Quantum Gates using a Pulsed Bias Scheme

P. Kumar, S.R. Skinner, C. Behrman, and J. E. Steck

Department of Electrical and Computer Engineering, College of Engineering

1. Introduction

As devices shrink to the size of a few atoms, quantum effects are becoming increasingly important. Indeed, the replacement for CMOS could easily be a quantum effect gate. We demonstrate here a new scheme to realize the NOT, the Controlled-NOT (C-NOT) and the Toffoli gates, all of which are essential elements for building a universal quantum computer [1]. Given the Hamiltonian of the quantum system, we show how parameter values describing the Hamiltonian can be chosen in order to realize these gates. Then we use a pulsed bias technique to implement these gate operations. To make explicit our proposed scheme and for deriving formulae for calculating the values of parameters related to the quantum system in order to realize the desired gate operations, we have used Superconducting Quantum Interference Devices (SQUIDs) as an example of a two-level system.

2. NOT, C-NOT and Toffoli Gates

A SQUID consists of a tiny loop of superconductor with a small discontinuity known as Josephson junction. The two basis states, $|\uparrow\rangle$ and $|\downarrow\rangle$, for the two-level single-SQUID system would be those having opposite flux, and thereby, having currents circulating in opposite directions in the superconducting loop. The 2-by-2 Hamiltonian matrix of a SQUID is given as

$$H = \Delta\sigma_X + \varepsilon\sigma_Z\,, \tag{1}$$

where ε is the bias, Δ is one half the uncoupled tunneling frequency, σ_X and σ_Z are the Pauli matrices.

Given an initial state of the system, the formal solution to the Schrödinger wave equation gives us the state of the system as a function of time. The probability of the SQUID in the $|\downarrow\rangle$ state can be written as an oscillatory function in terms of the parameters Δ (tunneling) and ε (bias) as:

$$P_\downarrow = \left(\frac{1}{2} \mp \frac{\varepsilon^2}{2\left(\Delta^2+\varepsilon^2\right)}\right) + \frac{\Delta^2}{2\left(\Delta^2+\varepsilon^2\right)}\cos\left(4\pi\sqrt{\left(\Delta^2+\varepsilon^2\right)}\,t\right), \tag{2}$$

the '-' and '+' signs being used when the system starts out initially in the $|\uparrow\rangle$ and $|\downarrow\rangle$ states, respectively.

A NOT gate can be realized for a single qubit by turning the bias on and off. When $\varepsilon >> \Delta$, the qubit remains in the state it has been initialized to and hence, a *memory* state can be realized. However, when the bias is turned off, the qubit oscillates between the two basis states and is in a *transitional* state. By controlling the frequency of oscillation to correspond to an odd or even integer multiple of half-cycles, the qubit, in its transitional state, can be made to either flip its state (NOT gate) or return to its initialized state.

To realize a C-NOT gate, we consider a coupled system of two SQUIDs, A and B, which has four basis states. The Hamiltonian is a 4-by-4 matrix given as

$$\begin{aligned} H &= \Delta_A\sigma_{XA} + \varepsilon_A\sigma_{ZA} + \Delta_B\sigma_{XB} + \varepsilon_B\sigma_{ZB} \\ &\quad + \xi\sigma_{ZA}\sigma_{ZB}\,, \end{aligned} \tag{3}$$

where ξ is the coupling factor and all the other terms have the usual meaning (subscripts A and B referring to SQUIDs A and B, respectively). Under a C-NOT gate operation, the control qubit A never changes its state. By maintaining a high bias [2] on the control qubit throughout the gate operation, we can force it to remain in its memory state. The 4-by-4 Hamiltonian matrix of the two-SQUID system can now be reduced to a 2-by-2 Hamiltonian matrix of a single SQUID system, given as

$$H_B = \Delta_B\,\sigma_{XB} + (\varepsilon_B \pm \xi)\sigma_{ZB}, \tag{3}$$

which describes the true evolution of the target qubit B only. Here, the coupling term, ξ, either adds or subtracts from the bias on qubit B, ε_B, depending on whether the expectation value of σ_{ZA} is +1 or -1 in the subspace of B (control qubit A in the $|\uparrow\rangle$ or $|\downarrow\rangle$ state, respectively). The equation for the probability in the $|\downarrow\rangle$ state for the target qubit B will be as given by Eq. (2) with the bias term ε replaced by the "effective" bias term ($\varepsilon_B \pm \xi$). Thus, there are two frequencies of oscillation for the probability in the $|\downarrow\rangle$ state for the target qubit B given as

$$f_{1,2} = 2\sqrt{\Delta_B^2 + (\varepsilon_B \pm \xi)^2}\,. \qquad (4)$$

We choose these frequencies such that in one case (control qubit A in the $|\uparrow\rangle$ state) the target qubit B in its transitional state undergoes an integer number of complete oscillations and returns to its initialized state, while in the other case (control qubit A in the $|\downarrow\rangle$ state), it undergoes an odd integer number of half cycles and flips its state. *To avoid attenuation to the amplitude of oscillations, by the term ($\Delta_B^2+(\varepsilon_B\pm\xi)^2$) in the denominator, when the target qubit flips its state, it is required that ε_B and ξ cancel each other.*

From our discussion it follows that if T is the time step and M is an integer number of complete cycles in the transitional state of target qubit B:

$$f_1 = 2\sqrt{\Delta_B^2 + (\varepsilon_B + \xi)^2} = \frac{M}{T}, \qquad (5)$$

$$f_2 = 2\sqrt{\Delta_B^2 + (\varepsilon_B - \xi)^2} = 2\Delta_B = \frac{M - \frac{1}{2}}{T}. \qquad (6)$$

For different values of M and T, Eqs. (5) and (6) are used to solve for Δ_B, ξ and ε_B. The solved value of ε_B is the value to which the bias must be pulsed low in order to bring the target qubit B to a transitional state. On completing the desired number of oscillations, the target qubit is then brought back to a memory state by pulsing the bias on it to an "arbitrarily" high value.

The reduced Hamiltonian approach is next used to implement the Toffoli gate, a three-qubit gate, two of which, A and B, are control qubits and the third, C, is the target. The target qubit flips its state only when both the control qubits are in the $|\downarrow\rangle$ state. The target qubit C is coupled to each of the control qubits, A and B, which are mutually uncoupled, through the coupling terms ξ_{AC} and ξ_{BC}, respectively.

Using our reduced Hamiltonian approach, by making ε_A and ε_B large, control qubits A and B are maintained in their initialization states ($|\uparrow\rangle$ or $|\downarrow\rangle$). The 8-by-8 Hamiltonian matrix of the system can be reduced to a 2-by-2 Hamiltonian matrix, for qubit C, as:

$$H_C = \Delta_C\,\sigma_{XC} + (\varepsilon_C \pm \xi_{AC} \pm \xi_{BC})\sigma_{ZC}. \qquad (7)$$

Here, the coupling term, ξ_{AC}, (ξ_{BC}) either adds or subtracts from ε_C depending on whether the expectation value of σ_{ZA} (σ_{ZB}) is +1 or -1 in the subspace of C.

The probability of the target qubit C in the $|\downarrow\rangle$ state is given by Eq. (2) with the bias term ε replaced by the "effective" bias term ($\varepsilon_C \pm \xi_{AC} \pm \xi_{BC}$). Therefore, there are four different frequencies as follows:

$$f = 2\sqrt{\Delta_C^2 + (\varepsilon_C \pm \xi_{AC} \pm \xi_{BC})^2}\,, \qquad (8)$$

which can be reduced to three if we choose ξ_{AC} and ξ_{BC} to be equal to each other.

We require that target qubit, C, flip its state only when each of the control qubits, A and B, are in the $|\downarrow\rangle$ state, which corresponds to an odd integer number of half cycle oscillations in the transitional state of qubit C. In order to avoid any attenuation at a half cycle, we need ε_C to cancel with the sum of the two coupling terms, ξ_{AC} and ξ_{BC}, i.e., $\varepsilon_C = (\xi_{AC} + \xi_{BC})$.

For a particular time step T, we will have three equations in three unknowns Δ_C , ε_C and ξ_{AC} (= ξ_{BC}) for the three frequencies of oscillation, two of which correspond to integer numbers of oscillation cycles and one corresponds to an odd integer number of half-cycles. These can be solved for to obtain parameter values for the Toffoli gate.

3. Conclusions

We have shown in this paper how to realize quantum gates by pulsing the bias of a quantum two level system. Using the reduced Hamiltonian, we were able to solve for parameter values in order to realize a C-NOT gate and a Toffoli gate.

4. References

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[2] Y. Nakamura, *et al.*,, Nature, **398**, 786 (1999).
[3] T. Yamamoto, *et al.*, Nature, **425**, 941 (2003).