynamic forces acting on the airfoil.

(A-C) Clap. During upstroke, the wing moves from the ventral to the dorsal side of the body (A). At the end of upstroke, the leading edges touch together first (B), then the wings rotate about the leading edge and the trailing edges approach each other, stopping vortices are formed as the vorticity shed rolls up from the trailing edge to the leading edge which then dissipated into the wake (C). (D-E) Fling. At the beginning of downstroke, the wings rotate about the trailing edge and the leading edges start to move away from each other first (D). Air rushes into the gap between the wing sections which produce high strength vortices. The vortex of one wing acts as the starting vortex of the other wing. Because of the symmetry of the wing motion, the magnitude of the resultant circulation about the two wings is zero. The wings move away from each other and the leading edge vortex forms on each wing (E).
1.2.4 Wake Capture

The wake capture mechanism is in fact the result of wing-wake interaction. Dickinson first observed this interaction in a 2-D motion of an inclined plate [17]. Similar phenomenon was observed by examining on a dynamically-scaled robotic fruit fly wing model [8]. As shown in Figure 1.4, as the wing approaches to the end of stroke (A), both the leading and the trailing edge vortices shed (B) which induce a strong velocity and acceleration field (dark blue arrows) (C). This can also be understood as the flow tends to maintain the velocity due to its inertia. When the wing reverses direction (D), it encounters the wake which therefore results in enhancing the aerodynamic forces (light blue arrows) immediately following the stroke reversal (E, F).

![Figure 1.4. Wake capture mechanism and vortex development [7]. Black lines with arrows show the flow; dark blue arrows show the induced velocities; light blue arrows show the resultant aerodynamic forces acting on the airfoil.](image)

1.3 Flapping Wing MAVs

Most commonly seen flapping wing MAVs can be classified into three categories: tail stabilized, passively stable and wing motion stabilized. The configuration of tail stabilized flapping wing MAV is similar to a conventional airplane, and it requires a tail to stabilize the vehicle and provide pitch and yaw control. One or two pairs of wings flap to generate lift and thrust and provide roll control. Usually this type of flapping wing MAV is restricted to forward flight. Passively stable flapping wing MAVs offer the ability to hover. They usually have two or more pairs of wings and take advantage of the clap and fling mechanism for lift enhancement. The passive stability can be achieved using two sails, one above the wings and
the other one below the wings at a longer distance, which act as dampers and function in a similar manner as a damped pendulum that returns to rest when disturbed to prevent unstable or oscillatory motions. The wing motion stabilized flapping wing MAVs require the wings to actively change position, flapping amplitude and frequency to stabilize the vehicle. Some existing flapping wing MAVs are reviewed next.

1.3.1 University of Toronto Mentor

Mentor at the University of Toronto is one of the first radio controlled flapping wing MAVs funded by DARPA program and demonstrated in 2002, as shown in Figure 1.5. Mentor has two pairs of wings at the top of the vehicle flapping in a clap and fling motion with 90° amplitude to achieve a hover flight at a frequency of 30 Hz. The stability and control of the vehicle are achieved by the fins exposed to the airflow coming from the wings. Mentor has two generations: the first one has a wingspan of 36 cm and weighs 580 g with an internal combustion engine (ICE); the second one has a wingspan of 30 cm and weighs 440 g with a battery powered brushless motor. [18]

![Figure 1.5 University of Toronto Mentor](image)

1.3.2 TU Delft DelFly

DelFly project started in 2005 at TU Delft. There have been four major generations, as shown in Figure 1.6, and their basic properties are summarized in Table 1.1. The DelFly models are able to achieve fast forward flight, slow backward flight and near hovering flight. They take advantage of the clap and fling mechanism and are controlled by the tail control surfaces. A camera vision-based stabilization is used to maintain the stability. In the Guinness book of records 2009, the DelFly Micro featured as the smallest airplane in the world equipped with a camera. DelFly Explorer is capable of fully autonomous flight. [19]
TABLE 1.1

PROPERTIES OF DelFly MODELS

<table>
<thead>
<tr>
<th>Model</th>
<th>Year</th>
<th>Weight</th>
<th>Span</th>
<th>Endurance</th>
</tr>
</thead>
<tbody>
<tr>
<td>DelFly I</td>
<td>2005</td>
<td>21 g</td>
<td>50 cm</td>
<td>Unknown</td>
</tr>
<tr>
<td>DelFly II</td>
<td>2007</td>
<td>16 g</td>
<td>28 cm</td>
<td>15 mins</td>
</tr>
<tr>
<td>DelFly Micro</td>
<td>2008</td>
<td>3.07 g</td>
<td>10 cm</td>
<td>3 mins</td>
</tr>
<tr>
<td>DelFly Explorer</td>
<td>2013</td>
<td>20 g</td>
<td>28 cm</td>
<td>9 mins</td>
</tr>
</tbody>
</table>

![Figure 1.6 DelFly Models [19]](image)

1.3.3 Nano Hummingbird

The Nano Hummingbird is a remote controlled flapping wing MAV developed by AeroVironment under the Nano Air Vehicle (NAV) development program funded by DARPA in 2011, and shown in Figure 1.7. It has a wingspan of 16.5 cm and weighs 19 g with a flight endurance of up to 4 minutes at an average speed of 18km/h. It is stabilized and controlled by adjusting the wing motion. It is capable of hovering and flight in any direction. It carries all the equipment required to fly including motors, batteries, control, communication system and a video camera. [20]
1.4 Thesis Outline

This thesis is organized as follows:

Chapter 2 contains a nonlinear dynamic model of a flapping wing MAV. Aerodynamic forces and moments are studied to derive the vehicle’s longitudinal equations of motion.

In Chapter 3, the nonlinear model is simplified and linearized about a hover condition. Averaging theory is applied to the linear model for stability analysis.

A linear controller is designed in Chapter 4 and applied to the nonlinear model to maintain a stable hover flight.

Chapter 5 contains the conclusion and future study.
CHAPTER 2

DYNAMIC MODELING

2.1 Reference Frames

A complete dynamic model of a flapping wing MAV is developed in this chapter. To derive the equations of motion, four reference frames are defined: Earth Frame, Body Frame, Stroke Plane Frame and Wing Frame, as shown in Figure 2.1.

a. The Earth Frame \((X_E, Y_E, Z_E)\) is an inertial frame. The vehicle’s absolute velocity, position and rotational angles will be described with respect to this frame.

b. The Body Frame \((X_B, Y_B, Z_B)\) is attached to the body with origin at the center of gravity. It is oriented with \(X_B\)-axis along the longitudinal axis of the central body with positive direction pointing backward and opposite to the direction of flight; \(Y_B\)-axis is perpendicular to the \(X_B\)-axis with positive direction pointing to the right side of the body; the \(Z_B\)-axis is perpendicular to the \(X_BY_B\)-plane with positive direction pointing upward. The body is assumed to be mass symmetric about \(X_BZ_B\)-plane.

c. The Stroke Plane Frame \((X_S, Y_S, Z_S)\) defines the flapping motion of the wing. It is attached to the right wing with origin at the wing joint and oriented with \(Z_S\)-axis in the same direction as the \(Z_B\)-axis; \(Y_S\)-axis is along the right wing leading edge with positive direction pointing to the wing tip; the \(X_S\)-axis is perpendicular to the \(Y_SZ_S\)-plane with positive direction pointing towards the dorsal side of the wing. The flapping motion of the wing occurs only on \(X_SY_S\)-plane.

d. The Wing Frame \((X_W, Y_W, Z_W)\) is defined with origin at the right wing joint. It is oriented with \(Y_W\)-axis along the right wing leading edge with positive direction pointing to the wing tip; \(Z_W\)-axis is along the right wing chord with positive direction pointing from the trailing edge to the leading edge; \(X_W\)-axis is perpendicular to the \(Y_WZ_W\)-plane with positive direction pointing normal to the dorsal side of the wing.

In this thesis, symmetric placement and flapping motion of the left and right wings are considered. As a result, the left wing is a mirror image of the right wing. The wing joints are placed at \((\Delta x, \pm \Delta y, \Delta z)\) in the Body Frame. For simplicity in the forces and moments calculations, we assume the wing joints lie in \(Y_BZ_B\)-plane; \(\Delta y\) is small and its value will be absorbed into the wing length. Therefore, \(\Delta x \approx 0, \Delta y \approx 0\) and \(\Delta z = h\) is used to define the wing joints’ location.
2.2 Wing Kinematics

As discussed in Chapter 1, the wing experiences various aerodynamic phenomenon while flapping. Since some of them are either hard to approximate analytically or have minor contribution to the lift generation, therefore in this thesis we will cover only the force generated due to wing translation (delayed stall of leading edge vortex) and rotation (rotational circulation) which consist the majority of lift generation.

In general, the wing motion can be divided into four kinematic portions: upstroke and downstroke translational phases and two rotational phases. It has been shown that most flapping wing animals tend to maintain at a near constant angle of attack during the mid part of each flapping stroke and rapidly rotate and reverse wing direction at the end of each stroke [8]. To explain this motion, the notions of flapping angle and inclination angle are introduced.

The flapping angle describes the wing angular position and it is defined as the angle between positive \( Y_S \)-axis and \( Y_BZ_B \)-plane as shown in Figure 2.2. The time behavior of the flapping angle can be described by a harmonic function as in equation (2.1). As a result, the expression of flapping angular velocity can be derived as in equation (2.2).

\[
\varphi = \varphi_0 + \varphi_m \cos(\omega t) \tag{2.1}
\]
\[ \dot{\phi} = -\phi_m \omega \sin(\omega t) \]  

(2.2)

where \( \phi \) is the flapping angle; \( \dot{\phi} \) is the flapping angular velocity; \( \phi_0 \) is the offset angle which defines the wing mean position; \( \phi_m \) is the flapping amplitude; \( \omega = 2\pi f \) and \( f \) is the flapping frequency.

Figure 2.2. Flapping angle.

The inclination angle is used to characterize the wing rotation and is defined as the angle between positive \( Z_W \)-axis and \( Y_SZ_S \)-plane as shown in Figure 2.3. The time behavior for a full stroke cycle (including upstroke and downstroke) is defined in equation (2.3) and shown in Figure 2.4.
\[ \alpha_{incl} = \begin{cases} 
\alpha_{incl_m} \sin(\omega t) & \text{for } 0 \leq t < \Delta t/2 \\
\alpha_{incl_m} & \text{for } \Delta t/2 \leq t < T/2-\Delta t/2 \\
-\alpha_{incl_m} \sin(\omega t) & \text{for } T/2-\Delta t/2 \leq t < T/2+\Delta t/2 \\
-\alpha_{incl_m} & \text{for } T/2+\Delta t/2 \leq t < T-\Delta t/2 \\
\alpha_{incl_m} \sin(\omega t) & \text{for } T-\Delta t/2 \leq t \leq T 
\end{cases} \] (2.3)

where \( \alpha_{incl_m} \) is the mid-stroke inclination angle; \( T \) is the period of a flapping cycle; \( \Delta t \) is the duration of wing rotation.

Figure 2.3. Inclination angle.
2.3 Aerodynamic Forces

The wing is assumed to be thin and rigid with a frictionless surface and negligible mass. A quasi-steady state aerodynamic modeling is applied which assumes the 3D time varying flapping wing force generation with some of the unsteady effects can be approximated by 2D thin airfoils translating with constant velocity and constant angle of attack [21].

Dickinson’s experimental results developed great empirical expressions for lift and drag coefficients due to the wing translation [22]. Deng et al. modified the results and decomposed the translational force into normal and tangential components using the blade element theory [23] which can be calculated as:

\[ F_{tr,N} = \frac{1}{2} \rho A U_{cp}^2 C_N(\alpha) \]  
\[ F_{tr,T} = \frac{1}{2} \rho A U_{cp}^2 C_T(\alpha) \]

where \( \rho \) is the density of air; \( A \) is the wing area; \( U_{cp} \) is the velocity at the wing centre of pressure. \( C_N(\alpha) \) and \( C_T(\alpha) \) are the translational force coefficients along the normal and tangential directions of the wing, and can be defined as a function of angle of attack as follows:

\[ C_N(\alpha) = 3.4 \sin(\alpha) \]
\[
C_T(\alpha) = \begin{cases} 
-0.4 \cos^2(2\alpha) & 0^\circ \leq |\alpha| \leq 45^\circ \\
0 & \text{Otherwise} \\
0.4 \cos^2(2\alpha) & 135^\circ \leq |\alpha| \leq 180^\circ 
\end{cases}
\] (2.7)

where \( \alpha \) is the angle of attack.

Theoretical estimation of forces due to wing rotation with a quasi-steady treatment was derived by Fung for small-amplitude flutter on thin, rigid wings [24]. Deng et al. applied the results and derived the rotational force expression using the blade element theory as follow [20]:

\[
F_{\text{rot}} = \frac{1}{2} \rho AC_{\text{rot}} \hat{c} c_m \hat{\alpha} U_{\text{cp}}
\] (2.8)

where \( C_{\text{rot}} \) is the rotational force coefficient; \( \hat{c} \) is the normalized rotational chord; \( c_m \) is maximum wing chord width; \( \hat{\alpha} \) is the wing rotational angular velocity with respect to the rotational axis. \( C_{\text{rot}} \) and \( \hat{c} \) are defined as:

\[
C_{\text{rot}} = 2\pi \left( \frac{3}{4} - \hat{x}_0 \right)
\] (2.9)

\[
\hat{c} = \frac{\int_0^\Lambda c(r) r dr}{\hat{r}_2 L A c_m}
\] (2.10)

\[
\hat{r}_2^2 = \frac{\int_0^\Lambda c(r) r^2 dr}{L^2 A}
\] (2.11)

where \( \hat{x}_0 \) is the dimensionless distance of the rotation axis from the leading edge; \( \hat{r}_2 \) is the dimensionless distance of center of pressure from wing base; \( L \) is the wing length as shown in Figure 2.5.

The value of \( \hat{x}_0 \) usually lies between 0.25 and 0.5 for most flapping wing insects [25]; while in [26], this value was found to have an approximately linear relationship with angle of attack. In this thesis we use the value of \( \hat{x}_0 = 0.25 \) which is the center of pressure location by thin airfoil theory. This value is also adopted in most studies. For simplicity, we further assume the center of pressure location remains constant during the entire flapping cycle, and is always on the rotational axis. Same assumption is made in [23, 27].

It’s been shown that the spanwise location of the center of pressure is approximately constant with respect to the changes in angle of attack. \( \hat{r}_2 \) and \( \hat{c} \) depend only on the wing morphology with a typical range of 0.6-0.7 and 0.5-0.75 respectively for most insects [28].
The velocity at wing center of pressure due to the wing flapping motion in the Stroke Plane Frame at hover condition can be approximated as follow:

\[
U_{cp} = \dot{r}_2 L \phi
\]  

(2.12)

The rotational force acts perpendicular to the wing surface and its direction can be determined according to Magnus effect and demonstrated in Figure 2.6. It can be concluded if the wing rotates in the same direction as it translates, a downward force is generated; if the wing rotates in the opposite direction as it translates, an upward force is generated.
At hover condition, the angle of attack can be measured as the angle between the wing chord line and stroke plane. It should be noticed that when the body is in motion, the velocity at wing center of pressure should be modified by the body velocity, which would lead to a change in angle of attack from hover condition, as shown in Figure 2.7. A more detailed discussion will be shown next.

Figure 2.7. Angle of attack modified by the body motion.

### 2.4 Nonlinear Model

To derive the equations of motion, it involves transformations of velocities and aerodynamic forces between the frames. Four transformation matrices are defined (note that the subscript defines the sequence of transformation, eg. BS indicates a transformation from the Body Frame to the Stroke Plane Frame).

\[
R_{BS} = \begin{bmatrix}
\cos(\varphi) & -\sin(\varphi) & 0 \\
\sin(\varphi) & \cos(\varphi) & 0 \\
0 & 0 & 1
\end{bmatrix} \quad (2.13)
\]

\[
R_{SB} = \begin{bmatrix}
\cos(\varphi) & \sin(\varphi) & 0 \\
-\sin(\varphi) & \cos(\varphi) & 0 \\
0 & 0 & 1
\end{bmatrix} \quad (2.14)
\]

\[
R_{WS} = \begin{bmatrix}
\cos(\alpha_{incl}) & 0 & -\sin(\alpha_{incl}) \\
0 & 1 & 0 \\
\sin(\alpha_{incl}) & 0 & \cos(\alpha_{incl})
\end{bmatrix} \quad (2.15)
\]
\[ R_{SW} = \begin{bmatrix} \cos (\alpha_{incl}) & 0 & \sin(\alpha_{incl}) \\ 0 & 1 & 0 \\ -\sin(\alpha_{incl}) & 0 & \cos (\alpha_{incl}) \end{bmatrix} \] (2.16)

From the previous discussion of the wing joint location and the wing parameters, the center of pressure location in the Body Frame can be found as:

\[ x_{cpB} = \hat{f}_2 L \sin(\varphi) + 0.25 \bar{c} \sin(\alpha_{incl}) \cos(\varphi) \] (2.17)

\[ z_{cpB} = h - 0.25 \bar{c} \cos(\alpha_{incl}) \] (2.18)

where \( x_{cpB} \) is the wing center of pressure position on \( X_B \)-axis; \( z_{cpB} \) is the wing center of pressure position on \( Z_B \)-axis.

Assume the body is in motion with linear velocities \([u, v, w]\) and angular velocities \([p, q, r]\) defined in the Body Frame. The velocity at wing center of pressure defined in equation (2.12) need to be modified with the body velocities’ effect added. By a transformation of the body velocities from the Body Frame to the Stroke Plane Frame, center of pressure velocity components along \( X_S \)-axis and \( Z_S \)-axis can be found as follows:

\[ U_{cp,XS} = \hat{f}_2 L \dot{\varphi} + \cos(\varphi) u + z_{cpB} \cos(\varphi) q \] (2.19)

\[ U_{cp,ZS} = w - x_{cpB} q \] (2.20)

Above expressions are then transformed into wing frame as:

\[ U_{cp,XW} = \cos(\alpha_{incl}) U_{cp,XS} + \sin(\alpha_{incl}) U_{cp,ZS} \] (2.21)

\[ U_{cp,ZW} = -\sin(\alpha_{incl}) U_{cp,XS} + \cos(\alpha_{incl}) U_{cp,ZS} \] (2.22)

The magnitude of velocity at center of pressure can be calculated as:

\[ U_{cp} = \sqrt{U_{cp,XW}^2 + U_{cp,ZW}^2} \] (2.23)

Now the resultant angle of attack with body velocities’ effect added can be found in equation (2.24). Note that the atan2 function returns a value in \([-\pi, \pi]\) which helps ensure the
correct force direction according to the frames and coefficients defined previously.

\[ \alpha = \text{atan2}(-U_{cp,z_w}, U_{cp,x_w}) \]  

(2.24)

The wing rotational angular velocity can be calculated as the time derivative of the angle of attack as:

\[ \dot{\alpha} = \frac{-U_{cp,z_w} U_{cp,x_w} + U_{cp,z_w}^2 U_{cp,x_w}}{U_{cp,x_w}^2 + U_{cp,z_w}^2} \]  

(2.25)

From equation (2.21) and equation (2.22), we can find

\[ \dot{U}_{cp,x_w} = -\sin(\alpha_{incl}) \dot{\alpha}_{incl} U_{cp,x_s} + \cos(\alpha_{incl}) \dot{U}_{cp,x_s} + \cos(\alpha_{incl}) \dot{\alpha}_{incl} U_{cp,z_s} \]

\[ + \sin(\alpha_{incl}) \dot{U}_{cp,z_s} \]

\[ \dot{U}_{cp,z_w} = -\cos(\alpha_{incl}) \dot{\alpha}_{incl} U_{cp,x_s} - \sin(\alpha_{incl}) \dot{U}_{cp,x_s} - \sin(\alpha_{incl}) \dot{\alpha}_{incl} U_{cp,z_s} \]

\[ + \cos(\alpha_{incl}) \dot{U}_{cp,z_s} \]

From equation (2.19) and equation (2.20), we can find

\[ \dot{U}_{cp,x_s} = \hat{r}_2 L \dot{\phi} - \sin(\phi) \dot{\phi} u + \cos(\phi) \dot{u} + \dot{z}_{cpB} \cos(\phi) q - z_{cpB} \sin(\phi) \dot{\phi} q + z_{cpB} \cos(\phi) \dot{q} \]

\[ \dot{U}_{cp,z_s} = \dot{w} - \dot{x}_{cpB} q - x_{cpB} \dot{q} \]

From equation (2.17) and equation (2.18), we can find

\[ \dot{x}_{cpB} = \hat{r}_2 L \cos(\phi) \dot{\phi} + 0.25 \epsilon \cos(\alpha_{incl}) \dot{\alpha}_{incl} \cos(\phi) - 0.25 \epsilon \sin(\alpha_{incl}) \sin(\phi) \dot{\phi} \]

\[ \dot{z}_{cpB} = 0.25 \epsilon \sin(\alpha_{incl}) \dot{\alpha}_{incl} \]

From equation (2.2) and equation (2.3), we can find

\[ \dot{\phi} = -\phi_m \omega^2 \cos(\omega t) \]
The translational force and rotational force can be calculated according to equation (2.4), (2.5) and (2.8). Since they act either normal or tangential to the wing surface which are in the Wing Frame, a series of transformations from the Wing frame to the Body Frame are required to derive the equations of motion.

The resultant forces in the Stroke Plane Frame can be calculated as:

\[
F_{X_S} = F_{tr,N,X_S} + F_{tr,T,X_S} + F_{rot,X_S} \quad (2.26)
\]

\[
F_{Z_S} = F_{tr,N,Z_S} + F_{tr,T,Z_S} + F_{rot,Z_S} \quad (2.27)
\]

where

\[
F_{tr,N,X_S} = F_{tr,N}\cos(\alpha_{incl})
\]

\[
F_{tr,N,Z_S} = F_{tr,N}\sin(\alpha_{incl})
\]

\[
F_{tr,T,X_S} = -F_{tr,T}\sin(\alpha_{incl})
\]

\[
F_{tr,T,Z_S} = F_{tr,T}\cos(\alpha_{incl})
\]

\[
F_{rot,X_S} = F_{rot}\cos(\alpha_{incl})
\]

\[
F_{rot,Z_S} = F_{rot}\sin(\alpha_{incl})
\]

Since we only consider a symmetric flapping motion of the wings, as a result, the lateral directional forces and moments will cancel out. The resultant forces and moment in
longitudinal direction in the Body Frame can be calculated as:

\[ F_{XB} = F_{XS} \cos (\varphi) \]  \hspace{1cm} (2.28)

\[ F_{ZB} = F_{ZS} \]  \hspace{1cm} (2.29)

\[ M_{YB} = -F_{ZB}x_{cpB} + F_{XB}z_{cpB} \]  \hspace{1cm} (2.30)

Standard 6-DOF equations of motion in longitudinal direction are applied to the vehicle as follows:

\[ \ddot{u} = \frac{2F_{XB}}{m} - qw + g\sin(\theta) \]  \hspace{1cm} (2.31)

\[ \ddot{w} = \frac{2F_{ZB}}{m} + qu - g\cos(\theta) \]  \hspace{1cm} (2.32)

\[ \dot{q} = \frac{2M_{YB}}{I_{yy}} \]  \hspace{1cm} (2.33)

\[ \dot{\theta} = q \]  \hspace{1cm} (2.34)

where \( m \) is the mass of the vehicle; \( g \) is the gravitational constant; \( \theta \) is the pitch angle defined as the angle between \( X_B \)-axis and \( X_E \)-axis; \( I_{yy} \) is the moment of inertia about \( Y_B \)-axis. The forces and moment get multiplied by a factor of 2, because of the two wings.

The nonlinear model of the vehicle defined by equation (2.31) to (2.34) is simulated with the numerical values defined in Table 2.1. Figure 2.8 shows the instantaneous aerodynamic forces and moment, and the open-loop simulation results over a flapping cycle.
TABLE 2.1
NUMERICAL VALUES OF PARAMETERS

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>25 Hz</td>
<td>M</td>
<td>0.03 kg</td>
</tr>
<tr>
<td>$\varphi_0$</td>
<td>0°</td>
<td>$I_{yy}$</td>
<td>$1.125 \times 10^{-5}$ kg*m²</td>
</tr>
<tr>
<td>$\varphi_m$</td>
<td>50°</td>
<td>$\alpha_{incl_m}$</td>
<td>45°</td>
</tr>
<tr>
<td>h</td>
<td>0.04 m</td>
<td>$\hat{x}_0$</td>
<td>0.25</td>
</tr>
<tr>
<td>L</td>
<td>0.12 m</td>
<td>$\hat{c}$</td>
<td>0.6</td>
</tr>
<tr>
<td>$c_m$</td>
<td>0.05 m</td>
<td>$\hat{r}_2$</td>
<td>0.65</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>0.0435 m</td>
<td>$T$</td>
<td>$1/f$</td>
</tr>
<tr>
<td>A</td>
<td>0.0054 m²</td>
<td>$\Delta t$</td>
<td>0.25 T</td>
</tr>
</tbody>
</table>

Figure 2.8. (a) Instantaneous aerodynamic forces and moment. (b) Open-loop simulation results of the states.
Figure 2.8. (continued)
In this chapter, the flight stability of a flapping wing MAV is studied. Averaging theory is applied to a simplified model together with the small perturbation theory to obtain the stability derivatives and a linearized model analytically.

As discussed in the previous chapter, the flight dynamics of a flapping wing vehicle constitute a nonlinear system with time periodic nature. One of the most outstanding strategies to deal with such a system is to apply the averaging theory. Averaging is the procedure of replacing a vector field by its average (over time or an angular variable) to obtain asymptotic approximations to the original system. For a periodic case, it’s essentially a singular perturbation problem. The classic applications of averaging theory to dynamical systems are in the context of studying vibrations [29, 30]. Averaging theory is also the most commonly used method in analysis of flapping wing flight. When applied to the linearized system, full cycle averaging converts it into a linear time invariant (LTI) system, while half or quarter cycle averaging converts it into a jump-style linear time varying (LTV) system.

Taylor and Thomas applied the theory to the study of flapping wing animal flight and discuss the effects of wing motion on the body motion [31]. They suggested that when compared to the body’s natural frequency of motion, if the frequency of the wing flapping motion is high enough, then the modes of the central body tend to not be excited by the wing motion, and the body will mostly be affected by the average forces.

3.1 Simplified Model

To derive an analytical expression for the averaged system, the aerodynamic model developed in chapter 2 is now simplified by assuming that the vehicle maintains a constant absolute value of angle of attack over a full flapping cycle; the wing flips and reverses direction instantaneously at the end of each half stroke. As a result, the contribution of rotational force is neglected. This assumption is previously applied in [32]. Therefore, the angle of attack can be now defined as:

\[ \alpha = -\text{sign}(\phi)\alpha_m \]  

(3.1)

Where \( \alpha_m \) is the constant absolute value of the angle of attack; \( \text{sign}(\phi) \) returns the sign of flapping angular velocity which will be discussed in section 3.2. It should be noticed that the use of sign function here and after is to ensure correct forces, moments and velocities’ directions based on the reference frames previously defined.

With the above assumption, geometrically we have \( \alpha_m + \alpha_{\text{incl}} = \pi/2 \). Therefore
some parameters previously defined can be modified using the angle of attack instead of the inclination angle as follows:

\[ x_{cp_b} = \hat{r}_2 L \sin(\phi) - 0.25 c \text{sign}(\phi) \cos(\alpha_m) \cos(\phi) \]  

(3.2)

\[ z_{cp_b} = h - 0.25 c \sin(\alpha_m) \]  

(3.3)

\[ C_N(\alpha) = -3.4 \text{sign}(\phi) \sin(\alpha) \]  

(3.4)

\[ C_T(\alpha) = -0.4 \text{sign}(\phi) \cos^2(2\alpha) \]  

(3.5)

\[ R_{WS} = \begin{bmatrix} \sin(\alpha) & 0 & -\cos(\alpha) \\ 0 & 1 & 0 \\ \cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \]  

(3.6)

\[ R_{SW} = \begin{bmatrix} \sin(\alpha) & 0 & \cos(\alpha) \\ 0 & 1 & 0 \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \]  

(3.7)

The normal and tangential components of the translational force are calculated using equation (2.4) and (2.5) with the new coefficients defined above. With a same procedure of coordinate transformation using equation (2.13), (2.14), (3.6) and (3.7), the resultant forces in the Stroke Plane Frame are obtained as:

\[ F_{x_S} = F_{tr,N,x_S} + F_{tr,T,x_S} \]  

(3.8)

\[ F_{z_S} = F_{tr,N,z_S} + F_{tr,T,z_S} \]  

(3.9)

Where

\[ F_{tr,N,x_S} = F_{tr,N} \sin(\alpha) \]

\[ F_{tr,N,z_S} = F_{tr,N} \cos(\alpha) \]

\[ F_{tr,T,x_S} = F_{tr,T} \cos(\alpha) \]

\[ F_{tr,T,z_S} = -F_{tr,T} \sin(\alpha) \]
The resultant forces and moment in the Body Frame and the equations of motion are the same expressions as in equation (2.28) to (2.34) but with the new parameters defined above. For completeness, they are still summarized here.

\[ F_{X_B} = F_{X_S} \cos (\varphi) \]  
(3.10)

\[ F_{Z_B} = F_{Z_S} \]  
(3.11)

\[ M_{Y_B} = -F_{Z_B} x_{c\theta_B} + F_{X_B} z_{c\theta_B} \]  
(3.12)

\[ \dot{u} = \frac{2F_{X_B}}{m} - qw + g \sin(\theta) \]  
(3.13)

\[ \dot{w} = \frac{2F_{Z_B}}{m} + qu - g \cos(\theta) \]  
(3.14)

\[ \dot{q} = \frac{2M_{Y_B}}{I_{yy}} \]  
(3.15)

\[ \dot{\theta} = q \]  
(3.16)

A comparison of the instantaneous aerodynamic forces and moment of the nonlinear model defined by equation (2.28) to (2.30) and the simplified model defined by equation (3.10) to (3.12) is given in Figure 3.1 (a). Figure 3.1 (b) shows the open-loop simulation results of the simplified model defined by equation (3.13) to (3.16).
Figure 3.1. (a) Comparison of the instantaneous aerodynamic forces and moment of the original model and the simplified model. (b) Open-loop simulation results of the simplified model.

3.2 Averaged Aerodynamic Forces and Moments

The average aerodynamic forces and moment over a full flapping cycle can be calculated according to [27] as:

\[
\bar{y} = \frac{1}{T} \int_0^T y(t) \, dt
\]  

(3.17)

where \(y\) represents the aerodynamic forces and moments.
For brevity in the following equations, two coefficients are defined as:

\[ c_1 = 1.7\rho A\sin(a_m)(\hat{r}_2 L\phi_m \omega)^2 \] \hspace{1cm} (3.18)

\[ c_2 = 0.2\rho A\cos^2(2a_m)(\hat{r}_2 L\phi_m \omega)^2 \] \hspace{1cm} (3.19)

And we further define:

\[ b_1 = c_1\sin(a_m) + c_2\cos(a_m) \] \hspace{1cm} (3.20)

\[ b_2 = c_1\cos(a_m) - c_2\sin(a_m) \] \hspace{1cm} (3.21)

By substituting equation (2.1), (2.2), (2.12), (3.4), (3.5) into equation (2.4) and (2.5), the expressions for \( F_{tr,N} \) and \( F_{tr,T} \) can be obtained. Then follow the frame transformations defined in equation (3.8) and (3.9), together with the center of pressure location defined in equation (3.2) and (3.3), equation (3.10) to (3.12) can now be fully expanded and rearranged as follows:

\[ F_{X_B}(t) = -b_1\cos(\phi_0)\sin^2(\omega t)\cos(\phi_m \cos(\omega t))\text{sign}(\phi) \]

\[ +b_1\sin(\phi_0)\sin^2(\omega t)\sin(\phi_m \cos(\omega t))\text{sign}(\phi) \] \hspace{1cm} (3.22)

\[ F_{Z_B}(t) = b_2 \sin^2(\omega t) \] \hspace{1cm} (3.23)

\[ M_{Y_B}(t) = (a_1 + a_3\text{sign}(\phi) + a_5\text{sign}(\phi))\sin^2(\omega t)\cos(\phi_m \cos(\omega t)) \]

\[ +(a_2 + a_4\text{sign}(\phi) + a_6\text{sign}(\phi))\sin^2(\omega t)\sin(\phi_m \cos(\omega t)) \] \hspace{1cm} (3.24)

where \( a_1 \) to \( a_6 \) are collections of constant terms and defined as:

\[ a_1 = -b_2\hat{r}_2 L\sin(\phi_0) \]

\[ a_2 = -b_2\hat{r}_2 L\cos(\phi_0) \]

\[ a_3 = 0.25\bar{c}b_2 \cos(a_m)\cos(\phi_0) \]
The cycle average forces and moment can be calculated according to equation (3.17) as:

\[
T\bar{F}_{X_B} = \frac{1}{T} \int_0^T F_{X_B}(t)dt
\]  

(3.25)

\[
T\bar{F}_{Z_B} = \frac{1}{T} \int_0^T F_{Z_B}(t)dt
\]  

(3.26)

\[
T\bar{M}_{Y_B} = \frac{1}{T} \int_0^T M_{Y_B}(t)dt
\]  

(3.27)

Note that the left superscript ‘T’ indicates the average is taken over a full flapping cycle, which is used to distinguish the quarter cycle averaging with left superscript ‘T/4’.

It can be observed that evaluating the integrals involves terms of the form \( \sin(\cos(x)) \) and \( \cos(\cos(x)) \) which have no indefinite integral solutions. However definite integrals exist and can be numerically solved by Bessel function \( J_{\nu}(z) \) and Struve function \( H_{\nu}(z) \) which are defined as follows [33]:

\[
J_{\nu}(z) = \frac{(\frac{z}{2})^{\nu}}{\pi^{\frac{1}{2}}\Gamma(\nu + \frac{1}{2})} \int_0^\pi \cos(z\cos\theta) (\sin\theta)^{2\nu}d\theta
\]  

(3.28)

\[
H_{\nu}(z) = \frac{2(\frac{z}{2})^{\nu}}{\pi^{\frac{1}{2}}\Gamma(\nu + \frac{1}{2})} \int_0^\pi \sin(z\cos\theta) (\sin\theta)^{2\nu}d\theta
\]  

(3.29)

Where \( \Gamma \) denotes the gamma function and is defined as:

\[
\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt
\]  

(3.30)

For clarity, it should be noticed that if we define \( \lambda = \omega t \), then as \( t \to 0 \), \( \lambda \to 0 \); as \( t \)
→ T, λ → 2π. Equation (3.22) to (3.27) can be expressed in an equivalent form over the interval of [0, 2π] as follows, which would fit the definition of Bessel function and Struve function.

\[ F_{XB}(\lambda) = -b_1 \cos(\phi_0) \cos^2(\lambda) \cos(\phi_m \cos(\lambda)) \text{sign}(\phi) \]

\[ + b_1 \sin(\phi_0) \sin^2(\lambda) \cos(\phi_m \cos(\lambda)) \text{sign}(\phi) \]  \hspace{1cm} (3.31)

\[ F_{ZB}(\lambda) = b_2 \sin^2(\lambda) \]  \hspace{1cm} (3.32)

\[ M_{YB}(\lambda) = (a_1 + a_3 \text{sign}(\phi) + a_5 \text{sign}(\phi)) \sin^2(\lambda) \cos(\phi_m \cos(\lambda)) \]

\[ + \left( a_2 + a_4 \text{sign}(\phi) + a_6 \text{sign}(\phi) \right) \sin^2(\lambda) \sin(\phi_m \cos(\lambda)) \]  \hspace{1cm} (3.33)

\[ T^{F_{XB}} = \frac{1}{2\pi} \int_{0}^{2\pi} F_{XB}(\lambda) d\lambda \]  \hspace{1cm} (3.34)

\[ T^{F_{ZB}} = \frac{1}{2\pi} \int_{0}^{2\pi} F_{ZB}(\lambda) d\lambda \]  \hspace{1cm} (3.35)

\[ T^{M_{YB}} = \frac{1}{2\pi} \int_{0}^{2\pi} M_{YB}(\lambda) d\lambda \]  \hspace{1cm} (3.36)

From equation (3.28) and (3.29) we can see the Bessel function is defined over the interval of [0, π] and the Struve function is defined over the interval of [0, \frac{\pi}{2}]. While our interval of interest is defined over [0, 2π]. So it’s necessary to break down the integral over [0, 2π] into the sum of integrals over smaller regions. To do so, two functions are defined as:

\[ F_1(\lambda) = \sin^2(\lambda) \cos(\phi_m \cos(\lambda)) \]  \hspace{1cm} (3.37)

\[ F_2(\lambda) = \sin^2(\lambda) \sin(\phi_m \cos(\lambda)) \]  \hspace{1cm} (3.38)

Equation (3.31) and (3.33) can now be written as:

\[ F_{XB}(\lambda) = -b_1 \cos(\phi_0) \text{sign}(\phi) F_1(\lambda) + b_1 \sin(\phi_0) \text{sign}(\phi) F_2(\lambda) \]  \hspace{1cm} (3.39)
\[ M_{Y_B}(\lambda) = (a_1 + a_3 \text{sign}(\hat{\phi}) + a_5 \text{sign}(\hat{\phi})) F_1(\lambda) + (a_2 + a_4 \text{sign}(\hat{\phi}) + a_6 \text{sign}(\hat{\phi})) F_2(\lambda) \quad (3.40) \]

The plots of \( F_1(\lambda) \) and \( F_2(\lambda) \) over a full flapping cycle are shown in Figure 3.2.

\[
\int_0^\pi F_1(\lambda) d\lambda = \int_\pi^{2\pi} F_1(\lambda) d\lambda = \int_{2\pi}^{3\pi} F_1(\lambda) d\lambda = \int_{3\pi}^{4\pi} F_1(\lambda) d\lambda \quad (3.41)
\]

\[
\int_0^\pi F_2(\lambda) d\lambda = -\int_\pi^{2\pi} F_2(\lambda) d\lambda = -\int_{2\pi}^{3\pi} F_2(\lambda) d\lambda = -\int_{3\pi}^{4\pi} F_2(\lambda) d\lambda \quad (3.42)
\]

Another time related term that needs to be evaluated is \( \text{sign}(\hat{\phi}) \). Based on the definition of \( \hat{\phi} \) from equation (2.2) and the reference frames previously developed, \( \text{sign}(\hat{\phi}) \) will return values as follow:

\[
\text{sign}(\hat{\phi}) = \begin{cases} 
-1 & 0 \leq \lambda < \frac{\pi}{2} \\
-1 & \frac{\pi}{2} \leq \lambda < \pi \\
1 & \pi \leq \lambda < \frac{3\pi}{2} \\
1 & \frac{3\pi}{2} \leq \lambda \leq 2\pi 
\end{cases} \quad (3.43)
\]
3.2.1 Full Cycle Averaged Force along X_B-axis

To evaluate the solution of $\overline{F_X}$, substitute equation (3.39) into equation (3.34) and split the integral associated with $F_1$ into two sub-integrals; integral associated with $F_2$ into four sub-integrals, according to Figure 3.2 and equation (3.41) and (3.42). We can obtain:

$$\overline{F_X} = \frac{1}{2\pi} \int_0^{2\pi} F_X(\lambda) d\lambda$$

$$\overline{F_X} = \frac{1}{2\pi} \left[ -\int_0^{2\pi} b_1 \cos(\varphi_0) \text{sign}(\phi) F_1(\lambda) d\lambda ight.$$  

$$+ \int_0^{2\pi} b_1 \sin(\varphi_0) \text{sign}(\phi) F_2(\lambda) d\lambda \right]$$

$$= \frac{1}{2\pi} \left[ -\int_0^{\pi} b_1 \cos(\varphi_0) \text{sign}(\phi) F_1(\lambda) d\lambda ight.$$  

$$- \int_0^{\pi} b_1 \cos(\varphi_0) \text{sign}(\phi) F_1(\lambda) d\lambda \right]$$

$$+ \frac{1}{2\pi} \left[ \int_0^{\pi} b_1 \sin(\varphi_0) \text{sign}(\phi) F_2(\lambda) d\lambda \right.$$  

$$+ \int_0^{\pi} b_1 \sin(\varphi_0) \text{sign}(\phi) F_2(\lambda) d\lambda \right.$$  

$$+ \int_0^{3\pi} b_1 \sin(\varphi_0) \text{sign}(\phi) F_2(\lambda) d\lambda$$

Then substitute equation (3.41) and (3.42) to get:

$$= \frac{1}{2\pi} \left[ \int_0^{\pi} b_1 \cos(\varphi_0) F_1(\lambda) d\lambda - \int_0^{\pi} b_1 \cos(\varphi_0) F_1(\lambda) d\lambda \right]$$

$$+ \frac{1}{2\pi} \left[ -\int_0^{\frac{\pi}{2}} b_1 \sin(\varphi_0) F_2(\lambda) d\lambda - \int_0^{\frac{\pi}{2}} b_1 \sin(\varphi_0) F_2(\lambda) d\lambda \right.$$

$$+ \int_0^{\frac{3\pi}{2}} b_1 \sin(\varphi_0) F_2(\lambda) d\lambda + \int_0^{\frac{3\pi}{2}} b_1 \sin(\varphi_0) F_2(\lambda) d\lambda$$

$$= 0$$
3.2.2 Full Cycle Averaged Force along $Z_B$-axis

By substituting equation (3.32) into equation (3.35), $\bar{F}_{Z_B}$ can be found as:

$$\bar{F}_{Z_B} = \frac{1}{2\pi} \int_{0}^{2\pi} F_{Z_B}(\lambda) d\lambda$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} b_2 \sin^2(\lambda) d\lambda$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} b_2 \left(1-\frac{(2\lambda)}{\pi}\right) d\lambda$$

$$= \frac{1}{4\pi} \int_{0}^{2\pi} b_2 d\lambda - \frac{1}{4\pi} \int_{0}^{2\pi} b_2 \cos{(2\lambda)} d\lambda$$

$$= \frac{1}{4\pi} \int_{0}^{2\pi} b_2 d\lambda - \frac{1}{4\pi} \int_{0}^{2\pi} b_2 \cos{(2\lambda)} d\lambda$$

$$= \frac{1}{2} b_2$$

3.2.3 Full Cycle Averaged Moment about $Y_B$-axis

$\bar{M}_{Y_B}$ can be obtained in the same way as finding $\bar{F}_{X_B}$ by dividing the integral into sub-integrals and evaluating the sign of each term according to equation (3.41) to (3.43) as:

$$\bar{M}_{Y_B} = \frac{1}{2\pi} \int_{0}^{2\pi} M_{Y_B}(\lambda) d\lambda$$

$$= \frac{1}{2\pi} \left[ \int_{0}^{\frac{\pi}{2}} (a_1 + a_3 \text{sign}(\phi) + a_5 \text{sign}(\phi)) F_1(\lambda) d\lambda + \int_{0}^{\frac{3\pi}{2}} (a_2 + a_4 \text{sign}(\phi) + a_6 \text{sign}(\phi)) F_2(\lambda) d\lambda \right]$$

$$= \frac{1}{2\pi} \left[ \int_{0}^{\frac{\pi}{2}} (a_1 + a_3 \text{sign}(\phi) + a_5 \text{sign}(\phi)) F_1(\lambda) d\lambda + \int_{0}^{\frac{3\pi}{2}} (a_2 + a_4 \text{sign}(\phi) + a_6 \text{sign}(\phi)) F_2(\lambda) d\lambda \right]$$

$$+ \frac{1}{2\pi} \left[ \int_{0}^{\frac{\pi}{2}} (a_1 + a_3 \text{sign}(\phi) + a_5 \text{sign}(\phi)) F_1(\lambda) d\lambda + \int_{0}^{\frac{3\pi}{2}} (a_2 + a_4 \text{sign}(\phi) + a_6 \text{sign}(\phi)) F_2(\lambda) d\lambda \right]$$

$$+ \frac{1}{2\pi} \left[ \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (a_2 + a_4 \text{sign}(\phi) + a_6 \text{sign}(\phi)) F_2(\lambda) d\lambda \right]$$

$$+ \frac{1}{2\pi} \left[ \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (a_2 + a_4 \text{sign}(\phi) + a_6 \text{sign}(\phi)) F_2(\lambda) d\lambda \right]$$

$$= \frac{1}{2\pi} \left[ \int_{0}^{\pi} (a_1 - a_3 - a_5) F_1(\lambda) d\lambda + \int_{\pi}^{2\pi} (a_1 + a_3 + a_5) F_1(\lambda) d\lambda \right]$$
\[\begin{align*}
+ \frac{1}{2\pi} \left[ \int_{0}^{\pi} (a_2 - a_4 - a_6) F_2(\lambda) d\lambda + \int_{\pi}^{2\pi} (a_2 - a_4 - a_6) F_2(\lambda) d\lambda \\
+ \int_{\pi}^{\frac{3\pi}{2}} (a_2 + a_4 + a_6) F_2(\lambda) d\lambda \\
+ \int_{\frac{3\pi}{2}}^{2\pi} (a_2 + a_4 + a_6) F_2(\lambda) d\lambda \right] \\
= \frac{1}{2\pi} \left[ \int_{0}^{\pi} (a_1 - a_3 - a_5) F_1(\lambda) d\lambda + \int_{0}^{\pi} (a_1 + a_3 + a_5) F_1(\lambda) d\lambda \\
+ \frac{1}{2\pi} \left[ \int_{0}^{\pi} (a_2 - a_4 - a_6) F_2(\lambda) d\lambda - \int_{0}^{\pi} (a_2 - a_4 - a_6) F_2(\lambda) d\lambda \\
- \int_{0}^{\pi} (a_2 + a_4 + a_6) F_2(\lambda) d\lambda \\
+ \int_{0}^{\pi} (a_2 + a_4 + a_6) F_2(\lambda) d\lambda \right] \\
= \frac{1}{\pi} \int_{0}^{\pi} a_1 F_1(\lambda) d\lambda
\end{align*}\]

According to equation (3.28), the numerical solution of the above integral can be found as:

\[
\tau_{\overline{M}_{Y_B}} = \frac{a_1 J_1(\phi_m) \Gamma(1.5)}{0.5\phi_m \pi^2}
\]

The final full cycle averaged aerodynamic forces and moment from previous analysis are summarized as follows:

\[
\tau F_{X_B} = 0 \quad (3.44)
\]

\[
\tau F_{Z_B} = \frac{1}{2} b_2 \quad (3.45)
\]

\[
\tau \overline{M}_{Y_B} = \frac{a_1 J_1(\phi_m) \Gamma(1.5)}{0.5\phi_m \pi^2} \quad (3.46)
\]

A comparison of the instantaneous non-averaged and full cycle averaged forces and moment are shown in Figure 3.3 for a full flapping cycle.
According to Orlowski and Girard’s work [34], an alternative method was applied using local averaging over a quarter flapping cycle. They proposed the results for a full cycle averaged system were not consistent with the solution to the full system. A quarter cycle averaged result would give a significant improvement in the accuracy of the position and orientation of the open loop simulations compared to full cycle averaged system. To verify their conclusion, quarter cycle averaged solution is derived here.

Equation (3.34) to (3.36) are modified by taking the integrals over each quarter cycle. The quarter cycle averaged forces and moment can then be written as piecewise functions as follows:

\[
\frac{T/4}{\pi} F_{X_B} = \begin{cases} 
\frac{2}{\pi} \int_0^{\pi/2} F_{X_B} (\lambda) d\lambda \\
\frac{2}{\pi} \int_{\pi/2}^{\pi} F_{X_B} (\lambda) d\lambda \\
\frac{2}{\pi} \int_{3\pi/2}^{2\pi} F_{X_B} (\lambda) d\lambda \\
\frac{2}{\pi} \int_{2\pi}^{3\pi/2} F_{X_B} (\lambda) d\lambda \\
\frac{2}{\pi} \int_{\pi}^{3\pi/2} F_{X_B} (\lambda) d\lambda 
\end{cases}
\] (3.47)
The above integrals will be evaluated in the same way as before by either directly solving the integrals, or transforming the integrals over the intervals where Bessel function and Struve function are defined and evaluating the sign(φ) term. With the help of the sign function (sign(φ) and sign(F_B)), equation (3.47) to (3.49) can be solved and written in a comprehensive form as follows:

\[
\frac{T}{4} \bar{F}_{Z_B} = \begin{cases} 
\frac{2}{\pi} \int_{0}^{\pi} F_B(\lambda) d\lambda \\
\frac{2}{\pi} \int_{\pi}^{3\pi} F_B(\lambda) d\lambda \\
\frac{2}{\pi} \int_{3\pi}^{2\pi} F_B(\lambda) d\lambda \\
\frac{2}{\pi} \int_{2\pi}^{\pi} F_B(\lambda) d\lambda 
\end{cases} (3.48)
\]

\[
\frac{T}{4} \bar{M}_{Y_B} = \begin{cases} 
\frac{2}{\pi} \int_{0}^{\pi} M_B(\lambda) d\lambda \\
\frac{2}{\pi} \int_{\pi}^{3\pi} M_B(\lambda) d\lambda \\
\frac{2}{\pi} \int_{3\pi}^{2\pi} M_B(\lambda) d\lambda \\
\frac{2}{\pi} \int_{2\pi}^{\pi} M_B(\lambda) d\lambda 
\end{cases} (3.49)
\]

\[
T/4 \bar{F}_{X_B} = -b_1 \cos(\varphi_0) \text{sign}(\dot{\varphi}) \frac{I_1(\nu_m)\Gamma(1.5)}{0.5\nu_m \pi^2} \\
-b_1 \sin(\varphi_0) \text{sign}(F_2) \frac{H_1(\nu_m)\Gamma(1.5)}{0.5\nu_m \pi^2} (3.50)
\]

\[
T/4 \bar{F}_{Z_B} = \frac{1}{2} b_2 (3.51)
\]

\[
T/4 \bar{M}_{Y_B} = (a_1 + a_3 \text{sign}(\dot{\varphi}) + a_5 \text{sign}(\dot{\varphi})) \frac{I_1(\nu_m)\Gamma(1.5)}{0.5\nu_m \pi^2} \\
+ (a_2 + a_4 \text{sign}(\dot{\varphi}) + a_6 \text{sign}(\dot{\varphi})) \text{sign}(F_2) \frac{H_1(\nu_m)\Gamma(1.5)}{0.5\nu_m \pi^2} (3.52)
\]

A comparison of the instantaneous non-averaged and quarter cycle averaged forces and moment are shown in Figure 3.4 for a full flapping cycle.
A comparison of open-loop simulations for non-averaged, full cycle averaged and quarter cycle averaged systems is shown Figure 3.5 for a full flapping cycle. It can be seen that the full cycle averaging ignores the oscillatory changes of force along $X_B$-axis and moment about $Y_B$-axis, and treats them as ‘self-balancing’ (zero in magnitude) which would result in loosing accuracy in $X_B$ directional velocity, pitch rate and pitch angle simulation results; while the quarter cycle averaged results fit the non-averaged system closely. But for the purpose of applying conventional stability analysis with quasi-steady assumption, the full cycle averaging will be used.
3.3 Linearization

The equations of motion developed in equation (3.13) to (3.16) can be linearized using small perturbation theory. Hover condition is taken as the equilibrium state where all the forces and moments are balanced and all states \([u_e, w_e, q_e, \theta_e]\) are zero. Note that the subscript ‘e’ here and after indicates the equilibrium value.

From the full cycle averaged solution we can see, after a full flapping cycle, the \(X_B\)-axis force and \(Y_B\)-axis moment will both be zero. For equilibrium, we need to have the averaged \(Z_B\)-axis aerodynamic force balances the weight of the vehicle, which leads to:

\[
T_{FZ_B} = mg
\]  

(3.53)

With the symmetric flapping assumption, we can infer that at equilibrium state the mean wing position will lie on \(Y_B\)-axis. As a result the mean equilibrium wing offset angle \(\varphi_{0e} = 0\). Then by choosing a value for the mean equilibrium flapping amplitude \(\varphi_{me}\), we can solve for the equilibrium hovering frequency from equation (3.53) as:

\[
f_e = \frac{2mg}{\sqrt{\rho \sin(\alpha_m)(\phi_{me} L_{m} \phi_{me}^2)} (1.7 \cos(\alpha_m) - 0.2 \cos^2(2\alpha_m))}
\]

(3.54)

The numerical values of parameters associated with hover condition are summarized in Table 3.1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_e)</td>
<td>17 Hz</td>
</tr>
<tr>
<td>(\varphi_{0e})</td>
<td>0°</td>
</tr>
<tr>
<td>(\varphi_{me})</td>
<td>50°</td>
</tr>
<tr>
<td>(\alpha_m)</td>
<td>45°</td>
</tr>
</tbody>
</table>

The model is assumed to be at near hover condition with small perturbations in the states \([\Delta u, \Delta w, \Delta q, \Delta \theta]\). The equations of motion can then be written as:
\[
\dot{u}_e + \Delta \dot{u} = \frac{2(F_{xB_e} + \Delta F_{x_B})}{m} - (q_e + \Delta q)(w_e + \Delta w) + g \sin(\theta_e + \Delta \theta)
\] (3.55)

\[
\dot{w}_e + \Delta \dot{w} = \frac{2(F_{z_B e} + \Delta F_{z_B})}{m} + (q_e + \Delta q)(u_e + \Delta u) - g \cos(\theta_e + \Delta \theta)
\] (3.56)

\[
\dot{q}_e + \Delta \dot{q} = \frac{2(M_{y_B e} + \Delta M_{y_B})}{I_{yy}}
\] (3.57)

\[
\dot{\theta}_e + \Delta \dot{\theta} = q_e + \Delta q
\] (3.58)

The perturbed aerodynamic forces and moment can be approximated by the first terms of Taylor’s theorem as [27]:

\[
\Delta F_{x_B} = X_u \Delta u + X_w \Delta w + X_q \Delta q
\] (3.59)

\[
\Delta F_{z_B} = Z_u \Delta u + Z_w \Delta w + Z_q \Delta q
\] (3.60)

\[
\Delta M_{y_B} = M_u \Delta u + M_w \Delta w + M_q \Delta q
\] (3.61)

By substituting equation (3.59) to (3.61) into equation (3.55) to (3.58) and eliminating the contributions for hover condition, we are able to get the perturbed equations of motion and write in matrix form as:

\[
\begin{bmatrix}
\Delta \dot{u} \\
\Delta \dot{w} \\
\Delta \dot{q} \\
\Delta \dot{\theta}
\end{bmatrix} = A
\begin{bmatrix}
\Delta u \\
\Delta w \\
\Delta q \\
\Delta \theta
\end{bmatrix}
\] (3.62)

where 

\[
A = \begin{bmatrix}
\frac{2X_u}{m} & \frac{2X_w}{m} & \frac{2X_q}{m} & g \cos(\theta_e) \\
\frac{2Z_u}{m} & \frac{2Z_w}{m} & \frac{2Z_q}{m} & g \sin(\theta_e) \\
\frac{M_u}{I_{yy}} & \frac{M_w}{I_{yy}} & \frac{M_q}{I_{yy}} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

### 3.3.1 Stability Derivatives with Full Cycle Averaging

The perturbed aerodynamic forces and moment would result from two aspects. First, the perturbed velocities would induce a change in wing center of pressure velocity. Second, the perturbed velocities would induce a change in angle of attack.
To investigate the effects of perturbed velocities on the aerodynamic forces and moment, replace \([u, w, q]\) with \([u_e + \Delta u, w_e + \Delta w, q_e + \Delta q]\) in equation (2.19) and (2.20), where \(u_e, w_e\) and \(q_e\) are all zero, we can get:

\[
U_{cp,x} = \hat{f}_2 L\phi + \cos(\phi)\Delta u + z_{cpb}\cos(\phi)\Delta q
\]

(3.63)

\[
U_{cp,z} = \Delta w - x_{cpb}\Delta q
\]

(3.64)

And the squared magnitude of center of pressure velocity can be obtained as:

\[
U_{cp}^2 = U_{cp,x}^2 + U_{cp,z}^2
\]

(3.65)

Substitute equation (3.63) and (3.64) into equation (3.65), neglect the \(\Delta^2\) term and eliminate the contribution of hovering flapping velocity, we can get:

\[
\Delta U_{cp}^2 = 2\hat{r}_2 L\phi \cos(\phi)\Delta u + 2\hat{r}_2 L\phi z_{cpb}\cos(\phi)\Delta q
\]

(3.66)

Equation (3.66) is then substituted into equation (2.4) and (2.5) to give the normal and tangential components of perturbed translational force. According to equation (3.8) to (3.12), the perturbed aerodynamic forces and moment in the Body Frame can be obtained. For brevity in the following equations, two coefficients are defined as:

\[
c_3 = 3.4\rho A\sin(\alpha_m)\hat{r}_2 L\phi_{me} \omega_e
\]

(3.67)

\[
c_4 = 0.4\rho A\cos^2(2\alpha_m)\hat{r}_2 L\phi_{me} \omega_e
\]

(3.68)

And we further define:

\[
b_3 = c_3\sin(\alpha_m) + c_4\cos(\alpha_m)
\]

(3.69)

\[
b_4 = c_3\cos(\alpha_m) - c_4\sin(\alpha_m)
\]

(3.70)

\[
F_3(t) = \sin(\omega_e t)\cos(\phi_{me}\cos(\omega_e t))
\]

(3.71)

\[
F_4(t) = \sin(\omega_e t)\sin(\phi_{me}\cos(\omega_e t))
\]

(3.72)
The perturbed aerodynamic forces and moment in the Body Frame can be obtained as:

\[ \Delta F_X^y = X^y_\Delta u + X^y_q \Delta q \] (3.77)

\[ \Delta F_Z^y = Z^y_\Delta u + Z^y_q \Delta q \] (3.78)

\[ \Delta M^y_B = M^y_\Delta u + M^y_q \Delta q \] (3.79)

Where

\[ X^y_\Delta u(t) = a_7 \text{sign}(\phi_e)F_7(t) - a_8 \text{sign}(\phi_e)F_6(t) + a_9 \text{sign}(\phi_e)F_8(t) \] (3.80)

\[ X^y_q(t) = (a_7 \text{sign}(\phi_e)F_7(t) - a_8 \text{sign}(\phi_e)F_6(t) + a_9 \text{sign}(\phi_e)F_8(t))(h - \frac{c}{4} \sin(\alpha)) \] (3.81)

\[ Z^y_u(t) = -a_{10}F_3(t) + a_{11}F_4(t) \] (3.82)

\[ Z^y_q(t) = (a_{10}F_3(t) - a_{11}F_4(t))(-h + \frac{c}{4} \sin(\alpha)) \] (3.83)

\[ M^y_U = a_{15}F_5(t) + (a_{16} + a_{17} \text{sign}(\phi_e))F_6(t) + a_{18} \text{sign}(\phi_e)F_7(t) + a_{19} \text{sign}(\phi_e)F_8(t) \] (3.84)

\[ M^y_q(t) = [ (a_{20} + a_{21} \text{sign}(\phi_e))F_7(t) + (a_{22} + a_{23} \text{sign}(\phi_e))F_6(t) + (a_{24} + a_{25} \text{sign}(\phi_e))F_8(t) ](-h + \frac{c}{4} \sin(\alpha)) \] (3.85)
Note that in the above equations the right superscript ‘v’ indicates that it is due to a change in wing center of pressure velocity, which is used to distinguish from the effect occurring due to the change in angle of attack written with superscript ‘α’. Here \( a_7 \) to \( a_{25} \) are collections of constant terms and defined as:

\[
\begin{align*}
    a_7 &= b_3 \cos^2(\varphi_{0_e}) \\
    a_8 &= 0.5b_3 \sin(2\varphi_{0_e}) \\
    a_9 &= b_3 \sin^2(\varphi_{0_e}) \\
    a_{10} &= b_4 \cos(\varphi_{0_e}) \\
    a_{11} &= b_4 \sin(\varphi_{0_e}) \\
    a_{12} &= b_4 \cos^2(\varphi_{0_e}) \\
    a_{13} &= 0.5b_4 \sin(2\varphi_{0_e}) \\
    a_{14} &= b_4 \sin^2(\varphi_{0_e}) \\
    a_{15} &= a_{13}r_2L \\
    a_{16} &= -0.5(a_{14} - a_{12})r_2L \\
    a_{17} &= a_{13}\frac{c}{4}\cos(\alpha_m) - a_8(h - \frac{c}{4}\sin(\alpha_m)) \\
    a_{18} &= -a_{12}\frac{c}{4}\cos(\alpha_m) + a_7(h - \frac{c}{4}\sin(\alpha_m)) \\
    a_{19} &= -a_{14}\frac{c}{4}\cos(\alpha_m) + a_9(h - \frac{c}{4}\sin(\alpha_m)) \\
    a_{20} &= -a_{13}r_2L
\end{align*}
\]
\[ a_{21} = a_{12} \frac{\vec{c}}{4} \cos(\alpha_m) - a_7 (h - \frac{\vec{c}}{4} \sin(\alpha_m)) \]

\[ a_{22} = 0.5(a_{14} - a_{12}) \hat{r}_2 L \]

\[ a_{23} = -a_{13} \frac{\vec{c}}{4} \cos(\alpha_m) + a_8 (h - \frac{\vec{c}}{4} \sin(\alpha_m)) \]

\[ a_{24} = a_{13} \hat{r}_2 L \]

\[ a_{25} = a_{14} \frac{\vec{c}}{4} \cos(\alpha_m) - a_9 (h - \frac{\vec{c}}{4} \sin(\alpha_m)) \]

The perturbed velocity will also induce a change in angle of attack which can be calculated as:

\[ \Delta \alpha = \tan^{-1} \left( \frac{U_{cp,z}}{U_{cp,x}} \right) \]  \hspace{1cm} (3.86)

Where

\[ U_{cp,x} = \hat{r}_2 L \dot{\phi} + \cos(\phi) \Delta u + z_{cpB} \cos(\phi) \Delta q \]

\[ U_{cp,z} = \Delta w - x_{cpB} \Delta q \]

By assuming the change in angle of attack is small, we can approximate the above equation as:

\[ \Delta \alpha \approx \frac{\Delta w - x_{cpB} \Delta q}{\hat{r}_2 L \dot{\phi} + \cos(\phi) \Delta u + z_{cpB} \cos(\phi) \Delta q} \]  \hspace{1cm} (3.87)

The effects of the perturbed angle of attack will show in the coefficients of normal and tangential components of translational force as follows:

\[ C_N(\alpha + \Delta \alpha) = -3.4 \text{sign}(\dot{\phi}) \sin(\alpha + \Delta \alpha) \]  \hspace{1cm} (3.88)

\[ C_T(\alpha + \Delta \alpha) = -0.4 \text{sign}(\dot{\phi}) \cos^2(\alpha + \Delta \alpha) \]  \hspace{1cm} (3.89)

Using small angle assumption, the above equations can be expanded as:

\[ C_N(\alpha + \Delta \alpha) = -3.4 \text{sign}(\dot{\phi}) \sin(\alpha) - 3.4 \text{sign}(\dot{\phi}) \cos(\alpha) \Delta \alpha \]

\[ C_T(\alpha + \Delta \alpha) = -0.4 \text{sign}(\dot{\phi}) \cos^2(2\alpha) + 0.8 \text{sign}(\dot{\phi}) \sin(4\alpha) \Delta \alpha \]
The first term in each of the above two equations contribute to the hover condition while the second terms contribute to the perturbation aerodynamic forces. We then define the perturbed coefficients as:

\[
\Delta C_N(\Delta \alpha) = -3.4 \text{sign}(\phi)\cos(\alpha)\Delta \alpha
\]

(3.90)

\[
\Delta C_T(\Delta \alpha) = 0.8 \text{sign}(\phi)\sin(4\alpha)\Delta \alpha
\]

(3.91)

Equation (3.90) and (3.91) are substituted into equation (2.4) and (2.5) to calculate the perturbation aerodynamic forces and moment due to change in angle of attack. For brevity in the following equations, four coefficients are defined as:

\[
c_5 = 1.7\rho \hat{A}_2 L q_m e \omega_e \cos(\alpha_m)
\]

(3.92)

\[
c_6 = 0.4\rho \hat{A}_2 L q_m e \omega_e \sin(4\alpha_m)
\]

(3.93)

\[
b_5 = c_5 \sin(\alpha_m) - c_6 \cos(\alpha_m)
\]

(3.94)

\[
b_6 = c_5 \cos(\alpha_m) + c_6 \sin(\alpha_m)
\]

(3.95)

By fully expanding and rearranging, the perturbed aerodynamic forces and moment in the Body Frame can be obtained as:

\[
\Delta F^g_{xB} = X^g_w \Delta w + X^g_q \Delta q
\]

(3.96)

\[
\Delta F^g_{ZB} = Z^g_w \Delta w + Z^g_q \Delta q
\]

(3.97)

\[
\Delta M^g_{YB} = M^g_w \Delta w + M^g_q \Delta q
\]

(3.98)

Where

\[
X^g_w(t) = a_{26}F_3(t) - a_{27}F_4(t)
\]

(3.99)

\[
X^g_q(t) = a_{28}F_5(t) + a_{29}F_6(t)
\]

(3.100)

\[
Z^g_w(t) = b_6 \text{sign}(\phi_e) \sin(\omega_e t)
\]

(3.101)

\[
Z^g_q(t) = a_{30} \text{sign}(\phi_e) F_3(t) + a_{31} \text{sign}(\phi_e) F_4(t)
\]

(3.102)
\[
M_{\phi}^m(t) = (a_{30}\text{sign}(\dot{\phi}_e) + a_{32} + a_{34})F_3(t)
+ (a_{31}\text{sign}(\dot{\phi}_e) + a_{33} + a_{35})F_4(t)
\]

(3.103)

\[
M_{\psi}^\rho(t) = a_{36}\text{sign}(\dot{\phi}_e)F_7(t)
+ (a_{37}\text{sign}(\dot{\phi}_e) + a_{38} + a_{42})F_6(t)
+ a_{39}\text{sign}(\dot{\phi}_e)F_0(t)
+ (a_{40} + a_{41})F_5(t)
\]

(3.104)

And \(a_{26}\) to \(a_{42}\) are collections of constant terms defined as:

\[
a_{26} = -b_5\cos(\varphi_{0_e})
\]

\[
a_{27} = -b_5\sin(\varphi_{0_e})
\]

\[
a_{28} = 0.5b_5\sin(2\varphi_{0_e})\hat{r}_2L
\]

\[
a_{29} = 0.5b_5\cos(2\varphi_{0_e})\hat{r}_2L
\]

\[
a_{30} = -b_6\sin(\varphi_{0_e})\hat{r}_2L
\]

\[
a_{31} = -b_6\cos(\varphi_{0_e})\hat{r}_2L
\]

\[
a_{32} = b_6\cos(\varphi_{0_e})\hat{r}_4\cos(\alpha_m)
\]

\[
a_{33} = -b_6\sin(\varphi_{0_e})\hat{r}_4\cos(\alpha_m)
\]

\[
a_{34} = a_{26}(h - \frac{\epsilon}{4}\sin(\alpha_m))
\]

\[
a_{35} = -a_{27}(h - \frac{\epsilon}{4}\sin(\alpha_m))
\]
\[ a_{36} = b_6 (r_2 L)^2 \sin^2 (\varphi_0 e) \]

\[ a_{37} = 0.5b_6 (r_2 L)^2 \sin (2\varphi_0 e) \]

\[ a_{38} = -0.5b_6 \hat{r}_2 L \frac{c}{4} \cos (\alpha_m) \cos (2\varphi_0 e) \]

\[ a_{39} = b_6 (r_2 L)^2 \cos^2 (\varphi_0 e) \]

\[ a_{40} = -0.5b_6 \hat{r}_2 L \frac{c}{4} \cos (\alpha_m) \sin (2\varphi_0 e) \]

\[ a_{41} = a_{28} (h - \frac{c}{4} \sin (\alpha_m)) \]

\[ a_{42} = a_{29} (h - \frac{c}{4} \sin (\alpha_m)) \]

The resultant stability derivatives are then calculated as the sum of the effects due to change in center of pressure velocity and change in angle of attack:

\[ X_u (t) = X_u^y (t) \]

\[ X_w (t) = X_w^\alpha (t) \]

\[ X_q (t) = X_q^y (t) + X_q^\alpha (t) \]

\[ Z_u (t) = Z_u^y (t) \]

\[ Z_w (t) = Z_w^\alpha (t) \]

\[ Z_q (t) = Z_q^y (t) + Z_q^\alpha (t) \]

\[ M_u (t) = M_u^y (t) \]

\[ M_w (t) = M_w^\alpha (t) \]

\[ M_q (t) = M_q^y (t) + M_q^\alpha (t) \]
### 3.3.2 Stability Analysis

As we can see from the previous result, the stability derivatives are functions of time. Averaging theory is applied over a full flapping cycle to transform the system into a LTI system. Then the pole location of the LTI system can be obtained for stability analysis.

The results of full flapping cycle averaged stability derivatives are summarized here. The detailed procedures can be found in Appendix.

\[
\begin{align*}
T_X^u & = - \frac{a_7}{\pi} \left( 1 + \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) - \frac{a_9}{\pi} \left( 1 - \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) \quad (3.111) \\
T_X^w & = 0 \quad (3.112) \\
T_X^q & = - \frac{a_7}{\pi} \left( h - \frac{\bar{c}}{4} \sin(\alpha_m) \right) \left( 1 + \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) \\
& \quad - \frac{a_9}{\pi} \left( h - \frac{\bar{c}}{4} \sin(\alpha_m) \right) \left( 1 - \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) \quad (3.113) \\
T_Z^u & = 0 \quad (3.114) \\
T_Z^w & = - \frac{2b_8}{\pi} \quad (3.115) \\
T_Z^q & = - \frac{2a_{30} \sin(\varphi_{me})}{\pi \varphi_{me}} \quad (3.116) \\
T_M^u & = - \frac{a_{18}}{\pi} \left( 1 + \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) - \frac{a_{19}}{\pi} \left( 1 - \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) \quad (3.117) \\
T_M^w & = - \frac{2a_{30} \sin(\varphi_{me})}{\pi \varphi_{me}} \quad (3.118) \\
T_M^q & = - \left( \frac{a_{21}}{\pi} \left( -h + \frac{\bar{c}}{4} \sin(\alpha_m) \right) + \frac{a_{36}}{\pi} \right) \left( 1 + \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) \\
& \quad - \left( \frac{a_{25}}{\pi} \left( -h + \frac{\bar{c}}{4} \sin(\alpha_m) \right) + \frac{a_{39}}{\pi} \right) \left( 1 - \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) \quad (3.119)
\end{align*}
\]

The LTI system is defined in equation (3.120). The longitudinal dynamic flight stability can be studied with the averaged system matrix \( \overline{T_A} \).
\[
\begin{bmatrix}
\Delta u \\
\Delta w \\
\Delta q \\
\Delta \theta
\end{bmatrix} = \tau_A \begin{bmatrix}
\Delta u \\
\Delta w \\
\Delta q \\
\Delta \theta
\end{bmatrix}
\]

where \( \tau_A = \begin{bmatrix}
\frac{2T_x}{m} & \frac{2T_y}{m} & \frac{2T_z}{m} & g \cos(\theta) \\
\frac{2T_y}{m} & \frac{2T_z}{m} & \frac{2T_x}{m} & g \sin(\theta) \\
\frac{2T_M}{m} & \frac{2T_M}{m} & \frac{2T_M}{m} & 0 \\
I_{yy} & I_{yy} & I_{yy} & 0
\end{bmatrix}\)

The eigenvalues of the LTI system are calculated and shown on a complex plane in Figure 3.6. It can be found there are a pair of complex conjugate eigenvalues with positive real part and two negative real eigenvalues.

![Figure 3.6. Pole location of the full cycle averaged linear system.](image)

The eigenvalues represent the natural modes of the system. A positive real part represents an exponential growth of the disturbance (unstable); a negative real part represents an exponential decay of the disturbance (stable). So in our case, the eigenvalues indicate that we an unstable oscillatory mode and two stable subsidence modes.

Equation (3.121) and (3.122) represent the time required for the initial perturbation to become half its initial value (for the stable subsidence mode) or double the amplitude of oscillation (for the unstable oscillatory mode), and the period of the oscillatory motion.

\[
t_{\text{half/double}} = \frac{0.693}{|\beta|}
\]
\[ T_{\text{oscillation}} = \frac{2\pi}{\bar{\omega}} \]  

(3.122)

Where \( \hat{n} \) is the real part of the eigenvalue, \( \bar{\omega} \) is the imaginary part of the eigenvalue (eg. \( \lambda = \hat{n} + \bar{\omega}i \) represents the eigenvalue).

The results are further non-dimensionalized with respect to the period of a full flapping cycle at hover condition. The properties of the system’s modes are summarized in Table 3.2.

<table>
<thead>
<tr>
<th>TABLE 3.2</th>
<th>PROPERTIES OF THE SYSTEM’S MODES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>Mode 2</td>
</tr>
<tr>
<td>Eigenvalue</td>
<td>2.0746±9.4103i</td>
</tr>
<tr>
<td>Stability</td>
<td>Unstable</td>
</tr>
<tr>
<td>Non-dimensional Time</td>
<td>( t^*_{\text{double}} = 5.68 )</td>
</tr>
<tr>
<td>Oscillation Time</td>
<td>( T^*_{\text{oscillation}} = 11.35 )</td>
</tr>
</tbody>
</table>

The eigenvectors associated with each eigenvalue are also calculated. The eigenvectors can be used to determine the magnitudes and phases of the disturbed states relative to each other. The results are summarized in Table 3.3 (Numbers in the parentheses are phase angles).

<table>
<thead>
<tr>
<th>TABLE 3.3</th>
<th>EIGENVECTORS AND PHASE ANGLE OF THE SYSTEM’S MODES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>Mode 2</td>
</tr>
<tr>
<td>( \Delta u )</td>
<td>0.92(−68.05°)</td>
</tr>
<tr>
<td>( \Delta w )</td>
<td>0(0°)</td>
</tr>
<tr>
<td>( \Delta q )</td>
<td>1.63(77.59°)</td>
</tr>
<tr>
<td>( \Delta \theta )</td>
<td>1(0°)</td>
</tr>
</tbody>
</table>

It can be seen from Table 3.2 and Table 3.3, the period of Mode 1 is 11.35 times a wingbeat period. The time of doubling the amplitude is 5.68 times a wingbeat period. \( \Delta u \), \( \Delta q \) and \( \Delta \theta \) are the main variables in this mode. \( \Delta q \) is about 10.5 times of \( \Delta u \). Horizontal and pitching oscillations constitute Mode 1. \( \Delta q \) is also the dominant variable in Mode 2. Mode 2 takes about 0.64 times a wingbeat for an initial perturbation to become half its initial value, which is about 10.8 times faster than Mode 3. Mode 3 is decoupled from the other modes. The only variable in Mode 3 is \( \Delta w \). It represents a descending or ascending motion of the vehicle.
It is known that the time response of a system may be represented as a weighted superposition of the system modes, where excitation of each mode is affected by the initial conditions [35], as shown in equation (3.123). So by appropriately choosing initial condition, we may see the system dominated by a single mode. The time responses of the three modes are shown in Figure 3.7.

\[ x(t) = V e^{\lambda(t-t_0)} W x(t_0) \]  

(3.123)

Where \( x(t) \) represents the time response of the states; \( \lambda \) is the eigenvalue; \( V \) is the right eigenvector; \( W \) is the left eigenvector; \( x(t_0) \) indicates the initial condition.

Figure 3.7. System’s modes decomposition.
The effects of wing joint vertical position and flapping amplitude on the system’s stability are also investigated.

The wing joint vertical position is critical to the system’s stability, since it directly affects the pitching moment due to $x_B$-axis force which acts through this arm; it also determines the effect of body pitching rate on the wing center of pressure velocity. Several wing joint vertical positions are considered at the hover condition. The eigenvalues are shown in Figure 3.8. It can be seen as lowering the joint position, the pole location associated with Mode 1 tends to move to the left half plane; the pole location associate with Mode 2 tends to move to the right half plane; the pole location associated with Mode 3 is independent of the joint position. As the wing joint position becomes lower than a particular value, Mode 1 becomes stable, while Mode 2 becomes unstable. This is caused by the sign change of $\mathbf{T}X_q$, $\mathbf{T}\bar{M}_u$ and $\mathbf{T}\bar{M}_q$ in the system matrix $\mathbf{T}A$. 

The effect of flapping amplitude change is shown in Figure 3.9. The result is obtained by changing the flapping amplitude while compensating with change in flapping frequency to maintain a hover condition. It can be seen that as the flapping amplitude increases, pole locations associated with both Mode 1 and Mode 2 are move to the left hand side; the pole location associated with mode 3 is independent to the flapping amplitude change.

**3.3.3 Stability Derivatives with Quarter Cycle Averaging**

As mentioned earlier, the system can also be modeled with quarter cycle averaging. The quarter cycle averaged stability derivatives are calculated and summarized as follows:
\[
\begin{align*}
T/4\dot{X}_u &= T/4\dot{X}_w = T/4\dot{X}_q = T/4\dot{X}_\alpha = T/4\dot{X}_\alpha + T/4\dot{X}_\alpha \\
T/4\dot{Z}_u &= T/4\dot{Z}_w = T/4\dot{Z}_q = T/4\dot{Z}_\alpha = T/4\dot{Z}_\alpha + T/4\dot{Z}_\alpha \\
T/4\dot{M}_u &= T/4\dot{M}_w = T/4\dot{M}_q = T/4\dot{M}_\alpha = T/4\dot{M}_\alpha + T/4\dot{M}_\alpha
\end{align*}
\]

Where

\[
\begin{align*}
T/4X_v &= \frac{a_1}{\pi} \left( 1 + \frac{\sin(2\varphi_m)}{2\varphi_m} \right) - \frac{a_2}{\pi} \left( \frac{1 - \cos(2\varphi_m)}{\varphi_m} \right) \text{sign}(\varphi) \text{sign}(F_6(t)) - \frac{a_3}{\pi} \left( 1 - \frac{\sin(2\varphi_m)}{2\varphi_m} \right) \\
T/4\dot{X}_v &= \frac{a_1}{\pi} \left( 1 + \frac{\sin(2\varphi_m)}{2\varphi_m} \right) - \frac{a_2}{\pi} \left( \frac{1 - \cos(2\varphi_m)}{\varphi_m} \right) \text{sign}(\varphi) \text{sign}(F_6(t)) \\
&- \frac{a_3}{\pi} \left( 1 - \frac{\sin(2\varphi_m)}{2\varphi_m} \right) \\
T/4Z_v &= 2a_{10} \frac{\sin(\varphi_m)}{\varphi_m} \text{sign}(\varphi) + 2a_{11} \left( \frac{1 - \cos(\varphi_m)}{\varphi_m} \right) \text{sign}(F_4(t)) \\
T/4\dot{Z}_v &= 2a_{10} \frac{\sin(\varphi_m)}{\varphi_m} \text{sign}(\varphi) (h - \frac{c}{4} \sin(\alpha_m)) + 2a_{11} \left( \frac{1 - \cos(\varphi_m)}{\varphi_m} \right) \text{sign}(F_4(t))(h - \frac{c}{4} \sin(\alpha_m)) \\
T/4\dot{M}_u &= -a_{15} \frac{\sin(2\varphi_m)}{\varphi_m} \text{sign}(\varphi) + (a_{16} + a_{17} \text{sign}(\varphi)) \left( \frac{1 - \cos(2\varphi_m)}{\varphi_m} \right) \text{sign}(F_6(t)) \\
&- \frac{a_{16}}{\pi} \left( 1 + \frac{\sin(2\varphi_m)}{2\varphi_m} \right) - \frac{a_{19}}{\pi} \left( 1 - \frac{\sin(2\varphi_m)}{2\varphi_m} \right) \\
T/4\dot{M}_q &= -a_{20} \frac{\text{sign}(\varphi)}{\pi} + a_{21} \left( -h + \frac{c}{4} \sin(\alpha_m) \right) \left( 1 + \frac{\sin(2\varphi_m)}{2\varphi_m} \right) \\
&+ a_{22} + a_{23} \frac{\text{sign}(\varphi)}{\pi} \left( -h + \frac{c}{4} \sin(\alpha_m) \right) \left( \frac{1 - \cos(2\varphi_m)}{\varphi_m} \right) \text{sign}(F_6(t)) \\
&- \frac{a_{24}}{\pi} \frac{\text{sign}(\varphi)}{\varphi_m} + a_{25} \left( -h + \frac{c}{4} \sin(\alpha_m) \right) \left( 1 - \frac{\sin(2\varphi_m)}{2\varphi_m} \right) \\
T/4\dot{X}_w &= -2a_{26} \frac{\sin(\varphi_m)}{\varphi_m} \text{sign}(\varphi) - 2a_{27} \left( \frac{1 - c}{\varphi_m} \right) \text{sign}(F_6(t))
\end{align*}
\]
\[
\frac{T}{4}X_q^\alpha = -a_{28} \frac{\sin(2\varphi_{me})}{\pi \varphi_{me}} \text{sign}(\dot{\varphi}) + a_{29} \left( \frac{1 - \cos(2\varphi_{me})}{\pi \varphi_{me}} \right) \text{sign}(F_6(t))
\]

\[
\frac{T}{4}Z_w^\alpha = -2 \frac{b_\delta}{\pi}
\]

\[
\frac{T}{4}Z_q^\alpha = -2a_{30} \frac{\sin(\varphi_{me})}{\pi \varphi_{me}} + 2a_{31} \left( \frac{1 - \cos(\varphi_{me})}{\pi \varphi_{me}} \right) \text{sign}(\dot{\varphi}) \text{sign}(F_4(t))
\]

\[
\frac{T}{4}M_w^\alpha = -2\left( a_{30} + a_{32} \text{sign}(\varphi) + a_{34} \text{sign}(\dot{\varphi}) \right) \frac{\sin(\varphi_{me})}{\pi \varphi_{me}} + 2\left( a_{31} \text{sign}(\dot{\varphi}) + a_{33} + a_{35} \right) \left( \frac{1 - \cos(\varphi_{me})}{\pi \varphi_{me}} \right) \text{sign}(F_4(t))
\]

\[
\frac{T}{4}M_q^\alpha = -\frac{a_{36}}{\pi} \left( 1 + \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) + (a_{37} \text{sign}(\dot{\varphi}) + a_{38} + a_{42}) \frac{1 - \cos(2\varphi_{me})}{\pi \varphi_{me}} \text{sign}(F_6(t)) - \frac{a_{39}}{\pi} \left( 1 - \frac{\sin(2\varphi_{me})}{2\varphi_{me}} \right) - (a_{40} + a_{41}) \frac{\sin(2\varphi_{me})}{\pi \varphi_{me}} \text{sign}(\dot{\varphi})
\]

With the quarter cycle averaged stability derivatives, the system is transformed into a jump-style linear time varying system defined in equation (3.124).

\[
\begin{bmatrix}
\Delta u \\
\Delta w \\
\Delta q \\
\Delta \theta
\end{bmatrix}
= \frac{T}{4}A
\begin{bmatrix}
\Delta u \\
\Delta w \\
\Delta q \\
\Delta \theta
\end{bmatrix} \tag{3.124}
\]

where \( \frac{T}{4}A = \begin{bmatrix}
2 \frac{T}{4}X_u & 2 \frac{T}{4}X_w & 2 \frac{T}{4}X_q & g\cos(\theta_e) \\
\frac{m}{2} & \frac{m}{2} & \frac{m}{2} & g\sin(\theta_e) \\
\frac{m}{2} & \frac{m}{2} & \frac{m}{2} & 0 \\
I_{yy} & I_{yy} & I_{yy} & 0
\end{bmatrix} \)
CHAPTER 4
CONTROLLER DESIGN

4.1 Control Derivative

The design objective in this paper is to design a control system for the vehicle to maintain a stable hover flight. A linear controller is designed and attached to the nonlinear system. The close-loop system is simulated at hover condition.

From Chapter 2 we can see, with a certain wing property, the aerodynamic forces and moment generated are essentially determined by the flapping frequency, flapping amplitude and mean wing position, which are chosen as the control inputs. Longitudinal maneuverability can be achieved by symmetrically varying left and right wing control inputs, while lateral maneuverability can be achieved by asymmetrically varying left and right wing control inputs.

For the purpose of controller design, the control derivatives need to be evaluated. The control derivative measures the change in the aerodynamic forces and moment for a unit change in the control inputs. As previously discussed, the dynamic system of the vehicle can be represented using averaged forces and moments. The control derivatives will then be calculated as the partial derivatives of the averaged forces and moment with respect to the flapping frequency, flapping amplitude and mean wing position.

Calculation of the control derivatives involves taking partial derivatives of each constant coefficient defined in the full cycle averaged and quarter cycle averaged forces’ and moment’s expressions with respect to the control inputs. The results are summarized as follows:

\[
\frac{\partial c_1}{\partial f} = 1.7 \rho \sin(\alpha_m)(\hat{f}_2 L \phi_m 2\pi)^2 2f \quad \frac{\partial c_1}{\partial \phi_m} = 1.7 \rho \sin(\alpha_m)(\hat{f}_2 L \omega)^2 2\phi_m
\]
\[
\frac{\partial c_2}{\partial f} = 0.2 \rho \cos^2(2\alpha_m)(\hat{f}_2 L \phi_m 2\pi)^2 2f \quad \frac{\partial c_2}{\partial \phi_m} = 0.2 \rho \cos^2(2\alpha_m)(\hat{f}_2 L \omega)^2 2\phi_m
\]
\[
\frac{\partial b_1}{\partial f} = \frac{\partial c_1}{\partial \alpha_m} \sin(\alpha_m) + \frac{\partial c_2}{\partial \alpha_m} \cos(\alpha_m) \quad \frac{\partial b_1}{\partial \phi_m} = \frac{\partial c_1}{\partial \alpha_m} \sin(\alpha_m) + \frac{\partial c_2}{\partial \alpha_m} \cos(\alpha_m)
\]
\[
\frac{\partial b_2}{\partial f} = \frac{\partial c_1}{\partial \alpha_m} \cos(\alpha_m) - \frac{\partial c_2}{\partial \alpha_m} \sin(\alpha_m) \quad \frac{\partial b_2}{\partial \phi_m} = \frac{\partial c_1}{\partial \alpha_m} \cos(\alpha_m) - \frac{\partial c_2}{\partial \alpha_m} \sin(\alpha_m)
\]
\[
\frac{\partial a_1}{\partial f} = - \frac{\partial b_2}{\partial \phi} \hat{f}_2 L \sin(\phi_0) \quad \frac{\partial a_1}{\partial \phi_0} = - \hat{b}_2 \hat{f}_2 L \cos(\phi_0) \quad \frac{\partial a_1}{\partial \phi_m} = - \frac{\partial b_2}{\partial \phi_m} \hat{f}_2 L \sin(\phi_0)
\]
\[
\frac{\partial a_2}{\partial t} = -\frac{\partial b_2}{\partial t} \hat{r}_2 \cos(\phi_0) \quad \frac{\partial a_2}{\partial \phi_0} = b_2 \hat{r}_2 \sin(\phi_0) \quad \frac{\partial a_2}{\partial \phi_m} = -\frac{\partial b_2}{\partial \phi_m} \hat{r}_2 \cos(\phi_0)
\]

\[
\frac{\partial a_3}{\partial t} = 0.25 c \frac{\partial b_2}{\partial t} \cos(\alpha_m) \cos(\phi_0) \quad \frac{\partial a_3}{\partial \phi_0} = -0.25 c b_2 \cos(\alpha_m) \sin(\phi_0)
\]

\[
\frac{\partial a_3}{\partial \phi_m} = 0.25 c \frac{\partial b_2}{\partial \phi_m} \cos(\alpha_m) \cos(\phi_0) \quad \frac{\partial a_4}{\partial t} = -0.25 c \frac{\partial b_2}{\partial t} \cos(\alpha_m) \sin(\phi_0)
\]

\[
\frac{\partial a_4}{\partial \phi_0} = -0.25 c b_2 \cos(\alpha_m) \cos(\phi_0) \quad \frac{\partial a_4}{\partial \phi_m} = -0.25 c \frac{\partial b_2}{\partial \phi_m} \cos(\alpha_m) \sin(\phi_0)
\]

\[
\frac{\partial a_5}{\partial t} = \left( -h + \frac{c}{4} \sin(\alpha_m) \right) \frac{\partial b_1}{\partial t} \cos(\phi_0) \quad \frac{\partial a_5}{\partial \phi_0} = \left( h - \frac{c}{4} \sin(\alpha_m) \right) b_1 \sin(\phi_0)
\]

\[
\frac{\partial a_5}{\partial \phi_m} = \left( -h + \frac{c}{4} \sin(\alpha_m) \right) \frac{\partial b_1}{\partial \phi_m} \cos(\phi_0) \quad \frac{\partial a_6}{\partial t} = \left( h - \frac{c}{4} \sin(\alpha_m) \right) \frac{\partial b_1}{\partial t} \sin(\phi_0)
\]

\[
\frac{\partial a_6}{\partial \phi_0} = \left( h - \frac{c}{4} \sin(\alpha_m) \right) b_1 \cos(\phi_0) \quad \frac{\partial a_6}{\partial \phi_m} = \left( h - \frac{c}{4} \sin(\alpha_m) \right) \frac{\partial b_1}{\partial \phi_m} \sin(\phi_0)
\]

It also involves taking partial derivatives of the Bessel and Struve function with respect to flapping amplitude. For brevity, following notations are used.

\[
S_{\text{Bessel}} = \frac{I_1(\phi_m) \Gamma(1.5)}{0.5 \phi_m \pi^2}
\]

\[
S_{\text{Struve}} = \frac{H_1(\phi_m) \Gamma(1.5)}{0.5 \phi_m \pi^2}
\]

\(S_{\text{Bessel}}\) and \(S_{\text{Struve}}\) can be written in terms of integrals using equation (3.28) to equation (3.30). Leibniz’s rule is applied to find the partial derivatives.

\[
\frac{\partial S_{\text{Bessel}}}{\partial \phi_m} = \frac{\partial}{\partial \phi_m} \frac{1}{\pi} \int_0^\pi \sin^2(\lambda) \cos(\phi_m \cos(\lambda)) d\lambda
\]

\[
= -\frac{1}{\pi} \int_0^\pi \sin^2(\lambda) \sin(\phi_m \cos(\lambda)) \cos(\lambda) d\lambda
\]

\[
\frac{\partial S_{\text{Struve}}}{\partial \phi_m} = \frac{\partial}{\partial \phi_m} \frac{2}{\pi} \int_0^\pi \sin^2(\lambda) \sin(\phi_m \cos(\lambda)) d\lambda
\]

55
\[= 2 \int_0^\pi \sin^2 (\lambda) \cos (\varphi_m \cos (\lambda)) \cos (\lambda) d\lambda\]

Since we are only interested in the hover condition, the above equations can be evaluated numerically using the equilibrium value of flapping frequency and amplitude.

The control derivatives of full cycle averaged aerodynamic forces and moment can be expressed as:

\[
\frac{\partial T F_B}{\partial f} = 0 \quad \frac{\partial T F_B}{\partial \phi_0} = 0 \quad \frac{\partial T F_B}{\partial \phi_m} = 0
\]

\[
\frac{\partial T F_B}{\partial f} = 0.5 \frac{\partial b_2}{\partial f} \quad \frac{\partial T F_B}{\partial \phi_0} = 0 \quad \frac{\partial T F_B}{\partial \phi_m} = 0.5 \frac{\partial b_2}{\partial \phi_m}
\]

\[
\frac{\partial T M_B}{\partial f} = \frac{\partial a_1}{\partial f} S_{Bessel} \quad \frac{\partial T M_B}{\partial \phi_0} = \frac{\partial a_1}{\partial \phi_0} S_{Bessel} \quad \frac{\partial T M_B}{\partial \phi_m} = \frac{\partial a_1}{\partial \phi_m} S_{Bessel} + \frac{a_1 S_{Bessel}}{\partial \phi_m}
\]

The control derivatives of quarter cycle averaged aerodynamic forces and moment can be expressed as:

\[
\frac{\partial T^{\text{qu}} F_B}{\partial f} = - \frac{\partial b_1}{\partial f} \cos (\phi_0) \text{sign}(\phi) S_{Bessel} + \frac{\partial b_1}{\partial f} \sin (\phi_0) \text{sign}(\phi) \text{sign}(F_2) S_{Struwe}
\]

\[
\frac{\partial T^{\text{qu}} F_B}{\partial \phi_0} = b_1 \sin (\phi_0) \text{sign}(\phi) S_{Bessel} + b_1 \cos (\phi_0) \text{sign}(\phi) \text{sign}(F_2) S_{Struwe}
\]

\[
\frac{\partial T^{\text{qu}} F_B}{\partial \phi_m} = - \frac{\partial b_1}{\partial \phi_m} \cos (\phi_0) \text{sign}(\phi) S_{Bessel} - b_1 \cos (\phi_0) \text{sign}(\phi) \frac{\partial S_{Bessel}}{\partial \phi_m}
\]

\[
+ \frac{\partial b_1}{\partial \phi_m} \sin (\phi_0) \text{sign}(\phi) \text{sign}(F_2) S_{Struwe} + b_1 \sin (\phi_0) \text{sign}(\phi) \text{sign}(F_2) \frac{\partial S_{Struwe}}{\partial \phi_m}
\]

\[
\frac{\partial T^{\text{qu}} F_B}{\partial f} = 0.5 \frac{\partial b_2}{\partial f} \quad \frac{\partial T^{\text{qu}} F_B}{\partial \phi_0} = 0 \quad \frac{\partial T^{\text{qu}} F_B}{\partial \phi_m} = 0.5 \frac{\partial b_2}{\partial \phi_m}
\]

\[
\frac{\partial T^{\text{qu}} M_B}{\partial f} = \left( \frac{\partial a_1}{\partial f} + \frac{\partial a_3}{\partial f} \text{sign}(\phi) + \frac{\partial a_5}{\partial f} \text{sign}(\phi) \right) S_{Bessel}
\]

\[
+ \left( \frac{\partial a_2}{\partial f} + \frac{\partial a_4}{\partial f} \text{sign}(\phi) + \frac{\partial a_6}{\partial f} \text{sign}(\phi) \right) \text{sign}(F_2) S_{Struwe}
\]

\[
\frac{\partial T^{\text{qu}} M_B}{\partial \phi_0} = \left( \frac{\partial a_1}{\partial \phi_0} + \frac{\partial a_3}{\partial \phi_0} \text{sign}(\phi) + \frac{\partial a_5}{\partial \phi_0} \text{sign}(\phi) \right) S_{Bessel}
\]

56
The control matrices for the full cycle averaged system and quarter cycle averaged system are defined as:

\[
T\mathbf{B} = \begin{bmatrix}
\frac{1}{m} \frac{\partial T_{FXB}}{\partial f} & \frac{1}{m} \frac{\partial T_{FXB}}{\partial \phi_0} & \frac{1}{m} \frac{\partial T_{FXB}}{\partial \phi_m} \\
\frac{1}{m} \frac{\partial T_{FYB}}{\partial f} & \frac{1}{m} \frac{\partial T_{FYB}}{\partial \phi_0} & \frac{1}{m} \frac{\partial T_{FYB}}{\partial \phi_m} \\
\frac{1}{m} \frac{\partial T_{M_YB}}{\partial f} & \frac{1}{m} \frac{\partial T_{M_YB}}{\partial \phi_0} & \frac{1}{m} \frac{\partial T_{M_YB}}{\partial \phi_m} \\
0 & 0 & 0
\end{bmatrix}
\]

(4.1)

\[
T/4\mathbf{B} = \begin{bmatrix}
\frac{1}{m} \frac{\partial T/4FXB}{\partial f} & \frac{1}{m} \frac{\partial T/4FXB}{\partial \phi_0} & \frac{1}{m} \frac{\partial T/4FXB}{\partial \phi_m} \\
\frac{1}{m} \frac{\partial T/4FYB}{\partial f} & \frac{1}{m} \frac{\partial T/4FYB}{\partial \phi_0} & \frac{1}{m} \frac{\partial T/4FYB}{\partial \phi_m} \\
\frac{1}{m} \frac{\partial T/4M_YB}{\partial f} & \frac{1}{m} \frac{\partial T/4M_YB}{\partial \phi_0} & \frac{1}{m} \frac{\partial T/4M_YB}{\partial \phi_m} \\
0 & 0 & 0
\end{bmatrix}
\]

(4.2)

4.2 Controller Design

In this thesis, the control design only covers the LTI system which is base on the full cycle averaging. The full cycle averaged closed-loop system under the influence of control inputs with full state feedback is represented as:

\[
\dot{x} = T\mathbf{A}x + T\mathbf{B}u
\]

(4.3)

\[
u = -Kx
\]

(4.4)

where \(x = [\Delta u \ \Delta w \ \Delta q \ \Delta \theta]^T\), \(u = [\Delta f \ \Delta \phi_0 \ \Delta \phi_m]^T\), and K is the feedback gain.
With the eigenstructure assignment technique, we’re able to assign the desired eigenvalues $\lambda_i$ and associated eigenvectors $v_i$ to the closed-loop system. By the definition of eigenvalue and eigenvector, we can write:

$$(\mathbf{T\bar{A}} - \mathbf{T\bar{B}K})v_i = \lambda_i v_i$$

$$[\lambda_i I - (\mathbf{T\bar{A}} - \mathbf{T\bar{B}K})]v_i = 0$$

$$(\lambda_i I - \mathbf{T\bar{A}})v_i + \mathbf{T\bar{B}Kv_i} = 0$$

$$[\lambda_i I - \mathbf{T\bar{A}} \mathbf{T\bar{B}}] [v_i] = 0$$

By solving the above equation with desired eigenvalues and eigenvectors, we are able to obtain the feedback gain $K$.

In general with this method, the desired eigenvalues are chosen to ensure specific transient response of the linear system. But in our design procedure, the linear controller is attached to the nonlinear system which is our interest. The Matlab command ‘place’ is used which allows us to only specify the desired eigenvalues, and the program picks the associated eigenvectors and calculates the gain according to the algorithm from [36] which minimizes the sensitivity of the closed-loop poles to perturbations.

### 4.3 Simulation Results

$\lambda = -20, -20, -5 \pm 15i$ are chosen as the desired eigenvalues. A perturbation of 1 m/s in $X_B$ and $Z_B$ directional velocities is assumed at the beginning of the simulation. Figure 4.1 (a) shows the simulation results of the close-loop linear system defined by equation (4.3) and (4.4), and Figure 4.1 (b) shows the control inputs required. Figure 4.2 (a) to (d) shows the simulation results of simplified nonlinear system defined by equation (3.13) to (3.16) with the linear controller attached. It is shown that the vehicle can be stabilized and reach the quasi steady state in less than 0.5s. Since the controller is achieved from a full cycle averaging which ignores the periodic change in aerodynamic forces and moment inside each cycle, therefore we have no control on the oscillatory motion of the vehicle inside each cycle. The vehicle will maintain about $\pm 0.1$ m/s and $\pm 0.03$ m/s oscillation in $X_B$ and $Z_B$ directional velocities, and about $\pm 5.7^\circ$ oscillation in pitch angle. The control inputs required to stabilize the vehicle are shown in Figure 4.2 (e) to (g). It can be seen that the major control inputs to balance the vehicle are the mean wing position and flapping amplitude which vary about $\pm 10^\circ$ and $\pm 5^\circ$ from the equilibrium value respectively. The change in flapping frequency is negligible.
Figure 4.1. (a) Simulation results of the close-loop linear system. (b) Control inputs required.
Figure 4.2. (a) to (d) Simulation results of the nonlinear model. (e) to (g) Control inputs required.
Figure 4.2. (continued)
Figure 4.2. (continued)
Figure 4.2. (continued)
CHAPTER 5

CONCLUSION

In this paper, a mathematical model of a flapping wing MAV is presented. Some key unsteady aerodynamic mechanisms were discussed. A nonlinear longitudinal-axis dynamic model of the vehicle is discussed. The model is simplified by ignoring the contribution from wing rotation and then linearized about a hover condition. By applying the averaging theory, a linear time invariant system is obtained and this is used for stability analysis and controller design. A linear controller with eigenstructure assignment technique is designed and attached to the nonlinear system. Simulations demonstrate that the controller can successfully stabilize the vehicle under perturbations about the hover condition.

Future study can be carried on in multiple directions involving more accurate flight dynamic model and more effective controller designs. Additional aerodynamic mechanisms need to be taken account to yield a more accurate model of the flight performance. From controller design aspect, current study can be carried on for different flight condition, such as forward/backward moving, turning and so on. A linear time varying jump-style system can be modeled with half or quarter cycle averaging, and this requires different controller design methods. Finally, nonlinear control design covering the entire flight envelope with robustness and optimization considerations would be the ultimate goal.
REFERENCES


[19] URL: [http://www.delfly.nl](http://www.delfly.nl) [cited May 2, 2016].


[33] “10 Bessel Functions, Bessel and Hankel Functions,” Digital Library of Mathematical Functions [online information], URL: [http://dlmf.nist.gov/10.9](http://dlmf.nist.gov/10.9) [cited April 14, 2016].


APPENDIXES
Calculate of the full cycle averaged stability derivatives involves evaluating of $\frac{1}{T}\int_0^T F_i(t)$, where $F_i(t)$ where $i = 3, 4, \ldots, 8$, as defined in equation (3.71) to (3.76). The plots of $F_3(t)$ to $F_8(t)$ over a full flapping cycle are shown in Figure A1.

Figure A1. Plots of $F_3$, $F_4$, $F_5$, $F_6$, $F_7$, $F_8$
By the geometric interpretation of integral, we can conclude that

\[ \int_{0}^{T} F_3(t) \, dt = \int_{T/4}^{T} F_3(t) \, dt = -\int_{T/2}^{T} F_3(t) \, dt = -\int_{3T/4}^{T} F_3(t) \, dt \]

\[ \int_{0}^{T} F_4(t) \, dt = -\int_{T/4}^{T} F_4(t) \, dt = \int_{T/2}^{T} F_4(t) \, dt = -\int_{3T/4}^{T} F_4(t) \, dt \]

\[ \int_{0}^{T} F_5(t) \, dt = \int_{T/4}^{T} F_5(t) \, dt = -\int_{T/2}^{T} F_5(t) \, dt = -\int_{3T/4}^{T} F_5(t) \, dt \]

\[ \int_{0}^{T} F_6(t) \, dt = -\int_{T/4}^{T} F_6(t) \, dt = \int_{T/2}^{T} F_6(t) \, dt = -\int_{3T/4}^{T} F_6(t) \, dt \]

\[ \int_{0}^{T} F_7(t) \, dt = \int_{T/4}^{T} F_7(t) \, dt = -\int_{T/2}^{T} F_7(t) \, dt = -\int_{3T/4}^{T} F_7(t) \, dt \]

\[ \int_{0}^{T} F_8(t) \, dt = \int_{T/4}^{T} F_8(t) \, dt = -\int_{T/2}^{T} F_8(t) \, dt = -\int_{3T/4}^{T} F_8(t) \, dt \]

By dividing the integrals over different intervals and evaluating the sign(\(\dot{\phi}\)) term, the full cycle averaged stability derivatives can be calculated as

Figure A1. (continued)
$$\hat{T}X_\gamma(t) = \frac{1}{T} \int_0^T X_\gamma(t) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} X_\gamma(\lambda_e) d\lambda_e$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[ a_7 \text{sign}(\phi_e) F_7(\lambda_e) - a_8 \text{sign}(\phi_e) F_6(\lambda_e) + a_9 \text{sign}(\phi_e) F_8(\lambda_e) \right] d\lambda_e$$

$$= \frac{1}{2\pi} \left[ \int_0^\pi a_7 \text{sign}(\phi_e) F_7(\lambda_e) d\lambda_e + \int_0^{2\pi} a_7 \text{sign}(\phi_e) F_7(\lambda_e) d\lambda_e \right]$$

$$- \frac{1}{2\pi} \left[ \int_0^\pi a_8 \text{sign}(\phi_e) F_6(\lambda_e) d\lambda_e + \int_0^{2\pi} a_8 \text{sign}(\phi_e) F_6(\lambda_e) d\lambda_e \right]$$

$$+ \int_0^{3\pi} a_8 \text{sign}(\phi_e) F_6(\lambda_e) d\lambda_e + \int_0^{2\pi} a_8 \text{sign}(\phi_e) F_6(\lambda_e) d\lambda_e$$

$$+ \frac{1}{2\pi} \left[ \int_0^\pi a_9 \text{sign}(\phi_e) F_8(t) d\lambda_e + \int_0^{2\pi} a_9 \text{sign}(\phi_e) F_8(t) d\lambda_e \right]$$

$$+ \int_0^{3\pi} a_9 \text{sign}(\phi_e) F_8(t) d\lambda_e + \int_0^{2\pi} a_9 \text{sign}(\phi_e) F_8(t) d\lambda_e$$

$$= \frac{1}{2\pi} \left[ - \int_0^\pi a_7 F_7(\lambda_e) d\lambda_e + \int_0^{2\pi} a_7 F_7(\lambda_e) d\lambda_e \right]$$

$$- \frac{1}{2\pi} \left[ \int_0^\pi a_8 F_6(\lambda_e) d\lambda_e + \int_0^{2\pi} a_8 F_6(\lambda_e) d\lambda_e \right]$$

$$+ \int_0^{3\pi} a_8 F_6(\lambda_e) d\lambda_e + \int_0^{2\pi} a_8 F_6(\lambda_e) d\lambda_e$$

$$+ \frac{1}{2\pi} \left[ \int_0^\pi a_9 F_8(t) d\lambda_e + \int_0^{2\pi} a_9 F_8(t) d\lambda_e + \int_0^{3\pi} a_9 F_8(t) d\lambda_e + \int_0^{2\pi} a_9 F_8(t) d\lambda_e \right]$$

$$= \frac{1}{2\pi} \left[ - \int_0^\pi a_7 F_7(\lambda_e) d\lambda_e - \int_0^{2\pi} a_7 F_7(\lambda_e) d\lambda_e \right]$$

$$- \frac{1}{2\pi} \left[ \int_0^\pi a_8 F_6(\lambda_e) d\lambda_e + \int_0^{2\pi} a_8 F_6(\lambda_e) d\lambda_e \right]$$

$$+ \int_0^{3\pi} a_8 F_6(\lambda_e) d\lambda_e + \int_0^{2\pi} a_8 F_6(\lambda_e) d\lambda_e$$

$$+ \frac{1}{2\pi} \left[ \int_0^\pi a_9 F_8(t) d\lambda_e + \int_0^{2\pi} a_9 F_8(t) d\lambda_e + \int_0^{3\pi} a_9 F_8(t) d\lambda_e + \int_0^{2\pi} a_9 F_8(t) d\lambda_e \right]$$

$$= \frac{1}{2\pi} \int_0^\pi a_7 F_7(\lambda_e) d\lambda_e - \frac{2}{\pi} \int_0^\pi a_9 F_8(t) d\lambda_e$$
\[ X_q(t) = \frac{1}{T} \int_0^T X_q(t) \, dt \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} X_q(\lambda_e) \, d\lambda_e \]

\[ = \frac{1}{2\pi} \left( h - \frac{c}{4} \sin(\alpha_m) \right) \left[ \int_0^{2\pi} a_7 \text{sign}(\phi_e) F_7(\lambda_e) \, d\lambda_e \right. \]

\[ - \int_0^{2\pi} a_8 \text{sign}(\phi_e) F_6(\lambda_e) \, d\lambda_e \]

\[ + \int_0^{2\pi} a_9 \text{sign}(\phi_e) F_8(\lambda_e) \, d\lambda_e \]

\[ = \frac{1}{2\pi} \left( h - \frac{c}{4} \sin(\alpha_m) \right) \left[ \int_0^{2\pi} a_7 \text{sign}(\phi_e) F_7(\lambda_e) \, d\lambda_e \right. \]

\[ + \int_{\pi}^{\pi} a_7 \text{sign}(\phi_e) F_7(\lambda_e) \, d\lambda_e \]

\[ - \int_{\pi}^{\pi} a_8 \text{sign}(\phi_e) F_6(\lambda_e) \, d\lambda_e \]

\[ + \int_{3\pi}^{3\pi} a_8 \text{sign}(\phi_e) F_8(\lambda_e) \, d\lambda_e \]

\[ + \int_{3\pi}^{3\pi} a_9 \text{sign}(\phi_e) F_9(\lambda_e) \, d\lambda_e \]

\[ + \int_{3\pi}^{3\pi} a_9 \text{sign}(\phi_e) F_9(\lambda_e) \, d\lambda_e \]

\[ = \frac{1}{2\pi} \left( h - \frac{c}{4} \sin(\alpha_m) \right) \left[ - \int_0^{\pi} a_7 F_7(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} a_7 F_7(\lambda_e) \, d\lambda_e \right] \]

\[ - \frac{1}{2\pi} \left( h - \frac{c}{4} \sin(\alpha_m) \right) \left[ - \int_0^{\pi} a_8 F_6(\lambda_e) \, d\lambda_e - \int_{\pi}^{2\pi} a_8 F_6(\lambda_e) \, d\lambda_e \right] \]

\[ + \int_0^{\pi} a_8 F_6(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} a_8 F_6(\lambda_e) \, d\lambda_e \]

\[ + \frac{1}{2\pi} \left( h - \frac{c}{4} \sin(\alpha_m) \right) \left[ - \int_0^{\pi} a_9 F_8(\lambda_e) \, d\lambda_e - \int_{\pi}^{2\pi} a_9 F_8(\lambda_e) \, d\lambda_e \right] \]

\[ + \int_0^{\pi} a_9 F_8(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} a_9 F_8(\lambda_e) \, d\lambda_e \]
\[ \begin{align*}
&= \frac{1}{2\pi} (h - \frac{c}{4} \sin(\alpha_m)) \left[ -\int_0^\pi a_7 F_7(\lambda_e) \, d\lambda_e - \int_0^\pi a_7 F_7(\lambda_e) \, d\lambda_e \right] \\
&\quad - \frac{1}{2\pi} (h - \frac{c}{4} \sin(\alpha_m)) \left[ -\int_0^\pi a_8 F_8(\lambda_e) \, d\lambda_e + \int_0^\pi a_8 F_8(\lambda_e) \, d\lambda_e \\
&\qquad + \int_0^\pi a_9 F_9(\lambda_e) \, d\lambda_e - \int_0^\pi a_9 F_9(\lambda_e) \, d\lambda_e \right] \\
&\quad + \frac{1}{2\pi} (h - \frac{c}{4} \sin(\alpha_m)) \left[ -\int_0^\pi a_9 F_9(\lambda_e) \, d\lambda_e - \int_0^\pi a_9 F_9(\lambda_e) \, d\lambda_e \\
&\qquad - \int_0^\pi a_9 F_9(\lambda_e) \, d\lambda_e - \int_0^\pi a_9 F_9(\lambda_e) \, d\lambda_e \right] \\
&\quad = -\frac{1}{\pi} (h - \frac{c}{4} \sin(\alpha_m)) \int_0^\pi a_7 F_7(\lambda_e) \, d\lambda_e - \frac{2}{\pi} (h - \frac{c}{4} \sin(\alpha_m)) \int_0^\pi a_9 F_9(\lambda_e) \, d\lambda_e
\end{align*} \]

\[ T \tilde{X}_w (t) = \frac{1}{\pi} \int_0^T X_w (t) \, dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} X_w^\alpha (\lambda_e) \, d\lambda_e \]
\[ = \frac{1}{2\pi} \left[ \int_0^{2\pi} a_{26} F_3(\lambda_e) \, d\lambda_e - \int_0^{2\pi} a_{27} F_4(\lambda_e) \, d\lambda_e \right] \\
\[ = \frac{1}{2\pi} \left[ \int_0^{\pi} a_{26} F_3(\lambda_e) \, d\lambda_e + \int_0^{\pi} a_{26} F_3(\lambda_e) \, d\lambda_e \\
&\quad - \frac{1}{2\pi} \left[ \int_0^{\pi} a_{27} F_4(t) \, d\lambda_e + \int_0^{\pi} a_{27} F_4(t) \, d\lambda_e \\
&\quad + \int_0^{\pi} a_{27} F_4(t) \, d\lambda_e + \int_0^{\pi} a_{27} F_4(t) \, d\lambda_e \right] \\
&\quad = \frac{1}{2\pi} \left[ \int_0^{\pi} a_{26} F_3(\lambda_e) \, d\lambda_e - \int_0^{\pi} a_{26} F_3(\lambda_e) \, d\lambda_e \\
&\quad - \int_0^{\pi} a_{27} F_4(t) \, d\lambda_e - \int_0^{\pi} a_{27} F_4(t) \, d\lambda_e \\
&\quad + \int_0^{\pi} a_{27} F_4(t) \, d\lambda_e - \int_0^{\pi} a_{27} F_4(t) \, d\lambda_e \right] \\
&\quad = 0 \]

\[ T \tilde{X}_q (t) = \frac{1}{\pi} \int_0^T X_q^\alpha (t) \, dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} X_q^\alpha (\lambda_e) \, d\lambda_e \]
\[ = \frac{1}{2\pi} \left[ \int_0^{2\pi} a_{28} F_5(\lambda_e) \, d\lambda_e + \int_0^{2\pi} a_{29} F_6(\lambda_e) \, d\lambda_e \right] \]
\[
\begin{align*}
&= \frac{1}{2\pi} \left[ \int_{0}^{\pi} a_{28} F_{5}(\lambda_{e}) \, d\lambda_{e} + \int_{\pi}^{2\pi} a_{28} F_{5}(\lambda_{e}) \, d\lambda_{e} \right] \\
&\quad+ \frac{1}{2\pi} \left[ \int_{0}^{\pi} a_{29} F_{6}(t) \, d\lambda_{e} + \int_{\pi}^{\pi} a_{29} F_{6}(t) \, d\lambda_{e} \right] \\
&\quad+ \int_{\frac{3\pi}{2}}^{2\pi} a_{29} F_{6}(t) \, d\lambda_{e} + \int_{\frac{3\pi}{2}}^{\pi} a_{29} F_{6}(t) \, d\lambda_{e} \\
&= \frac{1}{2\pi} \left[ \int_{0}^{\pi} a_{28} F_{5}(\lambda_{e}) \, d\lambda_{e} - \int_{0}^{\pi} a_{28} F_{5}(\lambda_{e}) \, d\lambda_{e} \right] \\
&\quad+ \frac{1}{2\pi} \left[ \int_{0}^{\pi} a_{29} F_{6}(t) \, d\lambda_{e} - \int_{0}^{\pi} a_{29} F_{6}(t) \, d\lambda_{e} \right] \\
&\quad+ \int_{\frac{3\pi}{2}}^{\pi} a_{29} F_{6}(t) \, d\lambda_{e} - \int_{\frac{3\pi}{2}}^{\pi} a_{29} F_{6}(t) \, d\lambda_{e} \\
&= 0
\end{align*}
\]

\[
\mathcal{T} Z^{\nu}_{T}(t) = \frac{1}{T} \int_{0}^{T} Z^{\nu}_{U}(t) \, dt
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} Z^{\nu}_{\lambda_{e}}(\lambda_{e}) \, d\lambda_{e}
\]

\[
= \frac{1}{2\pi} \left[ - \int_{0}^{2\pi} a_{10} F_{3}(\lambda_{e}) \, d\lambda_{e} + \int_{0}^{2\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e} \right]
\]

\[
= \frac{1}{2\pi} \left[ - \int_{0}^{\pi} a_{10} F_{3}(\lambda_{e}) \, d\lambda_{e} - \int_{\pi}^{2\pi} a_{10} F_{3}(\lambda_{e}) \, d\lambda_{e} \right]
\]

\[
+ \frac{1}{2\pi} \left[ \int_{0}^{\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e} + \int_{\pi}^{\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e} \right]
\]

\[
+ \int_{\frac{3\pi}{2}}^{2\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e} + \int_{\frac{3\pi}{2}}^{\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e}
\]

\[
= \frac{1}{2\pi} \left[ - \int_{0}^{\pi} a_{10} F_{3}(\lambda_{e}) \, d\lambda_{e} + \int_{0}^{\pi} a_{10} F_{3}(\lambda_{e}) \, d\lambda_{e} \right]
\]

\[
+ \frac{1}{2\pi} \left[ \int_{0}^{\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e} - \int_{\pi}^{2\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e} \right]
\]

\[
+ \int_{\frac{3\pi}{2}}^{\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e} - \int_{\frac{3\pi}{2}}^{\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e}
\]

\[
= 0
\]

\[
\mathcal{T} Z^{\nu}_{Q}(t) = \frac{1}{T} \int_{0}^{T} Z^{\nu}_{Q}(t) \, dt
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} Z^{\nu}_{Q}(\lambda_{e}) \, d\lambda_{e}
\]

\[
= \frac{1}{2\pi} \left[ (-h + \frac{5}{4} \sin (\alpha_{m})) \left[ \int_{0}^{2\pi} a_{10} F_{3}(\lambda_{e}) \, d\lambda_{e} - \int_{0}^{2\pi} a_{11} F_{4}(\lambda_{e}) \, d\lambda_{e} \right] \right]
\]

75
\[
\mathbf{Z}_q^a(t) = \frac{1}{T} \int_0^T \mathbf{Z}_q^a(t) \, dt
\]

\[
\mathbf{Z}_q^a(t) = \frac{1}{2\pi} \left[ \int_0^{2\pi} a_{30} \text{sign}(\varphi_e) F_3(\lambda_e) \, d\lambda_e + \int_0^{2\pi} a_{31} \text{sign}(\varphi_e) F_4(\lambda_e) \, d\lambda_e \right]
\]

\[
\mathbf{M}_u^\gamma(t) = \frac{1}{T} \int_0^T \mathbf{M}_u^\gamma(t) \, dt
\]

\[
\mathbf{M}_u^\gamma(t) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{M}_u^\gamma(\lambda_e) \, d\lambda_e
\]
\[
\frac{1}{2\pi}[\int_0^{2\pi} a_{15} F_5(\lambda_e) \, d\lambda_e + \int_0^{2\pi} (a_{16} + a_{17} \text{sign}(\phi_e)) F_6(\lambda_e) \, d\lambda_e \\
+ \int_0^{2\pi} a_{18} \text{sign}(\phi_e) F_7(\lambda_e) \, d\lambda_e + \int_0^{2\pi} a_{19} \text{sign}(\phi_e) F_8(\lambda_e) \, d\lambda_e]
\]

\[
= \frac{1}{2\pi}[\int_0^{2\pi} a_{15} F_5(\lambda_e) \, d\lambda_e + \int_0^{2\pi} a_{15} F_5(\lambda_e) \, d\lambda_e]
\]

\[
+ \frac{1}{2\pi}[\int_0^{2\pi} (a_{16} + a_{17} \text{sign}(\phi_e)) F_6(\lambda_e) \, d\lambda_e \\
+ \int_0^{2\pi} (a_{16} + a_{17} \text{sign}(\phi_e)) F_6(\lambda_e) \, d\lambda_e \\
+ \int_0^{2\pi} (a_{16} + a_{17} \text{sign}(\phi_e)) F_6(\lambda_e) \, d\lambda_e \\
+ \int_0^{2\pi} (a_{16} + a_{17} \text{sign}(\phi_e)) F_6(\lambda_e) \, d\lambda_e]
\]

\[
+ \frac{1}{2\pi}[\int_0^{\pi} a_{18} \text{sign}(\phi_e) F_7(\lambda_e) \, d\lambda_e + \int_0^{2\pi} a_{18} \text{sign}(\phi_e) F_7(\lambda_e) \, d\lambda_e] \\
+ \frac{1}{2\pi}[\int_0^{\pi} a_{19} \text{sign}(\phi_e) F_8(\lambda_e) \, d\lambda_e + \int_0^{2\pi} a_{19} \text{sign}(\phi_e) F_8(\lambda_e) \, d\lambda_e]
\]

\[
= \frac{1}{2\pi}[\int_0^{\pi} a_{15} F_5(\lambda_e) \, d\lambda_e + \int_0^{\pi} a_{15} F_5(\lambda_e) \, d\lambda_e] \\
+ \frac{1}{2\pi}[\int_0^{\pi} (a_{16} - a_{17}) F_6(\lambda_e) \, d\lambda_e + \int_0^{\pi} (a_{16} - a_{17}) F_6(\lambda_e) \, d\lambda_e \\
+ \int_0^{\pi} (a_{16} + a_{17}) F_6(\lambda_e) \, d\lambda_e + \int_0^{\pi} (a_{16} + a_{17}) F_6(\lambda_e) \, d\lambda_e] \\
+ \frac{1}{2\pi}[-\int_0^{\pi} a_{18} F_7(\lambda_e) \, d\lambda_e + \int_0^{\pi} a_{18} F_7(\lambda_e) \, d\lambda_e] \\
+ \frac{1}{2\pi}[-\int_0^{\pi} a_{19} F_8(\lambda_e) \, d\lambda_e - \int_0^{\pi} a_{19} F_8(\lambda_e) \, d\lambda_e] \\
+ \int_0^{\pi} a_{19} F_8(\lambda_e) \, d\lambda_e + \int_0^{\pi} a_{19} F_8(\lambda_e) \, d\lambda_e]
\]

\[
= \frac{1}{2\pi}[\int_0^{\pi} a_{15} F_5(\lambda_e) \, d\lambda_e - \int_0^{\pi} a_{15} F_5(\lambda_e) \, d\lambda_e] \\
+ \frac{1}{2\pi}[\int_0^{\pi} (a_{16} - a_{17}) F_6(\lambda_e) \, d\lambda_e - \int_0^{\pi} (a_{16} - a_{17}) F_6(\lambda_e) \, d\lambda_e \\
+ \int_0^{\pi} (a_{16} + a_{17}) F_6(\lambda_e) \, d\lambda_e - \int_0^{\pi} (a_{16} + a_{17}) F_6(\lambda_e) \, d\lambda_e] \\
+ \frac{1}{2\pi}[-\int_0^{\pi} a_{18} F_7(\lambda_e) \, d\lambda_e - \int_0^{\pi} a_{18} F_7(\lambda_e) \, d\lambda_e] \\
+ \frac{1}{2\pi}[-\int_0^{\pi} a_{19} F_8(\lambda_e) \, d\lambda_e - \int_0^{\pi} a_{19} F_8(\lambda_e) \, d\lambda_e \\
- \int_0^{\pi} a_{19} F_8(\lambda_e) \, d\lambda_e - \int_0^{\pi} a_{19} F_8(\lambda_e) \, d\lambda_e] \\
77
\[
\begin{align*}
&= -\frac{1}{\pi} \int_0^\pi a_{18} F_7(\lambda_e) d\lambda_e - \frac{2}{\pi} \int_0^{\frac{3\pi}{2}} a_{19} F_8(\lambda_e) d\lambda_e \\
T \mathbf{M}_q^\gamma(t) &= \frac{1}{T} \int_0^T M_q^\gamma(t) dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} M_q^\gamma(\lambda_e) d\lambda_e \\
&= \frac{1}{2\pi} (-h + \frac{\bar{c}}{4} \sin (\alpha_m)) \left[ \int_0^{2\pi} (a_{20} + a_{21} \text{sign}(\phi_e)) F_7(\lambda_e) d\lambda_e \\
&\quad + \int_0^{2\pi} (a_{22} + a_{23} \text{sign}(\phi_e)) F_6(\lambda_e) d\lambda_e \\
&\quad + \int_0^{2\pi} (a_{24} + a_{25} \text{sign}(\phi_e)) F_8(\lambda_e) d\lambda_e \right] \\
&+ \frac{1}{2\pi} (-h + \frac{\bar{c}}{4} \sin (\alpha_m)) \left[ \int_0^{\pi} (a_{20} + a_{21} \text{sign}(\phi_e)) \frac{\pi}{2} F_7(\lambda_e) d\lambda_e \\
&\quad + \int_0^{\pi} (a_{22} + a_{23} \text{sign}(\phi_e)) \frac{\pi}{2} F_6(\lambda_e) d\lambda_e \\
&\quad + \int_0^{\pi} (a_{24} + a_{25} \text{sign}(\phi_e)) \frac{\pi}{2} F_8(\lambda_e) d\lambda_e \right] \\
&+ \frac{1}{2\pi} (-h + \frac{\bar{c}}{4} \sin (\alpha_m)) \left[ \int_0^{\frac{3\pi}{2}} (a_{20} + a_{21} \text{sign}(\phi_e)) \frac{3\pi}{2} F_7(\lambda_e) d\lambda_e \\
&\quad + \int_0^{\frac{3\pi}{2}} (a_{22} + a_{23} \text{sign}(\phi_e)) \frac{3\pi}{2} F_6(\lambda_e) d\lambda_e \\
&\quad + \int_0^{\frac{3\pi}{2}} (a_{24} + a_{25} \text{sign}(\phi_e)) \frac{3\pi}{2} F_8(\lambda_e) d\lambda_e \right] \\
&+ \frac{1}{2\pi} (-h + \frac{\bar{c}}{4} \sin (\alpha_m)) \left[ \int_0^{\frac{5\pi}{2}} (a_{20} + a_{21} \text{sign}(\phi_e)) \frac{5\pi}{2} F_7(\lambda_e) d\lambda_e \\
&\quad + \int_0^{\frac{5\pi}{2}} (a_{22} + a_{23} \text{sign}(\phi_e)) \frac{5\pi}{2} F_6(\lambda_e) d\lambda_e \\
&\quad + \int_0^{\frac{5\pi}{2}} (a_{24} + a_{25} \text{sign}(\phi_e)) \frac{5\pi}{2} F_8(\lambda_e) d\lambda_e \right]
\end{align*}
\]
\[
\frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^\pi (a_{20} - a_{21}) F_7(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^{\pi/2} (a_{22} - a_{23}) F_6(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_{\pi/2}^{\pi} (a_{22} + a_{23}) F_6(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_{\pi/2}^\pi (a_{22} + a_{23}) F_6(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^{\pi/2} (a_{24} - a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_{\pi/2}^{\pi} (a_{24} - a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_{\pi/2}^\pi (a_{24} + a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^{\pi/2} (a_{24} + a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_{\pi/2}^\pi (a_{24} + a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
= \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^\pi (a_{20} - a_{21}) F_7(\lambda_e) \, d\lambda_e \right] \\
- \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^\pi (a_{22} + a_{23}) F_6(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^\pi (a_{22} - a_{23}) F_6(\lambda_e) \, d\lambda_e \right] \\
- \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^\pi (a_{23} + a_{22}) F_6(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^{\pi/2} (a_{24} - a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_{\pi/2}^{\pi} (a_{24} + a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_{\pi/2}^\pi (a_{24} - a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
+ \frac{1}{2\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_{\pi/2}^\pi (a_{24} + a_{25}) F_8(\lambda_e) \, d\lambda_e \right] \\
= -\frac{1}{\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^\pi a_{21} F_7(\lambda_e) \, d\lambda_e \right] \\
- \frac{2}{\pi} (-h + \frac{c}{4} \sin(\alpha_m)) \left[ f_0^\pi a_{25} F_8(\lambda_e) \, d\lambda_e \right]
\]
\[ T\mathcal{M}_w(t) = \frac{1}{T} \int_0^T M_w^\alpha(t) \, dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} M_w^\alpha(\lambda_e) \, d\lambda_e \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} (a_{30} \text{sign}(\phi_e) + a_{32} + a_{34}) F_3(\lambda_e) \, d\lambda_e \]
\[ + \int_0^{2\pi} (a_{31} \text{sign}(\phi_e) + a_{33} + a_{35}) F_4(\lambda_e) \, d\lambda_e \]
\[ = \frac{1}{2\pi} \left[ \int_0^{\pi} (a_{30} \text{sign}(\phi_e) + a_{32} + a_{34}) F_3(\lambda_e) \, d\lambda_e \right. \]
\[ + \left. \int_{\pi}^{2\pi} (a_{31} \text{sign}(\phi_e) + a_{33} + a_{35}) F_4(\lambda_e) \, d\lambda_e \right] \]
\[ = \frac{1}{2\pi} \left[ \int_0^{\pi} (-a_{30} + a_{32} + a_{34}) F_3(\lambda_e) \, d\lambda_e + \int_0^{2\pi} (a_{30} + a_{32} + a_{34}) F_3(\lambda_e) \, d\lambda_e \right] \]
\[ + \frac{1}{2\pi} \left[ \int_0^{\pi} (-a_{31} + a_{33} + a_{35}) F_4(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} (-a_{31} + a_{33} + a_{35}) F_4(\lambda_e) \, d\lambda_e \right] \]
\[ = \frac{1}{2\pi} \left[ \int_0^{\pi} (-a_{30} + a_{32} + a_{34}) F_3(\lambda_e) \, d\lambda_e - \int_0^{\pi} (a_{30} + a_{32} + a_{34}) F_3(\lambda_e) \, d\lambda_e \right] \]
\[ + \frac{1}{2\pi} \left[ \int_0^{\pi} (-a_{31} + a_{33} + a_{35}) F_4(\lambda_e) \, d\lambda_e - \int_{\pi}^{2\pi} (-a_{31} + a_{33} + a_{35}) F_4(\lambda_e) \, d\lambda_e \right] \]
\[ = -\frac{1}{\pi} \int_0^{\pi} a_{30} F_3(\lambda_e) \, d\lambda_e \]

\[ T\mathcal{M}_q(t) = \frac{1}{T} \int_0^T M_q^\alpha(t) \, dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} M_q^\alpha(\lambda_e) \, d\lambda_e \]
\[
\begin{align*}
&= \frac{1}{2\pi} \left[ \int_{0}^{2\pi} a_{36}\text{sign}(\phi_e) F_{7}(\lambda_e) \, d\lambda_e \\
&\quad + \int_{0}^{2\pi} (a_{37}\text{sign}(\phi_e) + a_{38} + a_{42}) F_{6}(\lambda_e) \, d\lambda_e \\
&\quad + \int_{0}^{2\pi} a_{39}\text{sign}(\phi_e) F_{8}(\lambda_e) \, d\lambda_e + \int_{0}^{2\pi} (a_{40} + a_{41}) F_{5}(\lambda_e) \, d\lambda_e \right] \\
&= \frac{1}{2\pi} \left[ \int_{0}^{\pi} a_{36}\text{sign}(\phi_e) F_{7}(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} a_{36}\text{sign}(\phi_e) F_{7}(\lambda_e) \, d\lambda_e \right] \\
&\quad + \frac{1}{2\pi} \left[ \int_{0}^{\pi} (a_{37}\text{sign}(\phi_e) + a_{38} + a_{42}) F_{6}(\lambda_e) \, d\lambda_e \\
&\quad + \int_{\pi}^{2\pi} (a_{37}\text{sign}(\phi_e) + a_{38} + a_{42}) F_{6}(\lambda_e) \, d\lambda_e \\
&\quad + \int_{\pi}^{2\pi} (a_{37}\text{sign}(\phi_e) + a_{38} + a_{42}) F_{6}(\lambda_e) \, d\lambda_e \right] \\
&\quad + \frac{1}{2\pi} \left[ \int_{0}^{\pi} a_{39}\text{sign}(\phi_e) F_{8}(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} a_{39}\text{sign}(\phi_e) F_{8}(\lambda_e) \, d\lambda_e \right] \\
&\quad + \frac{1}{2\pi} \left[ \int_{0}^{\pi} (a_{40} + a_{41}) F_{5}(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} (a_{40} + a_{41}) F_{5}(\lambda_e) \, d\lambda_e \right] \\
&= \frac{1}{2\pi} \left[ - \int_{0}^{\pi} a_{36} F_{7}(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} a_{36} F_{7}(\lambda_e) \, d\lambda_e \right] \\
&\quad + \frac{1}{2\pi} \left[ \int_{0}^{\pi} (-a_{37} + a_{38} + a_{42}) F_{6}(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} (-a_{37} + a_{38} + a_{42}) F_{6}(\lambda_e) \, d\lambda_e \\
&\quad + \int_{\pi}^{2\pi} (a_{37} + a_{38} + a_{42}) F_{6}(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} (a_{37} + a_{38} + a_{42}) F_{6}(\lambda_e) \, d\lambda_e \right] \\
&\quad + \frac{1}{2\pi} \left[ - \int_{0}^{\pi} a_{39} F_{8}(\lambda_e) \, d\lambda_e - \int_{\pi}^{\pi} a_{39} F_{8}(\lambda_e) \, d\lambda_e \\
&\quad + \int_{\pi}^{2\pi} a_{39} F_{8}(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} a_{39} F_{8}(\lambda_e) \, d\lambda_e \right] \\
&\quad + \frac{1}{2\pi} \left[ \int_{0}^{\pi} (a_{40} + a_{41}) F_{5}(\lambda_e) \, d\lambda_e + \int_{\pi}^{2\pi} (a_{40} + a_{41}) F_{5}(\lambda_e) \, d\lambda_e \right]
\end{align*}
\]
\[
\frac{1}{2\pi} \left[ -\int_0^\pi a_36 F_7(\lambda_e) d\lambda_e - \int_0^\pi a_36 F_7(\lambda_e) d\lambda_e \right] \\
+ \frac{1}{2\pi} \left[ \int_0^\pi (-a_{37} + a_{38} + a_{42}) F_6(\lambda_e) d\lambda_e - \int_0^\pi (-a_{37} + a_{38} + a_{42}) F_6(\lambda_e) d\lambda_e + \int_0^\pi (-a_{37} + a_{38} + a_{42}) F_6(\lambda_e) d\lambda_e \right] \\
+ \frac{1}{2\pi} \left[ \int_0^\pi a_{39} F_8(\lambda_e) d\lambda_e - \int_0^\pi a_{39} F_8(\lambda_e) d\lambda_e - \int_0^\pi a_{39} F_8(\lambda_e) d\lambda_e - \int_0^\pi a_{39} F_8(\lambda_e) d\lambda_e \right] \\
+ \frac{1}{2\pi} \left[ \int_0^\pi (a_{40} + a_{41}) F_5(\lambda_e) d\lambda_e - \int_0^\pi (a_{40} + a_{41}) F_5(\lambda_e) d\lambda_e \right] \\
= \frac{-1}{\pi} \int_0^\pi a_36 F_7(\lambda_e) d\lambda_e - \frac{2}{\pi} \int_0^\pi a_{39} F_8(\lambda_e) d\lambda_e
\]

\[T_{2\pi}^{\alpha}(t) = \frac{1}{T} \int_0^T T_{2\pi}^{\alpha}(t) dt\]

\[= \frac{1}{2\pi} \int_0^{2\pi} Z_{2\pi}^{\alpha}(\lambda_e) d\lambda_e\]

\[= \frac{1}{2\pi} \int_0^{2\pi} b_6 \text{sign}(\phi_e) \sin(\lambda_e) d\lambda_e\]

\[= \frac{1}{2\pi} \int_0^{\pi} b_6 \text{sign}(\phi_e) \sin(\lambda_e) d\lambda_e + \frac{1}{2\pi} \int_0^{2\pi} b_6 \text{sign}(\phi_e) \sin(\lambda_e) d\lambda_e\]

\[= -\frac{1}{2\pi} \int_0^{\pi} b_6 \sin(\lambda_e) d\lambda_e + \frac{1}{2\pi} \int_0^{2\pi} b_6 \sin(\lambda_e) d\lambda_e\]

\[= -\frac{1}{2\pi} \int_0^{\pi} b_6 \sin(\lambda_e) d\lambda_e - \frac{1}{2\pi} \int_0^{\pi} b_6 \sin(\lambda_e) d\lambda_e\]

\[= -\frac{1}{\pi} \int_0^{\pi} b_6 \sin(\lambda_e) d\lambda_e\]

\[= \frac{b_6}{\pi} \left[ \cos(\lambda_e) \right]_0^\pi\]

\[= -\frac{2b_6}{\pi}\]
The solutions to integrals involved in the above equations can be calculated as follow

\[ \int_0^\pi F_3(\lambda_e) \, d\lambda_e = \int_0^\pi \sin \lambda_e \cos(\varphi_{me} \cos \lambda_e) \, d\lambda_e \]

let \( \cos \lambda_e = \tau_e \), as \( \lambda_e \to 0, \tau_e \to 1 \); as \( \lambda_e \to \pi, \tau_e \to -1 \). \( d\lambda_e = -\frac{1}{\sin \lambda_e} \, d\tau_e \)

\[ = -\int_1^{-1} \sin \lambda_e \cos(\varphi_{me} \tau_e) \frac{1}{\sin \lambda_e} \, d\tau_e \]
\[ = \int_{-1}^1 \cos(\varphi_{me} \tau_e) \, d\tau_e \]
\[ = \left[ \frac{\sin(\varphi_{me} \tau_e)}{\varphi_{me}} \right]_1^{-1} \]
\[ = \frac{2\sin(\varphi_{me})}{\varphi_{me}} \]

\[ \int_0^\pi F_4(\lambda_e) \, d\lambda_e = \int_0^\pi \sin \lambda_e \sin(\varphi_{me} \cos \lambda_e) \, d\lambda_e \]

let \( \cos \lambda_e = \tau_e \), as \( \lambda_e \to 0, \tau_e \to 1 \); as \( \lambda_e \to \pi, \tau_e \to 0 \). \( d\lambda_e = -\frac{1}{\sin \lambda_e} \, d\tau_e \)

\[ = -\int_1^{0} \sin \lambda_e \sin(\varphi_{me} \tau_e) \frac{1}{\sin \lambda_e} \, d\tau_e \]
\[ = \int_0^1 \sin(\varphi_{me} \tau_e) \, d\tau_e \]
\[ = -\left[ \frac{\cos(\varphi_{me} \tau_e)}{\varphi_{me}} \right]_0^1 \]
\[ = \frac{1-\cos(\varphi_{me})}{\varphi_{me}} \]

Similarly, we can get

\[ \int_0^\pi F_5(\lambda_e) \, d\lambda_e = \frac{\sin(2\varphi_{me})}{\varphi_{me}} \]

\[ \int_0^\pi F_6(\lambda_e) \, d\lambda_e = \frac{1-\cos(2\varphi_{me})}{2\varphi_{me}} \]

\[ \int_0^\pi F_7(\lambda_e) \, d\lambda_e = \int_0^\pi \sin \lambda_e \cos^2(\varphi_{me} \cos \lambda_e) \, d\lambda_e \]

let \( \cos \lambda_e = \tau_e \), as \( \lambda_e \to 0, \tau_e \to 1 \); as \( \lambda_e \to \pi, \tau_e \to -1 \). \( d\lambda_e = -\frac{1}{\sin \lambda_e} \, d\tau_e \)

\[ = -\frac{1}{2} \int_{-1}^{1} \sin \lambda_e [1 + \cos(2\varphi_{me} \tau_e)] \frac{1}{\sin \lambda_e} \, d\tau_e \]
\[
\begin{align*}
&= \frac{1}{2} \int_{-1}^{1} [1 + \cos(2\varphi_m \tau_e)] d\tau_e \\
&= \frac{1}{2} \left[ \tau_e + \frac{\sin(2\varphi_m \tau)}{2\varphi_m} \right]_{-1}^{1} \\
&= 1 + \frac{\sin(2\varphi_m \tau)}{2\varphi_m} \\
\int_{0}^{\pi} F_\theta(\lambda_e) \, d\lambda_e &= \int_{0}^{\pi} \sin\lambda_e \sin^2(\varphi_m \cos\lambda_e) \, d\lambda_e \\
&= \frac{1}{2} \int_{0}^{\pi} \sin\lambda_e [1 - \cos(2\varphi_m \cos\lambda_e)] \, d\lambda_e \\
\text{let } \cos\lambda_e &= \tau_e, \text{ as } \lambda_e \rightarrow 0, \tau_e \rightarrow 1; \text{ as } \lambda_e \rightarrow \pi, \tau_e \rightarrow 0. \ \text{d}\lambda_e = -\frac{1}{\sin\lambda_e} \text{d}\tau_e \\
&=-\frac{1}{2} \int_{1}^{0} \sin\lambda_e [1 - \cos(2\varphi_m \tau_e)] \frac{1}{\sin\lambda_e} \, d\tau_e \\
&= \frac{1}{2} \left[ \tau_e - \frac{\sin(2\varphi_m \tau_e)}{2\varphi_m} \right]_{0}^{1} \\
&= \frac{1}{2} \left[ 1 - \frac{\sin(2\varphi_m \tau_e)}{2\varphi_m} \right] \\
\text{By substituting the solutions of the integrals, the resultant full cycle averaged stability derivatives can be obtained and summarized as follow}
\end{align*}
\]

\[
\begin{align*}
\overline{T_{Xu}} &= -\frac{a_\varphi}{\pi} \left( 1 + \frac{\sin(2\varphi_m \tau)}{2\varphi_m} \right) - \frac{a_\varphi}{\pi} \left( 1 - \frac{\sin(2\varphi_m \tau)}{2\varphi_m} \right) \\
\overline{T_{Xw}} &= 0 \\
\overline{T_{Xq}} &= -\frac{a_\varphi}{\pi} \left[ h - \frac{c}{4} \sin(\alpha_m) \right] \left( 1 + \frac{\sin(2\varphi_m \tau)}{2\varphi_m} \right) \\
&\quad - \frac{a_\varphi}{\pi} \left[ h - \frac{c}{4} \sin(\alpha_m) \right] \left( 1 - \frac{\sin(2\varphi_m \tau)}{2\varphi_m} \right) \\
\overline{T_{Zu}} &= 0 \\
\overline{T_{Zw}} &= -\frac{2b_\varphi}{\pi} \\
\overline{T_{Zq}} &= -\frac{2a_\varphi \sin(\varphi_m \tau)}{\pi \varphi_m \\
\end{align*}
\]
\[
\begin{align*}
\tilde{T}_u &= -\frac{a_{18}}{\pi} \left( 1 + \frac{\sin(2\varphi_m)}{2\varphi_m} \right) - \frac{a_{19}}{\pi} \left( 1 - \frac{\sin(2\varphi_m)}{2\varphi_m} \right) \\
\tilde{T}_w &= -\frac{2a_{30}\sin(\varphi_m)}{\pi\varphi_m} \\
\tilde{T}_q &= -\left( \frac{a_{21}}{\pi} \left( -h + \frac{\bar{c}}{4} \sin(\alpha_m) \right) + \frac{a_{36}}{\pi} \left( 1 + \frac{\sin(2\varphi_m)}{2\varphi_m} \right) \\
&\quad - \left( \frac{a_{25}}{\pi} \left( -h + \frac{\bar{c}}{4} \sin(\alpha_m) \right) + \frac{a_{39}}{\pi} \left( 1 - \frac{\sin(2\varphi_m)}{2\varphi_m} \right) \right)
\end{align*}
\]